NUMBERS AND SETS [1]

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RESUMO *Neste artigo discuto a intuição subjacente à definição de números como conjuntos proposta por Frege e Russell, assim como a crítica de Benacerraf a esta definição. Eu tento mostrar que o argumento de Benacerraf não é tão forte como alguns filósofos o tomaram. Adicionalmente, examino uma alternativa à definição de Frege e Russell proposta por Maddy, e indico algumas dificuldades encontrada pela mesma.*

ABSTRACT *In this paper I discuss the intuition behind Frege's and Russell's definitions of numbers as sets, as well as Benacerraf's criticism of it. I argue that Benacerraf's argument is not as strong as some philosophers tend to think. Moreover, I examine an alternative to the Fregean-Russellian definition of numbers proposed by Maddy, and point out some problems faced by it.*

Palavras-Chave *Números, conjuntos, logicismo, ontologia, aritmética*

In this paper I shall be concerned with a fundamental ontological question about mathematical objects, namely, the relation between numbers and sets. I will first discuss the natural intuition for identifying numbers with sets: I will review some well known aspects of Frege's and Russell's logicism, and

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how this natural suggestion inspires both philosophers. Moreover, I will briefly explain how numbers are treated within the axiomatic set theory of Zermelo Fraenkel (ZF). Next, I will explain a particular argument made by Paul Benacerraf against the identification in question. As I shall argue, Benacerraf's argument is not as decisive as some philosophers have taken it to be, and it seems to beg the question against the realist in some relevant respects. At any rate, his criticism inspired an alternative approach for numbers developed in the work of Penelope Maddy, which I shall briefly review. I will argue that this alternative approach has some problems of its own, which makes it less attractive than the Fregean-Russellian set theoretical definition.

I-Numbers as Sets

As I said above, there is a natural suggestion, which was the basic intuition behind the definition of numbers proposed in Frege's and Russell's logicism, namely, that numbers just *are* sets. It is well known that Frege's main philosophical project in *Grundlagen der Arithmetik* (1884, from now on simply *Grundlagen*) is showing that arithmetic as such is nothing but a further developed branch of logic. Most of his arguments for this view come from the observation that arithmetic has a range of applicability that far surpasses the range of any other science. Any empirical science is restricted in its applicability to things that are real, and that are located in space and time. Geometry has a wider applicability, since it applies not only to what is real, but also to anything that might be conceived of, as long as it is spatial. But it is also restricted to things that are possible objects of a spatial intuition. Arithmetic, on the contrary, applies also to things that are not in space: we can count things like concepts, dreams, ideas, souls, God, and so on. In a word: arithmetic is applicable to everything that is thinkable. Hence, if it goes as far as thought goes, and if logic gives the basic laws of thought, then arithmetic is as widely applicable as logic.

One consequence of this view is that the basic entities of arithmetic, if there are any, must be of a logical nature. I shall not go into all details of Frege's arguments in *Grundglagen* here. But it seems clear to him that there are some basic objects of arithmetic, namely, numbers, since we use numerals in some basic sentences (i.e., equalities) that are true. Now one of the principles of Frege's thought, the so-called context principle, says that questions of existence of abstract entities have to be solved by paying attention to sentential contexts where putative names for these entities are being employed: if we employ numerals as singular terms in sentences that are true, then we have to admit that the corresponding entities (numbers) exist, and that they are objects. There is actually a different sort of grammatical evidence for Frege's claim that numerals are singular terms, namely, the fact that they are usually preceded by the definite article like, e.g., in 'the number 3'. As Frege sees it, the presence of the definite article indicates that the expression is meant as referring to one and only one object. I shall skip here a deeper discussion of whether Frege is right in this assumption, or whether he unduly dismisses other equally appealing grammatical evidence for treating numbers as second order concepts rather than objects.[2]

Now there was a challenge for Frege: On the one hand, he had to say which objects numbers are in such a way that their logical nature had to become evident. On the other hand, the account of numbers that Frege was after had to provide an explanation of why numbers are universally applicable. If Frege had simply stated that numbers are logical objects, there would be, strictly speaking, no novelty in his philosophy in comparison to Leibniz's philosophical views. It follows that there must be some logical objects to which numbers are reducible, and this reduction must be well motivated. Now, for Frege, concepts are the most basic kind of logical entities. However, due to their predicative nature, they are essentially distinct from objects, and hence cannot be the adequate candidates for numbers. Truth-values are objects in Frege's ontology, but it is hard to imagine that numbers could be reduced to them. For this reason Frege says in a note to an article by Jourdain from 1910 that "our first aim, then, was to obtain objects out of concepts, namely, extents of concepts or classes" (*Kleine Schriften*, p. 339).[3] That is to say, according to him, we must consider numbers as being reducible to the extension of concepts.

