MAKING THE MATHEMATICAL WORLD: ON JULIAN COLE'S INSTITUTIONAL ACCOUNT OF MATHEMATICS*

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Abstract: Even though it is obvious that mathematics involves social activities, this rather trivial fact is rarely considered as important for its subject matter, mostly due to its undesired ontological consequences. An attempted solution for this tension was developed by Julian Cole's institutional account of mathematics, named Practice-Dependent Realism. In the present paper, Cole's account is evaluated, and its lights and shadows assessed concerning the ontological problem that he seeks to solve. I argue that his

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institutional account, although failing in delivering a sufficient ontological account of mathematics, still opens an important linguistic route for explaining its practice.

1. Introduction: The Tension

Consider the following claim, which at first sight might seem rather uncanny: *vector spaces and the Supreme Court of the United States are, ontologically speaking, very much similar*. Or, more generally, *mathematical objects resemble social institutions*. The claim has occasionally surfaced in the philosophical debate, receiving little defense. One reason for this lack of support is that the claim apparently conflicts with the standard platonist view of mathematics, where the role of mathematicians is often limited to report (rather than create) mathematical facts[1]. The reason, so it goes, is that accounting for an active role of mathematical practice would undermine the pretensions for the necessary and universal character of mathematics. While the practice is contingent, mathematics is not. Call this problem as *the tension* between mathematical practice and ontology.

On the one side of the debate we find that mathematical platonists normally shows little interest in mathematical

[1] There is, of course, no agreement between philosophers on the best theoretical foundation for mathematics. But as far as the practice is concerned, platonism seem to be the underlying spirit, as famously reported by Hardy: "For me, and I suppose for most mathematicians, there is another reality, which I will call *Mathematical reality*; [...] I believe that mathematical reality lies outside us, that our function is to discover or observe it, and that the theorems which we prove, and which we describe grandiloquently as our 'creations', are simply our notes of our observations." (HARDY, 1992, p.123)

practice. If mathematical reality is an independent realm of abstract objects, then one has to believe that mathematician's actions are entirely collateral. They might be epistemically relevant for providing different means for mathematical discovery, but ontologically they are irrelevant. The metaphysical nature of numbers, geometric figures, vectors spaces, sets, etc. must be independent of contingent factors.

But, on the other side, we find an obvious problem with such a platonist picture: it is alien to the fact that mathematics *is*, nonetheless, a social practice. Of course, questions about the metaphysical nature of mathematical objects or the epistemic status of mathematical truth are the cornerstone of any philosophy of mathematics. But questions about the mathematical progress, the nature of mathematical proofs, the role of heuristics, mathematical explanation, and so many others, are usually ignored. The turning point against this neglect is commonly set on Imre Lakatos's (1976) work, the accepted birth of the Maverick tradition that takes mathematical practice as the forefront for philosophical research. This trend continued in the works of Philip Kitcher (1984), Paul Ernest (1998), Reuben Hersh (1997), among others.

The ontological consequences of such questions are possibly what platonists most fear in adopting a more substantial view about the practice, as mathematicians' actions and choices would directly settle existence matters about mathematical objects. Nonetheless, Maverick authors such as Kitcher, Hersh and Ernest did accept that any reasonable account of the mathematical practice is incompatible with theories that explain mathematics - its objects and truths - in aprioristic terms. Hersh and Ernest, for example, argued that mathematics is entirely constituted by mathematicians' social practices. Hersh goes even further, claiming that mathematical objects are also processes that are subject to change[2], thus accepting the ontological consequences feared by the platonists.

