

A General-Equilibrium Closed-Form Solution to the Welfare Costs of Inflation*

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Summary: 1. Introduction; 2. The model; 3. A closed-form solution for the welfare costs of inflation; 4. A direct comparison with Bailey's measure; 5. Comparing the general-equilibrium and the partial-equilibrium measures.

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This article presents a closed-form solution to Lucas's (2000) general-equilibrium expression for the welfare costs of inflation. The formula applies when the money demand is bi-logarithmic. An analytical solution for the difference between Bailey's (1956) partial-equilibrium measure and Lucas's general-equilibrium measure is also provided. In Lucas's original work, only numerical solutions are offered to these questions.

Este artigo apresenta uma fórmula fechada para o cálculo do custo de bem estar da inflação proposto por Lucas (2000) em um modelo de equilíbrio geral. A fórmula aplica-se quando a demanda por moeda é do tipo bi-logarítmica. O artigo deduz também uma fórmula que permite calcular analiticamente a diferença entre os custos de bem estar da inflação em equilíbrio parcial (fórmula de Bailey) e em equilíbrio geral. O trabalho original de Lucas prevê apenas soluções numéricas para estes cálculos.

1. Introduction

In this paper I derive a closed-form solution to Lucas's general-equilibrium expression for the welfare costs of inflation when the money demand function is double-logarithmic.¹ Next, I use this closed-form solution to derive an expression

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¹Lucas (2000) argues that this is the functional specification of the money demand that best fits the United States historical time series.

which delivers, also in closed-form, the difference between the general-equilibrium and Bailey's (1956) partial-equilibrium measure of the welfare costs of inflation.

In Lucas's (2000) original paper, both the solution of the underlying nonlinear differential equation leading to the general-equilibrium welfare figures, as well as the comparison with Bailey's estimates, are based only on numerical methods.

This article is divided as follows. Section 2 presents a continuous-time, no-growth version of Lucas's shopping time model. Given the correspondent interpretation of the variables in each case, both the discrete and the continuous approach, with or without growth, lead to the same non-linear differential equation describing the welfare costs of inflation (equation 5.8 in the original paper and equation (6) in section 2 of this article). We therefore present the continuous-time no-growth model for the sake of simplicity in the exposition, with no loss in generality.

2. The Model

In Lucas's (2000, sec. 5) analysis of the welfare costs of inflation the representative consumer is supposed to maximize utility from the consumption (c):

$$\int_0^{\infty} e^{-gt} U(c) dt \quad (1)$$

subject to the households budget constraint (2) and to the transactions-technology constraint (3):

$$\dot{m} = 1 - (c + s) + h - \pi m \quad (2)$$

$$-c + m\phi(s) \geq 0 \quad (3)$$

In these equations, s stands for the fraction of the initial endowment spent as transacting time (the total endowment of time being equal to the unity), m for the real quantity of money, π for the rate of inflation, $U(c)$ for a concave utility function, h for the (exogenous) real value of the flow of money transferred to the household by the government, $g > 0$ for a continuous-time discount factor (Lucas uses $1/(1 + \rho)$ for the discrete case) and $F(m, s) = m \phi(s)$, $\phi'(s) > 0$, for the transacting technology.

Intertemporal optimization leads to the first order condition:

$$\phi(s) = rm\phi'(s) \quad (4)$$

Equilibrium in the goods market reads:

$$1 - s = m\phi(s) \quad (5)$$

In the steady-state solution m converges to a constant figure, the rate of interest r equals the rate of inflation plus the discount factor ($r = \pi + g$), the inflation equals the rate of monetary expansion and the real transfers (h) equal the inflation tax ($h = \sigma m$, σ standing for the rate of monetary expansion).

Solving the system given by (4) and (5) for $s = s(r)$ and $m = m(r)$ yields $s'(r) > 0$ and $m'(r) < 0$. The problem of deriving $s(r)$ from $m(r)$ without knowing $\phi(s)$ is solved by eliminating $\phi(s)$ and $\phi'(s)$ using (4) and (5). The result is the differential equation (Lucas (2000, equation 5.8)):

$$s' = - \frac{r(1-s)}{1-s+r} \frac{m'}{m} \quad (6)$$

which determines the welfare cost $s(r)$ as a function of the money-demand $m(r)$.

Lucas (2000) argues that the double-logarithmic functional specification fits the United States data better than the alternative semi-log specification. Making $m(r) = Ar^{-a}$, $0 < a < 1$, $A > 0$, (6) leads to:

$$\frac{ds}{dr} = v(r, s) = \frac{(1-s)(aAr^{-a})}{1-s+Ar^{1-a}} \quad (7)$$

$$s(r_0) = s_0, \quad r_0 > 0 \quad (8)$$

Lucas does not provide a closed-form solution to this equation. His welfare figures, as well as his comparison with Bailey's measure, are based on numerical calculations.

3. A Closed-Form Solution for the Welfare Costs of Inflation

I start the formal analysis by demonstrating existence and uniqueness.