Frege's notion of extension of concepts corresponds in some aspects to the modern idea of sets, but there are some important differences. For example, in the most popular set theories nowadays there is an implicit notion of sets according to which theses are any collections of previously given objects, while for Frege a set can only be seen as the extension of some previously given concept. Another difference is that, for Frege, any concept has an extension, while today, after Russell's paradox, we know that some concepts have no extension. It follows that, in Frege's conception of extensions, the axiom of foundation, which we take nowadays to be true of sets, is almost everywhere violated.

It seems that the broader reasons for identifying numbers with extensions (sets) in Frege's work is relatively clear. But which extensions? In a letter

2 See Dummett (1991), chapter 9, for a discussion of this point.
3 I quote here from Jourdain's own translation of Frege's notes.

to Karl Zsigmondy,[4] we find the clearest exposition of the basic intuition that led Frege to the choice that he ultimately made:

> Generally each number belongs to several aggregates. Hence, a natural suggestion is to divide the aggregates in classes, so that all aggregates that have the same number are gathered together in the same class. In this way, to each number corresponds a class of aggregates, and to each of our classes a number. Different numbers correspond to different classes as we have them, and different classes correspond to different numbers. What else do we know about the numbers except the fact that we can recognize the same number again, and that we can distinguish different numbers? The same is true of our classes. It is very compelling to say: our classes are numbers and numbers are classes of aggregates. We completely eliminate in this way the distinction between numbers and our classes. Don't we have in this way everything that we need? (*Wissenschaftlicher Briefwechsel*, p. 271)

We have the strong feeling that these aggregates have something in common, and this feeling comes from the fact that any two of them can be put into a one-to-one correspondence. So, they are all equinumerous. Hence we could, on the one hand, choose anyone as having the same number of objects as any other but, on the other hand, we also feel that there is no one in particular that we could choose as being *the* number 10. So, none of them in particular is the number 10, but all of them have something to do with the number 10. This strongly suggests, as Frege concludes, that the number 10 is all of them at once, i.e., the number 10 is the set of all 10-membered sets. Frege's definition of number is formulated in *Grundlagen* § 68 in the following way: the number that belongs to the concept *F* is the extension of the concept *equinumerous with F*. The individual numbers are so defined: 0 is the number that belongs to the concept $x \vee x$; $\vee$1 is the number that belongs to the concept $x = 0$; 2 is the number that belongs to the concept $(x = 0 \vee x = 1)$; ... n+1 is the number that belongs to the concept $(x = 0 \vee x = 1 \vee ... \vee x = n)$.

From our modern perspective, there is no such thing as the set of all 10-membered sets, as there is no set of all singleton sets, as there is no set of all n-membered sets for any finite or infinite n. (If these collections were sets, then by the union axiom, their arbitrary union would be a set. But this is simply the set of all sets, which cannot be a set, for it would have to include its own power set, and hence be larger than it actually is, but no set is larger than the set of all sets.) These collections would be too large to be sets, and they are more appropriately treated as proper classes. I will come back to this point later.

4 As the editors of Frege's philosophical correspondence explain, the letter has no unequivocal date, although it was dated as "after 1918" by Scholz.

For reasons that need not concern us here, Frege's definition did not work well, since the formal development of it in *Grundgesetze der Arithmetik* (from 1893, from now on simply *Grundgesetze*) was based on an axiom that turned out to be inconsistent.[5] But it would be inaccurate to conclude, from Frege's failure, that numbers are not to be seen as sets (or, as he would prefer, as extensions). In the appendix to *Grundgesetze* vol. II (1903), written shortly after the discovery of Russell's paradox, Frege still proclaims that, even if this particular reduction of numbers to extensions may not have proven as successful as he thought, the way was open for an adequate "scientific foundation" of arithmetic, i.e., he thought that some other reduction of numbers to sets should be sought.