Hersh and Ernest's position was labeled as **Social Constructivism**, an account that has recently received an updated version by Julian Cole (2008, 2009, 2013, 2015). Cole argues that mathematical objects are best understood as existing institutional entities that are dependent upon the practice, thus putting him in the center of the tension above. But Cole also affords a solution for the tension between mathematical ontology and its practice. This paper seeks to evaluate Cole's thesis. I'll first detail his position, as developed in the last decade, that mathematical domains are institutional entities, to then present Jill Dieterle's (2010) objection, namely the Contingency Problem. I'll argue that Cole's attempted answer to Dieterle's objection is insufficient for solving the tension. Then, I'll focus on the use of declarative acts and collective agreements in mathematics, arguing that Cole fails to properly solve the tension with them, although rightfully stressing its importance for the practice. I'll conclude by hinting the benefits of taking his position further and adopting a Searlian perspective about mathematics, as suggested by recent authors. Even though Cole's work is still inconclusive concerning its upshots, it offers valuable insights that the philosophical community must take notice.

2. Practice-Dependent Realism

Cole's account attempts to reconcile some of the realist intuitions about mathematics with a more active role of the practice. The resulting view is that mathematical domains,

[2] See chapter 5 in (HERSH, 1997).

being constituted of abstract entities, exist only *dependently* from our practices. He labels this weak form of realism as **Practice-Dependent Realism** (**PDR**). Cole (2008, 2009) argues that the realism of **PDR** is consistent with the *creativity*, *freedom*, and *authority* that mathematicians enjoy in the practice of their work.

Practically, **PDR** views mathematical domains and objects as the results of social constructions that are performed by means of constitutive declarations. It means that the existence of mathematical domains and objects is dependent on the stipulation performed by a group of individuals having the normative power to declare them to exist. Since the conditions of success of such declarations are identified with those that we usually use for creating institutions (like, *e.g.*, universities), at the heart of Cole's proposal, we find the identification of mathematical domains with institutional objects.

In (COLE, 2013), the main tenet of **PDR** is summarized as follows.

> **Main Thesis of PDR**: *Mathematical domains are freestanding institutional entities that, at least typically, are introduced to serve representational functions.* (COLE, 2013, p.9)

To understand this thesis properly, we have to spell out some details on what institutional entities are and what is a representational function.

2.1. Institutional Entities

Following John Searle's seminal works (1995, 2010), an institutional entity is one that exists in virtue of a collective agreement, as the Supreme Court of the United States, that some pieces of paper count as Money or that Julian Cole is a Professor at Buffalo State College. Common to all these cases is that the corresponding institutional entities have been introduced to serve kinds of functions. Julian Cole's position as a Professor has been granted to fulfill some duties associated with Buffalo State College. In other terms, by collective agreements, Julian Cole has been invested with the rights and obligations specific to his role as part of a framework of deontic relationships that characterize the Buffalo State College. For what concerns Money, a twenty-dollar bill grants deontic powers to her owner, enabling her to use it to buy a product of equal value.

According to Searle, the roles that we collectively agree to assign to people or objects are called **Status Functions** (**SF**). These functions can normally be expressed as "*X* count as *Y* in *C*", which can be read as we collectively imposing the status of *Y* to some *X* within the context *C*, *e.g.*, that Julian Cole (*X*) count as Professor (*Y*) at Buffalo State College (*C*). *X* may be an existing object or can be created to fulfill the status function, as in the case of corporations. In (SEARLE, 2010) these are also called Constitutive Rules, that is, a rule that is responsible not only for regulating how a given action must be performed but create the very action in question, thus putting some institutional facet of reality into existence. The rules of Chess, for example, not only regulate how a given move must be performed, they also constitute the game as an institution. Formally, an institution is a system of such constitutive rules, while an institutional facet of reality is the entity that exists in virtue of such rules, according to Cole (2013, §1.3).

If the **SF**s are an essential aspect of the institutional entities that populate our everyday lives, it is not easy to see how a given mathematical object or domain might be able to perform such a function. This is why, in **PDR**, we find that mathematical objects play a different institutional role: they serve **Representational Functions** (**RF**). The **RF**s are explained by noting that some institutional entities have no deontic powers in themselves, but they are only auxiliary entities that help us navigate social reality. In other terms, some institutional entities serve the purpose to represent, analyze, reason, discover, etc. some facets of reality. For example, a border is an institutional entity whose function is solely to allow us to reason about transitions of land ownership. It does not have deontic powers in itself, as, on the contrary, has the owner of that piece of land demarcated by such border[3].