Proposition 1 *Consider s and r in a closed, bounded and convex region $D \subset \mathbb{R}_{++}^2$, with r bounded away from zero. Then there exists a unique solution to (7) and (8).*

Proof It is easy to see that, with r bounded away from zero, $v(r, s) \in C^1$, and, by the mean-value theorem, and for a certain constant $L > 0$, satisfies the Lipschitz condition $|v(r, s_1) - v(r, s_2)| \leq L |s_1 - s_2|$ for each pair $(r, s_1), (r, s_2)$ in D .

It follows from a standard result in ordinary differential equations based on the contraction mapping theorem (see, e.g., Coddington and Levinson (1955)) that there exists an interval containing r such that a solution to (7) exists, and that this solution is unique. It is then immediate that such a solution can be continued to the right to a maximal interval of existence $[r_0, +\infty)$. ■

Even though existence has been easily proved in Proposition 1, it is by no means clear that this non-separable, non-linear differential equation presents a closed-form solution. For example, it is well known that a simple equation like $\frac{ds}{dr} = w(r, s) = s^2 - r$ cannot be expressed as a finite combination of elementary functions or algebraic functions and integrals of such functions. I shall show, next, that such a problem does not happen with (7) and (8).

Proposition 2 *The solution to (7) and (8) is given by*

$$r = \left[\frac{a-1}{A} (1-s) \left[1 - (1-s)^{-1/a} \right] \right]^{\frac{1}{1-a}} \quad (9)$$

Proof Start by considering $r_0 > 0$ and the initial condition

$$s(r_0) = s_0 \quad (10)$$

Suppose $s(r)$ is a solution to (7), given (10). Then, since $s'(r_0) > 0$, the inverse function $r = r(s)$ is defined in a sufficiently small neighborhood of the point s_0 and:

$$\frac{dr}{ds} + \frac{-1}{a(1-s)}r = \frac{1}{aA}r^a \quad (11)$$

This type of equation is generally called a Bernoulli equation, which can be easily solved by an adequate change of coordinates. Consider the diffeomorphism that associates with each $r > 0$, $t = r^{1-a}$. Then (11) is equivalent to the equation:

$$\frac{dt}{ds} - \frac{(1-a)}{a(1-s)}t = \frac{1-a}{aA}$$

Multiplying both sides of this equation by the integration factor $\exp(-\int_0^s \frac{1-a}{a(1-\phi)}d\phi)$:

$$\frac{d}{ds} \left[t \exp(-\int_0^s \frac{1-a}{a(1-\phi)}d\phi) \right] = \frac{1-a}{aA} \left[\exp(-\int_0^s \frac{1-a}{a(1-\phi)}d\phi) \right]$$

Integrating in s and using the fact that $t(0) = 0$:

$$t \exp\left(-\int_0^s \frac{1-a}{a(1-\phi)} d\phi\right) = \int_0^s \frac{1-a}{aA} \exp\left(-\int_0^g \frac{1-a}{a(1-\phi)} d\phi\right) dg$$

Solving for the integral of $1/(1-\phi)$:

$$t = \frac{a-1}{A} (1-s) + \frac{1-a}{A} (1-s)^{\frac{a-1}{a}}$$

Use the fact that $t = r^{1-a}$ to get (9). ■

4. A Direct Comparison with Bailey's Measure

Lucas provides numerical simulations in order to compare his general-equilibrium measure (6) and Bailey's partial-equilibrium measure (B) of the welfare costs of inflation. Having obtained a closed-form solution for the former allows us to provide a closed-form expression for the difference between these two measures.

Proposition 3 *The difference between the general-equilibrium (s) and Bailey's partial-equilibrium (B) measure of the welfare costs of inflation is given by:*

$$B - s = a(1-s)\left(-1 + (1-s)^{\frac{-1}{a}}\right) - s \quad (12)$$

Proof Bailey's measure, in differential form, is given by the area-under-the-inverse-demand-curve:

$$dB = -rm'(r)dr, \quad B(0) = 0$$

By substituting the double-logarithmic money demand function into the above expression and integrating:

$$r = \left(\frac{B(1-a)}{aA}\right)^{\frac{1}{1-a}} \quad (13)$$

Solve (13) for B and use the value of r given by (9) to obtain (12). ■

5. Comparing the General-Equilibrium and the Partial-Equilibrium Measures

Both Lucas (2000), through numerical simulations, and Simonsen and Cysne (2001), analytically, have shown that Bailey's measure is an upper bound to Lucas' general-equilibrium measure, and that the difference between B and s in an increasing function of s . A final Proposition shows that both conclusions are consistent with equation (12).

Proposition 4 $B(s) \geq s$ and the difference $B(s) - s$ is an increasing function of s .

Proof Make $B(s) - s = g(s)$. Then, $g(0) = 0$ and

$$g'(s) = \left[(1-s)^{-\frac{1}{a}} - 1 \right] (1-a)$$

Hence, $g'(s) > 0$ for any $s > 0$. It follows that $B > s$ for any strictly positive values of s and that the difference $B - s$ increases with s . ■

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