As Frege sometimes indicates (even before the discovery of Russell's paradox), it was actually not without some reluctance that he was led to regard numbers as extensions, and to introduce the infamous Axiom V of *Grundgesetze*. But there was also a general methodological imperative behind his choice of extensions as the ontological basis of arithmetic. This was seen by him as the only possible way of giving a logical foundation to arithmetic, as he explains in his letter to Russell from July 28, 1902 (*Wissenschaftlicher Briefwechsel*, p. 223). Indeed, as it seems, Frege might have preferred defining numbers as concepts, since this would not need the introduction of any sort of object whatsoever as numbers.[6] But this way of proceeding was not seen by him as methodologically safe. This stems from his preference for the extensionalist over the intensionalist approach to logic. He says in several places (e.g., in *Nachgelassene Schriften*, p. 133) that, although extensionalist logicians are wrong when they identify concepts with their extensions, they are right, however, in showing a preference for extensions. Why should this be seen as a methodological advantage? There is a historical reason for this view. By the time Frege formulated his logicism, there was an intense debate among German logicians and mathematicians between the so-called *Umfangslogiker* (extensionalists) and the so-called *Inhaltslogiker* (intensionalists). *Umfangslogiker* (like, e.g., Schröder) were those who advocated a way of doing logic that was very close to a pure algebra of classes. The *Inhaltslogi-*

5 This is Frege's Axiom V, which says that the extension (or, more generally, value range) of two concepts (functions) is identical if and only if they yield the same value for any object as argument. As Russell communicated the discovery of the paradox in a letter in 1902, Frege immediately recognized that it could be derived within his own system, and that Axiom V was responsible for it.

6 This is actually not quite as simple as I put it here. If numbers are second order concepts, then Frege needs no objects playing the role of numbers, but there is no guarantee that any number $n+1$ exists unless there are $n+1$ objects in the universe. That is to say, if numbers are not themselves objects, then the existence of infinite natural numbers would presuppose the existence of countably infinite objects in the universe. For a discussion of this point, see Dummett (1991, p. 132) and my Ruffino (1998, p. 157).

ker (like Lotze and Husserl) advocated the thesis that logic is concerned with more than an algebra of classes, i.e., logic is concerned with the content specific to concepts. Now in a way Frege endorses the intensionalists' position, since logic for him is a science dealing primarily with concepts. But, on the other hand, the notion of content implicit in most of the intensionalists' works was most of the time strongly psychological. But logic should by all means be kept apart from psychology for him. Hence, according to Frege, although logic deals primarily with concepts and their contents, the safest way of doing so without slipping into psychologism is to treat concepts via their extensions.[7]

Russell has essentially the same intuition as Frege. In chapter two of *Introduction to Mathematical Philosophy* (1919, from now on simply *IMP*) he searches for the correct ontological nature of numbers by paying attention to some general aspects of them. As he explains, a number *n* is a way of bringing together all classes of *n* things, i.e., the number two brings together all pairs, the number three all trios, and so on. We know that all things in these classes are equinumerous without knowing previously what the numbers two, three, and so on are, since being equinumerous, for two given classes *a* and *b*, simply means that there is a one-to-one correlation between the elements of *a* and the elements of *b*. But despite the similarity of positions, Russell is not as resolute as Frege in defining the number *n* as the class of all classes of n things. He makes the following somewhat obscure remark:

> It is [...] more prudent to content ourselves with the class of couples, which we are sure of, than to hunt for a problematical number 2 which must always remain elusive. (*IMP*, p. 18)

That is to say, Russell does not seem to think that he has grasped the real nature of numbers by defining them as classes, as Frege does. But, according to him, this is the best approximation that we can get to the nature of numbers that is philosophically respectable.

We have nowadays, within the context of axiomatic set theories, some alternative definitions of natural numbers as sets different from those definitions proposed by Frege and Russell. The best known treatment is the standard one in ZF, in which the natural numbers are defined as follows (where '∅' stays for the empty set, and 'S ' for successor):

7 The titles of the papers published in two of the most influential journals of philosophy in Germany in the 1890s, the *Vierteljahrschrift für wissenschaftliche Philosophie* and the *Zeitschrift für Philosophie und philosophische Kritik*, very often suggest some sort of psychologistic approach to concepts. This is the sort of approach that, I think, Frege was trying to avoid by endorsing the extensionalist methodology.