These are functions that we can attribute to all institutional entities, may they be universities, states, or football teams. But how can we attribute these functions to such entities? Following Searle once again, institutional entities are made existent by means of *declarative speech acts*: acts of speech that under specific conditions of felicity can shape reality by the simple act of declaring it as such. In other terms, a declaration attributes a function to a given object or person by the simple act of saying that object or person serves that function[4]. Moreover, the collective agreement on this imposition of function put into existence the network of deontic relationships that we grant to that function.

[3] As Cole (2015, p.1109) mentions, "[...] borders serve as surrogate objects for deontic transitions," *i.e.*, for representational functions.

[4] But this may not involve linguistic utterances, as Searle (2010, p.13-14) clarifies. They are called *Status Functions Declaration* or *SF-Declarations*. Cole labels them as *Representational Acts*.

2.2. From Institutions to Mathematics

Cole's account can now be easily presented as saying that, much as the Buffalo State College or the United States Supreme Court, mathematical domains are institutional entities. However, differently than other social accounts of mathematical reality, Cole claims that **PDR** considers mathematical domains as *genuinely existent*. Generally, "[...] someone's undertaking an ontological commitment to a mathematical domain during the course of an investigation suffices for the existence of the domain in question," (COLE, 2013, p.27) granting that enough deontic powers are in place for the collective agreement on the declarative performance.

Institutional entities can also be dependent on some other facets of reality, either brute or even institutional. For example, the 20 dollar bill is dependent on a piece of paper and the Federal Reserve System for carrying the deontic powers associated with its value. Cole call's such as non-freestanding institutional entities, *i.e.*, those "[...] 'identical to' or 'constituted by' some entity or collection of entities that is identifiable independently of the institution responsible for the existence of the institutional entity in question." (COLE, 2013, p.16). If a given institutional entity is independent of other facets of reality, such is said to be *freestanding*. As the central thesis of **PDR** stated, mathematical domains are freestanding. This is consistent with the expected abstractedness of mathematical objects, as they are not likely to be reduced or constituted by brute or physical facets of reality, or even regulated by some institution. **PDR** sees mathematical objects as abstract entities, or pure constitutive social constructs, that is, those who "[...] exist wholly in virtue of the undertaking of certain acts, decisions, or practices of social significance." (COLE, 2008, p.115) With these features in mind, Cole can describe

PDR in terms of abstract objects that are dependent upon social practices, thus still distinct from standard platonism. In comparison, and following the definition of mathematical platonism in Linnebo (2018), we may say that Cole's **PDR** satisfies the Existence and Abstractness conditions (that mathematical objects exist as abstract entities, a position also called Object Realism), but not the Independence condition (that they exist independently from our language, thought and practices). However, as we shall see, **PDR** attempts to deny Independence while keeping some of its features.

The reason for introducing mathematical entities is, in Cole's view, to serve representational functions. Numbers were first introduced to serve **RF**s for reasoning about finite collections. The concept of number then evolved to the point of having a full ω-sequence of entities that perform these **RF**s for reasoning about all finite collections. As another example, possible worlds were introduced to simplify our reasoning about modalities. Thus, we treat possible worlds as surrogates, or *representational objects*, for matters of possibility and necessity in the same way we treat numbers as surrogates[5] for matters involving the cardinality of finite pluralities.

This picture can be stratified by iterating the attribution of **RF** to more and more abstract pieces of mathematics. According to Cole, mathematical reality can be thought of as constructed in layers. First, we construct mathematical objects to fulfill representational functions of facets of reality that are not mathematical; for example, we introduce numbers to count and geometrical figures to measure things. Next, the second layer of mathematical objects is introduced to serve representational functions about the lower layers, that is, about the mathematical domains and objects already

[5] In (COLE, 2015), **RF**s are named **Surrogacy Functions**.

constructed. This is made possible by the fact that mathematical objects, being institutional objects, can serve as representational functions of any given facet of reality, be it physical or institutional.