0 is the set $\varnothing$
1 is the set $\{\varnothing\}$
2 is the set $\{\varnothing,\{\varnothing\}\}$
3 is the set $\{\varnothing,\{\varnothing\}, \{\varnothing,\{\varnothing\}\}\}$
...
S(n) is the set $n\cup\{n\}$

We can prove induction and recursion theorems for the set ϖ of all natural numbers from the axioms of ZF. We can also define a linear ordering on ϖ in the following way: $m<n$ iff $m\in n$. Using recursion, we define addition and multiplication on natural numbers. We can also define the integers as equivalence classes of pairs of natural numbers, the rational as equivalence classes of pairs of integers, and finally the real numbers as Dedekind cuts, with all corresponding operations. In a word, we can formulate (and proof) the whole arithmetic and analysis within ZF (actually, in ZFC, which is ZF plus axiom of choice).

It is worth noticing a fundamental difference between the set theoretical approaches reviewed here. For Frege, the existence of infinite numbers was a consequence only of the way numbers are defined. No previous existence had to be assumed as a guarantee that infinite numbers exist. Indeed, if there are zero objects, then the number zero exists, since it is defined as the number belonging to the concept $x \vee x$, i.e., as the extension of the concept *equinumerous with* $x \vee x$. By Axiom V, this extension exists. If zero exists, by the same reasoning 1 exists, since it is defined as the extension of the concept *equinumerous with the concept* $x = 0$. And so on. In Russell, numbers are not so defined as implying their own existence. Indeed, as he explains in chapter XIII of *IMP*, there is no guarantee that, for an arbitrary n, there are classes with n elements. But if there are no such classes then number n is the empty class, and the number $n+1$ must also be the empty class, and therefore $n=n+1$, which violates Peano's axioms. Hence, in order to guarantee the existence of infinite natural numbers with the desired properties, Russell needs the Axiom of Infinity, which postulates the existence of infinite objects in the universe. In ZF the existence of an infinity of numbers is also guaranteed by an axiom, which says that there is at least one inductive set, i.e., a set containing $\varnothing$ as element, and for every x, if x is an element of this set, then the successor of x (i.e., $x \cup \{x\}$) is also an element of it. The set of natural numbers is then defined as the intersection of all inductive sets. The reason why Frege did not need anything like an axiom of infinity is that his Axiom V allowed the "transformation" of concepts into objects, and thereby he has a supply of as many objects as there are concepts. The hardest task involved in Frege's definition was that of finding concepts that are adequate from a mathematical point of view; but as to their existence, he could get it for free from Axiom V.

II-Benacerraf's Problem of Multiple Reductions

In a sequence of two famous articles, Paul Benacerraf has posed some challenges for the realist account of numbers as sets. In the first one (1965), Benacerraf raises some difficulties for the idea that there is one particular correct identification of numbers as sets. Actually, as we shall see, Benacerraf wants his argument to have a broader impact, and to challenge the very idea that there is a correct ontological reduction of numbers at all. In the second article (1973), Benacerraf argues that there are two apparently incompatible tasks to be fulfilled by any philosophical account of mathematics: The first task is that of providing a correct account of truth for mathematical statements that does justice to the fact that they have, at least on the surface, the same syntactical form assumed by ordinary statements. (If we say, for example, 'There are at least two prime numbers between four and ten', this sentence seems to call for the same kind of explanation of its truth-conditions that we would provide for a sentence like 'There are at least tree streets named 'Broadway' in Boston'.) The other task is an account of our knowledge of these statements. If we explain the truth of mathematical statements in terms of their correspondence with an arrangement of objects and relations in the world (as we do with ordinary empirical statements), then there is a deficit on the epistemic side, for we are, according to Benacerraf, incapable of giving an account of our knowledge of these mathematical facts. On the other hand, if we account for our mathematical knowledge in terms of things that are familiar to us (proofs, conventions, intuitions, etc.) then there is a difficulty in explaining why the statements that we take to be true are *true*. Although I find it flawed in some fundamental aspects, I shall forgo a deeper discussion of this argument here. I want to concentrate instead on the points that Benacerraf makes in the first article, since they are more directly relevant to the main question of this paper, namely, the ontological relation between numbers and sets.

Benacerraf's argument in the first article starts with the claim that, if numbers are sets, there must be an answer as to *which* sets they are. Now it is well known from set theory that there are some possible reductions of numbers to sets that are satisfactory. There is, for example, Zermelo's account, for which

0 = Ø
1 = { Ø }
2 = {{ Ø }}
...
n+1 = {n}

with the corresponding definitions of elementary operations; and there is also the possibility of identifying numbers with von Neumann's ordinals, which is essentially the standard definition in ZF that we reviewed in section I. But if we consider, say, the number 3 in each of these accounts, we get different sets. There is further disagreement. For one of these approaches (von Neumann's), a number *m* is a member of any larger number *n*, while for the other (Zermelo's), *m* is a member only of its successor. Successor itself has different definitions: $S(n) = n \cup \{n\}$ for one approach, and $S(n) = \{n\}$ for the other. The explanation of cardinality is also different for each one of the approaches. In von Neumann's approach, a set has cardinality *n* if and only if it can be put into a one-to-one correspondence with the number *n*, but this explanation would be wrong in Zermelo's approach, since here all numbers are singleton sets.

Now, Benacerraf claims, both accounts seem to be correct in that they both satisfy conditions that seem to be necessary (and possibly) sufficient for correctness. These conditions are the following, according to him: (i)-a correct account should provide definitions of '1', 'number' and 'successor', and of the operations '+', 'x', so that the basic laws of arithmetic can be derived; (ii)-it should also provide an explanation of the applications of numbers to other non-numerical entities, that is to say, an explanation of cardinality and cardinal numbers (1965, p. 277). But if both accounts are right from this perspective, and if they are nevertheless different, then the difference must be in some aspect that is non-essential. Hence, as Benacerraf concludes, to be identified with this or that sequence of sets is not essential for numbers.

From these considerations, Benacerraf concludes that numbers are not sets, for if they were sets we should be able to say *which* sets they are. Moreover, according to him, numbers are nothing at all in particular. Any ϖ-sequence of objects can be the sequence of natural numbers, as long as the successor relation, and the other relevant operations on numbers, are properly defined. The only major restriction that Benacerraf places is that the ordering defined over the elements of the arbitrary ϖ-sequence should be recursive. In a more recent text (1996), Benacerraf changes his mind and drops even this minimal requirement: there is no reason anymore, so he thinks, for the ordering to be recursive. Any ϖ-sequence would do it, according to him.

III-Numbers as Properties of Sets

There is an alternative approach that, on the one hand, combines set theoretical realism with mathematical Platonism and, on the other hand, seems to be able to avoid Benacerraf's problem of multiple reductions. This approach was first proposed by Penelope Maddy in a paper (1981) and later developed in her book (1990), in the context of a reconstruction of set theoretical realism in naturalistic terms. Maddy is fully convinced of the force of Benacerraf's argument as showing that, since numbers can be identified with more than one ϖ-sequence, we cannot hold anymore that they are sets. But she still wants to retain part of the intuition that guided Frege in his definition of cardinal numbers, namely, the idea that a numerical statement says something about a concept. According to Frege, when we say 'There are three chairs in this room', we are saying something not about the objects in this room, but about the concept *chair in this room*, namely, that it has three instances (*Grundlagen* §§ 46-52). Based on this observation, Frege toyed with the idea that, since this is the case, numbers may be second order properties (concepts) after all. Indeed, before proposing the definition of numbers in terms of extensions of concepts in *Grundlagen* § 68, he presents in § 55 an attempted definition of numbers as second order concepts, or, better said, as part of numerical quantifiers (i.e., of expressions of the form 'there are *n xs* such that... '), which he ends up rejecting as inadequate.[8] But instead of properties of concepts, Maddy believes that numbers should rather be seen as properties of sets. Consider three different sets: the set of five books on the table in front of me, the set of fingers in my right hand, and finally the set of stars in the Southern Cross. Maddy claims that, although it is metaphysically wrong to say that these sets taken altogether are the number five, they have, nevertheless, something to do with the number five: being five-numbered is instantiated by all of them. Hence, Maddy suggests that the number five should rather be seen as a property shared by all these sets. Numbers are indeed, according to her, fundamental properties of sets, and set theory involves the study of properties of sets in the same way that physics, for instance, involves the study of fundamental properties of physical bodies like length and temperature.