3. The Contingency Problem

Following Jill Dieterle (2010), one problem that **PDR** faces is the *Contingency Problem*: given that collective agreements are essentially a contingent phenomenon, mathematical objects would exists only contingently. If that's so, how is it possible to accommodate the universality and necessity of mathematics within the practice-dependent ontology? In her words:

> [...] if one claims that X is socially constructed, then one must believe that X is not inevitable. If X is the product of social forces, then X exists, or exists in the way it does, only contingently. (DIETERLE, 2010, p.322-3)

This doesn't seem to be the case for mathematics since we do not expect that mathematical objects exist contingently, or that mathematical statements would be only contingently true.

The contingency problem relates directly to the tension above presented, as one cannot solve the tension without giving a negative answer to the contingency problem. The tension asks whether it is possible to provide a substantial account of the practice that is consistent with the necessity and universality of mathematics, while the contingency problem seems to offer a negative answer. A realist answer would simply deny that we can account for the practice with such high standards. But Cole is trying to have the cake and eat it too.

3.1. The Answer

Cole's answer to Dieterle's objection, in (COLE, 2013), is based on a closer analysis of the representational functions that mathematical objects concretely perform. First, he noticed that the **RF**s served by mathematical objects are universal, as they are about "all (external) possibilities of some type." (COLE, 2013, p.30) This roughly means that numbers, for example, must be general enough so that any collection could be counted. Cole offers an illustration of how such initial arithmetical domains might have been constructed. Initially, our predecessors introduced natural numbers as tools for representing the cardinality of physical facets of reality. Eventually, they "[...] recognized that they wished to use natural numbers to perform their **RF**s with respect to all (externally) possible finite collections." (COLE, 2013, p.29) Thus, these entities were expanded over time, and their applicability grew, to the point of a full ω-sequence of entities being introduced: the natural numbers.

The idea is that arithmetic started with basic operations (as counting, adding, subtracting) in what we may call applied arithmetic, and evolved into abstract notions that could represent those same operations in a generalized setting. In this new generalized setting, numbers were introduced so that any aggregate could be counted, either in the past, present, or future and no matter the metaphysical category they belong. This is why the existence of mathematical objects must have no restriction since we expect to make use of mathematical concepts in a way that does not depend on any spatio-temporal or modal conditions. In other terms, we expect them to have their **RF**s regardless of the time, or space, or conditions of application[6].

But how can an institutional entity exist without such restrictions? Cole defines the *modal profile* of a given institutional facet of reality as the set of worlds in which that facet of reality exists and the *temporal profile* as the set of times in which a facet of reality exists. Following this, it is expected to be perfectly possible to a given institutional entity to have the modal profile that best fits the **RF**s that we want them to perform[7].

The only requirement to create an institutional facet of reality with amodal or atemporal profiles --- that is, that exists without modal and temporal restrictions, holding in every world and at every time --- is to recognize the deontic powers relevant at the time of the declaration. Just as we now recognize human rights as an institution holding retroactively

[6] It is questionable whether we should restrain mathematics to such conditions. In a way, alternative systems are intelligible, given that mathematics can be at least conceivable beyond the laws of logic. But Cole's point is not about intelligibility but applicability, given that mathematical objects are supposed to perform representational functions in the first place. In this case, the modal and temporal conditions are set as having usage in mind. Cole's point is that any object that may exist only contingently cannot represent its function in every possible and required scenario. For that reason, if numbers are to exist and fulfill their representational functions, they must exist necessarily. Of course, Cole is not claiming that numbers necessarily exist because they are logical in character, but that *we declare* them so to fulfill the representational functions that they are designed to perform. What is left to be answered is why such representational functions are so much so constrained. Although the answer may vary, the outcome seems to be that **PDR** is not practice-dependent after all, as it will be discussed in the next section.