8 The reasons for Frege's rejection of this idea are not quite clear. He concentrates on the claim that this approach does not do justice to the fact that numerals are employed in equalities, and hence must be proper names. But there seems to be a further reason for Frege's rejection, even if he is not explicit about it: If numbers were second-order concepts, then the existence of a number *n+1* would depend on the existence of n objects. And since numbers are not themselves these objects, then arithmetic would depend, for its truth, on the previous existence of non-arithmetical (i.e., non-logical) objects, which would be unacceptable for Frege. For a more detailed discussion of Frege's reasons, see Dummett (1991) and Ruffino (1998).

Maddy's alternative seems to avoid Benacerraf's difficulty, in that there is no question of identifying numbers with particular sets anymore. Instead, sequences of sets like von Neumann's ordinals or Zermelo's numbers can be seen as different standards for measuring the number-property of sets, in the same way that different yardsticks can be seen as different (though equivalent) instruments for measuring length, and there is no saying that one of them "is" length. As she comments,

> [W]hen Benacerraf tells the story of two youngsters who learn von Neumann and Zermelo versions of number theory, their dispute over whether or not 3Î17 is analogous to an imagined argument over whether an inch is plastic or wooden. (1981, p. 507)

Now Maddy's program involves some immediate difficulties, as she herself recognizes. The first difficulty is that, as it seems, if numbers are properties of sets, and not sets themselves, then we have to recognize two kinds of basic entities in set theory (sets and properties), and not just one (sets), and this seems to require the introduction of a different kind of variable, as well as of new axioms for properties. The second difficulty arises when it comes to the individuation of these properties, for although there is no correct identification of numbers as sets anymore, there is nevertheless a correct identification of numbers as properties. So we have to find out what are the correct properties, and how they are to be distinguished from other properties. How are properties to be individuated from one another if not through their extensions? Frege thought that the analogue of identity for concepts is given by coextensiveness, i.e., two concepts are "identical" if and only if they have the same extension, but Maddy rejects this alternative claiming that it is usually wrong that coextensional properties are the same. How can we say, for example, that the property of being equinumerous with {Ø,{ Ø}, {{ Ø}}} is the same as the property of being equinumerous with { Ø, { Ø}, { Ø,{ Ø}}}? Maddy appeals at this point to the notion of nomological coextensiveness, which the above mentioned predicates provably have.

But a more serious worry arises in connection with the picture that Maddy has in mind regarding number theory as a science. How are the number-properties to be investigated? This is the question raised and answered in the following passage:

> How do properties usually appear within formal set theory? Answer: via their extensions. One doesn't speak of 'being a prime' or 'being a real number', one speaks of 'the set of primes' and 'the set of real numbers'. This is why I said I was inclined to agree with Benacerraf that a property view lends some support to Frege's identification of numbers with sets of equinumerous sets. Of course, the problem is that the

> extension of being 3-membered is not a set, but a proper class. So, I think that the problem of understanding the role of number properties within formal set theory is a special case of the problem of understanding the role of proper classes. (1981, p. 508)

As she remarks here, number properties are actually to be studied via their extensions, and this can be carried out by developing a theory of proper classes. (Maddy herself developed such a theory in Maddy (1983).) Her strategy seems to be very much in the spirit of Frege's defense of extensionalism in logic. As we saw, for Frege, the extensional approach to concepts is not just an option, but a condition for being scientific. Maddy's remark here seems actually to reinforce the Fregean alternative for a definition of numbers (if the definition is properly amended with the qualification that numbers are not to be seen as sets of sets, but as proper classes of sets) instead of carrying some strength to her own alternative. For her point was that we are better off if, instead of treating numbers as sets, we regard them as properties of sets. But now the suggestion is that these properties have to be studied through their extensions, and their identity is to be given by nomological coextensiveness. But then it is not clear exactly what is gained by preferring this detour, instead of simply identifying numbers with proper classes, and claiming that understanding the role of numbers is a special case of understanding the role of proper classes. In other words, it is not clear why Maddy's approach should be more attractive than the amended Fregean approach (except for the fact that is apparently avoids Benacerraf's problem).