[7] See (COLE, 2013, p.20-21).

for every human person in previous times, our predecessors might have recognized the existence of numbers in the same unrestricted modal and temporal profiles. As Cole puts it:

> [...] in the actual world, at time *t*, a mathematical domain, *X*, exists as an abstract, atemporal, and amodal entity if and only if, in this world, at *t*, there are people who collectively recognize the existential DPs carried by *X*. (COLE, 2013, p.31)

Cole concludes that even if mathematical domains and objects are introduced (or constructed) by collective agreements on specific declarations, they exist necessarily and atemporally. Furthermore, truths concerning such objects are regarded as necessary either.

Of course, in this scenario, any arithmetical domain would necessarily exist. But Cole argues that such domains are not arbitrarily constructed. So he advances a similar argument for the objectivity of mathematics. Earlier, in (COLE, 2009), it is argued that mathematics is only epistemically objective. Once a given domain is fixed, truths regarding such domain are objective, given the objective fact that we had stipulated such domain as a genuinely existing entity. Dieterle (2010, p.235-6) also objected that the notion of epistemic objectivity is not strong enough for characterizing mathematical truth.

As an answer and attempted solution, Cole put forward an argument for a more robust notion of objectivity. The core idea is that the **RF**s being performed do impose some restrictions on the domain of objects created to fulfill them. For example, it is expected that whatever Numbers may be, they must obey Hume's Principle[8]. And given the universality of the **RF**s that mathematical domains are supposed to perform, these restrictions are fairly strong. So much so that, as he concludes, "[...] arithmetical truths, such as 7 + 5 = 12, could not be any different from how they are and the natural numbers still serve their **RF**s." (COLE, 2013, p.33-34) With such strong conditions, one might conclude that there must exist a canonical structure of objects that serve these functions[9].

To summarize Cole's Answer, even if mathematical domains (in the example, arithmetical) are declared to exists in virtue of collective agreements, they still exist (1) necessarily, since they are created with unrestricted modal and temporal profiles; and (2) objectively, since they obey the restrictions necessary for their **RF**s to be fulfilled, and which could not be otherwise.

3.2. The Verdict

The necessity and objectivity of mathematical knowledge are standards that are hard to abandon. They are not only celebrated features that the philosophical tradition has embraced, but they also fulfill the intuitive ideas we have about mathematical objects and mathematical knowledge. Thus, it is to no surprise that Cole wants to retain both with **PDR**. Platonists have an easier job explaining it as consequences of the Abstractness and Independence

[8] Given the close connection between counting and Hume's Principle, it is hard to think that one can serve any representational functions involved in counting and deny Hume's Principle.

[9] If this assessment of Cole's account is correct, then it is hard not to paint him as a Structuralist in disguise.

ascribed to mathematical objects[10]. If they are in fact acausal abstract entities that exist independently from our thoughts and practices, statements about such objects must be both objectively and necessarily true as well. However, things are not that easy if the independence condition is dropped, given that, as Dieterle's objection have already pointed out, to claim that mathematical objects and domains are dependent on the practice is *ipso facto* to claim that they could be different. In contrast, Cole's answer differs from other social constructivists. Hersh and Ernest bit the bullet on the contingency problem, dropping any hope to account for mathematical indubitability in the first place[11]. But this does not make justice to our basic intuitions about mathematical truth. Arithmetical truths, for example, are not simply contingent. They are true in a robust way, and this is also the case for their applicability.

It is usual for social constructivists to model the constructed domains in the informal mathematical notions that are used on our daily needs[12]. Cole has a similar starting point: the representational functions that numbers suppose to perform, such as "[...] representing the cardinality of various collections." (COLE, 2013, p.29) Since our constructions suppose to model such representational functions (the counting of finite aggregates at first), we

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[10] I'm not claiming that platonists got it right, only that their position is designed to explain such standards to the cost of mathematical practice, following the Tension above discussed.

[11] See chapter 4 in (HERSH, 1997) and chapter 1 in (ERNEST, 1998).

[12] See, for example, Kitcher's (1984) view that Arithmetic is a theory about the operational activities of an ideal subject or Hersh's (1997) claim that pure numbers are abstractions from adjectival numbers.

might ask what the proper connection between both is. How can we choose between different constructed domains? Can we have different arithmetics equally consistent with our representational functions?

For Jill Dieterle, a social constructivist might answer the contingency problem by suggesting that some domains are more likely to be constructed than others, following the **RF**s that we need to perform. It would suggest the idea that "The natural numbers are so deeply grounded in our representation of the world that it is almost unimaginable that mathematicians would not have constructed that domain." (DIETERLE, 2010, p.323) In his answer, Cole offered a similar answer. In fact, in his picture, it is unimaginable that mathematicians could have constructed the mathematical domains differently and *still* preserve the relevant representational functions.

In the case of natural numbers, Cole starts from premises similar to Frege's motivation for logicism. Frege thought that arithmetic, having the most inclusive possible domain of application, should not rest on an intuitive foundation[13]. If numbers were intuitively defined, they could not be used to count those facets of reality that are not intuitively given. In **PDR**'s case, since we start with the intended representational functions, our constructions should be broad enough to embrace all such applications. Their **RF**s shouldn't be restricted given the possibility of applying numbers to anything that can be counted. Their modal and temporal profiles cannot have such constraints.

His conclusion is that arithmetical domains are objectively constructed in such a way that all relevant features, and consequently, all pure arithmetical statements, are fixed. Another way to put it is that we cannot collectively

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[13] See (FREGE, 1953, §14).

agree on some exotic arithmetical domain in which 7+5=13 holds, as such odd arithmetic would run against the priority given to the representational functions, namely to count finite collections --- and we know that seven objects together with five objects make a collection of twelve objects. Cole (2013, p.10) assumes that "[...] the fact that when you combine a collection with cardinality 7 and a collection with cardinality 5 you typically obtain a collection with cardinality 12 does not obtain in virtue of collective agreement," meaning that only pure mathematical statements are consequences of collective agreements. In other words, they are constructed as idealized models for our basic arithmetical applications.

The problem with such an argument is that it does not really explain why it cannot be the case that 7+5=13, without appealing to another objective realm that is independent of our constructions. If the objectivity of mathematics is discharged on its representational functions, what makes the latter objective?

In this point, Cole is not entirely free from the Independence thesis. For if we take seriously the representational function of counting that numbers serve, there are constraints that we have to consider. For example it is expected that a bijection must exist between each aggregate and the numbers (whatever we take them to be) representing their cardinalities, as no aggregate can have two different numbers as its size. But why should this be the case? How can we explain the nature of the constraints that **RF**s have? This could be answered in a number of different ways, none favorable to **PDR**, as the following three options show.

1. **Metaphysical Constraints**: We might advance a metaphysical explanation. It could be the case that the very nature of the relevant facets of reality under consideration is intrinsically mathematical. This

would explain why our representational functions demand the restrictions that they impose on the domains of objects constructed. This would, however, shift the priority from practice to reality itself, given that 7+5=12 would be a necessary truth as it would model a necessary feature of the world. But then, the practical side of **PDR** would seem superfluous.

2. **Cognitive Constraints**: Or we could take a Kantian-like route. It could be the case that our representational capabilities are demanding such a mathematical 'way of thinking'. But in this case, the constraints in our representational functions would end up being constraints about our representations themselves. In this scenario, 7+5=12 would be a necessary truth because there could be no possible way our cognitive capabilities to think of it differently. If so, why **PDR** would count as a realist position?

3. **Logical Constraints**: It could be the case that either reality or our means to represent it are logically constrained. This would imply a return to logicism. If our representational functions are logically determined, then numbers are logically determined as well. Thus, 7+5=12 would be necessarily true since it could not be different without implying some contradiction[14]. But now, **PDR** would need to

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[14] For the sake of the argument, assume $m=n\rightarrow (m-t)=(n-t)$ as an axiom. Then, from 7+5=13, one gets that (7+5)-12=(13)-12, which in turn yields that 0=1. In this case, we can follow Frege' thesis that number statements are ascriptions about concepts. Thus, the sentence "the number of *F*'s is 0" is equivalent to "the

propose an explanation of the objectivity of the logical rules. Given the anti-psychological stance of logicism, this task is expected to be difficult under social constructivism.