III-Benacerraf's Argument Reconsidered

Now I want to go back to Benacerraf's argument and critically ask how much it really achieves. The conclusion of his argument seems *prima facie* too strong: From the fact that there are more then one successful reduction of numbers to sets he concludes that numbers could not be sets. But it is not exactly clear why he thinks that two successful definitions of numbers to sets prevent them from being sets. Independently of this, however, I think that Benacerraf's argument is not as impressive or conclusive as some philosophers (like Maddy) have taken it. In order to better understand the possibility of multiple reductions of numbers to sets and what exactly this amounts to, it might be of some help to compare this case with the analogous one of ordered pairs. Ordered pairs are entities, and we find it compelling that they should be reducible to sets. (At least this is how we learn about ordered pairs in ZF.) Now it is well known that there are more than one possible definition of ordered pairs as sets that satisfy the condition of adequacy given by

<a,b>=<c,d> iff a=c and b=d.

We have, for example, Kuratowski's definition (which became standard), i.e., <a,b>={{a}, {a,b}}, but also Wiener's definition, i.e., <a,b>={{{a},Ø }, {{b}}}. This seems to present an analogous case to the one of numbers as sets, and if we were to apply Benacerraf's reasoning here, we should say that this possibility of multiple reductions is sufficient to show that ordered pairs are not sets.[9] And, if ordered pairs cannot be sets, following Benacerraf's conclusion, neither can functions and relations be sets, for they are defined in standard ZF set theory as sets of ordered pairs, neither can other entities like Peano's systems be sets, etc. At this point I just want to call attention for the fact that Benacerraf's conclusion, if correct, would have a far greater impact on our beliefs than the restricted one about numbers as sets, since a whole group of things that are normally treated as sets could simply not be sets.

It is doubtful, however, that Benacerraf's argument of multiple reductions is a decisive argument against a realist philosopher that wants to hold on to the idea that numbers are sets. I do not intend here to defend the view that Frege's intuition was right and numbers are indeed sets, but rather to point out what seems to me to be a basic weakness of Benacerraf's argument. It seems that his argument gains its apparent force from not taking the realist's perspective seriously enough. But it is not hard to see how a realist could resist Benacerraf's claim in a surprisingly simple way. Faced with the possibility of multiple reductions that Benacerraf mentions, two different reactions are possible: one is to discredit, as Benacerraf and Maddy do, the idea that numbers are particular sets. The other one is to consider the different possible reductions as different working hypotheses, each one trying to describe a reality of numbers as sets existing independently of our theories. There might be small differences between the approaches that are not, strictly speaking, essential for each one of them to derive the laws of arithmetic. But these small differences make one of them more practical, simple and elegant. We know that von Neumann's definition has several advantages over Zermelo's. Now Benacerraf is certainly aware of this fact. But he does not consider these advantages as being something that matters. For all that matters for him is that the different approaches satisfy the criteria of correctness, namely, they both provide definitions of the basic predicates ('number', 'successor', 'one', etc.) and an explanation of cardinality.

9 Kitcher (1978) concludes from this fact that set theory needs two kinds of entities, namely, sets and also functions. Maddy (in personal communication) also thinks that this fact alone is enough to show that ordered pairs are not sets, although we may use set theoretical counterparts of ordered pairs that satisfy the condition of adequacy for mathematical purposes.

And, in so doing, Benacerraf disregards a way of thinking characteristic of realist philosophers. In what follows, I will present three examples that illustrate this way of thinking.

We could contrast Benacerraf's view with Frege's pragmatic view expressed in two different passages of the introduction of *Grundgesetze.* The first passage is one in which Frege justifies his highly controversial thesis that complete sentences are semantically analogous to proper names, and the objects to which they refer are the truth values true and false. As he explains, one of the facts that delayed the publication of the book was that he introduced some technical novelties in his old system of the *Begriffsschrift,* and these changes led him to dismiss an almost completed early version of his book. One of this changes, as he explains, was the introduction of truth-values as objects. As Frege comments,

> Only a detailed acquaintance with this book can show how much simpler and sharper everything becomes by the introduction of truth-values. These advantages alone put a great weight in balance in favor of my own conception, which indeed may seem strange at first sight. (*Grundgesetze,* p. ix)

Now it is interesting to compare Frege's attitude towards truth-values with his attitude towards extensions of concepts. As he comments in the same introduction, extensions were necessary from the beginning, and we cannot, according to him, build anything without them. But with truth-values the situation was different. They were not necessary from the beginning, since a version of *Grundgesetze* (the neglected manuscript mentioned in the introduction) was actually prepared by Frege without them. So the justification for the introduction of truth values as objects is mainly pragmatic, i.e., the technical advantages brought by this move is a good sign that a theory that introduces truth values is closer to the truth than a theory (let's say, the old version of *Grundgesetze*) that dispenses with these objects.

Another sign of this pragmatic attitude can be found in the closing remarks of the introduction of *Grundgesetze*. Frege recognizes that his system might not be the only possibly one. Than he adds:

> Anyone who holds other convictions has only to try to erect a similar structure upon them, and I think he will perceive that it does not work, or at least does not work so well.

It is not quite clear what Frege is referring to when he says that other system do not work "as well" as his. But in view of his comment that I quoted earlier, it seems plausible to assume that working well for him means not just being able to get all theorems derivable in his system, but also deriving it in a short, simple and elegant way.[10] Although simplicity, practicality, etc., are not essential aspects of a formal system, they are, nevertheless, to be taken into account in the dispute between rival theories.

We find something quite similar in Church's defense of the same thesis (in Church 1956). In the opening sections of the book, he says that a great advantage of regarding sentences as names of truth-values is that we can, in this case, apply the semantic theory that was independently developed for ordinary names and predicates, thereby producing a unified theory. And he adds:

> Else we should have to develop independently a theory of the meaning of sentences; and in the course of this, it seems, the development of these three sections [on sense and reference of names, on constants and variables, and on functions] would be so closely paralleled that in the end the identification of sentences as a kind of names (though not demonstrated) would be very forcefully suggested as a means of simplifying and unifying the theory. (1956, 24)

That is to say, the acceptance of truth-values as objects referred to by sentences is not intrinsically necessary for Church, but this acceptance is justified because it greatly simplifies the semantical theory.

As a third illustration of this typically realistic attitude, I want to quote a passage from Gödel's discussion of the continuum problem (1947). Gödel famously claims that the axioms of set theory up to that point are insufficient to settle the question one way or another, and that new axioms are necessary. Moreover, the system of set theory can be "supplemented without arbitrariness" by new axioms, so as to better capture the concept of set. And Gödel adds the following remark regarding the acceptability of new axioms:

> [E]ven disregarding the intrinsic necessity of some new axiom, and even in case it has no intrinsic necessity at all, a probable decision about its truth is possible also in another way, namely, inductively by studying its "success." Success here means fruitfulness in consequences, in particular in "verifiable" consequences, i.e., consequences demonstrable without the new axiom, whose proofs with the help of the new axiom, however, are considerably simpler and easier to discover, and make it possible to contract into one proof many different proofs. (1947, p. 477)

That is to say, in Gödel's perspective, practicality plays an important role in the recognition of some axioms as true, even if these axioms are not in

10 There is some biographical evidence that, by the time he wrote *Grundgesetze*, Frege was well acquainted with Dedekind's work in "*Was sind und was sollen die Zahlen.*" Lothar Kreiser registers in his recent biography that, in the winter semester of 1889/1990, Frege offered a seminar on Dedekind's monograph (Kreiser 2001, p. 295). That is to say, he was aware that there were alternative ways of arriving at essentially the same results that he obtained in *Grundgesetze*.

principle indispensable, and other proofs for the same results might be produced without them.

Now why are things like practicality, simplicity, and so on, important (though not essential) for Frege, Church, and Gödel, while they do not seem to be so for someone like Benacerraf? I suspect that part of what is involved here is a fundamental difference between the realist's perspective of regarding mathematical theories as working hypothesis, as an attempt to describe an independent mathematical reality, and the attitude of someone like Benacerraf, for whom any theory that produces such and such results is in principle correct. If my perspective is correct here, there is an important sense in which Benacerraf's argument begs the question against the realist, for it disregards a typically realistic way of looking at matters of truth and existence. Indeed his argument takes for granted that some non-essential aspects of mathematical theories do not count, while realists like Frege, Gödel and Church would tend to see these secondary aspects as signs that a working hypotheses is closer to the truth than another one, even if both hypotheses can yield essentially the same results.

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