Regardless of the choice, our declarative acts would be just means to fix different ways of representing either the mathematical aspects of reality, our cognitive capabilities, or the logical structure of mathematical objects. In all these cases, mathematical truth would be independent from our practices and collective agreements.

As we saw, for **PDR**, an ontological commitment is what it takes for a domain to exist. It is, however, unclear how inconsistent domains (like Russell's set or those used in proofs by contradictions) would fit this picture. Cole (2013, p.27) claims that "[...] an individual is able to undertake such a commitment only if he or she has constructed an appropriate concept that, at least roughly, coherently characterizes the said domain." But it is unclear which notion of coherence we should appeal to discriminate between existent and nonexistent totalities.

In (2008, p.125, ft.37), Cole claims to import Shapiro's conception of coherence, as stated in (SHAPIRO, 1997). In Shapiro's structuralist position, it is pressing to consider how implicit definitions are able to characterize structures coherently. The first option is to explain coherence in terms of deductive consistency. In a first-order scenario, one would only require completeness, as first-order languages that are deductively complete have a model and, therefore, are consistent. But completeness requires some mathematics (particularly, Set Theory). In Shapiro's structuralism, this is

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number of F's is 1", which is a plain contradiction since both are true if, for some object a, both $F(a)$ and $\sim F(a)$ holds.

acceptable. But in the case of **PDR**, one cannot introduce sets if, in principle, set theory is needed for deciding whether the intended domain is consistent.

Moreover, as Shapiro continues, things are worse with second and higher-order logics, where no completeness is available. In this case, one can have consistent theories that have no models[15]. Thus, a consistent domain characterization is not a sufficient condition for making a successful declaration of existence, as **PDR** wants. A second attempted solution, rejected by Shapiro, is to define coherence in terms of satisfiability. But satisfiability requires the existence of a model. Given that "exists" is here understood as being a member of the set-theory hierarchy, one would need to accept the structure of sets and the coherence of set-theory. For **PDR**, this would mean, once again, to already require mathematics in order to provide successful declarations within it.

To be fair, Cole also rejects coherence in terms of deductive consistency or satisfiability. Shapiro opted in taking it as a primitive intuitive notion that can, at best, be explained through satisfiability. But in **PDR**'s case, this is not entirely satisfactory. Even if we depart from any deductive consistency requirement, a successfully declared system for arithmetic would need to be at least non-trivial. But the statement that, for instance, Peano Arithmetic is non-trivial is already an arithmetic statement (following Gödel's arithmetization) that is consistent to deny, granting the non-triviality of PA[16]. Even in an informal scenario, a coherently

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[15] For instance, the second-order Peano Axioms in conjunction with the negation of Gödel Sentence is consistent, but has no models, as Shapiro explains.

[16] It follows from Gödel's incompleteness theorem that, if PA is non-trivial, then it cannot prove its own non-triviality.

characterized system would need some arithmetic just to make sense of its non-triviality[17]. Thus, deductive consistency, satisfiability or even informal coherence are not free from some level of mathematical explanation.

Nonetheless, one has to recognize that what makes a coherent characterization of a domain cannot be a matter of conventions themselves, even in the intuitive and informal sense. By simply agreeing beforehand, we cannot declare what coherence means, safe from circularity[18]. And in this case, any explanation on what a coherent domain consists of, or why the representational functions are constrained the way they are, would rely on constraints that are prior to our very acts of declaration, just as the three options above state. If we assume that coherence is not something dependent from our stipulations, whatever it may be, then the whole agenda of **PDR** seems bankrupted[19]. To rephrase Gödel's famous realist stance, it seems that the representational constraints force themselves on us, making any creative component superfluous.

The three options above also make it hard to accept that one can reconcile a realist ontology with a defense of the mathematical practice, as **PDR** wants to have it. If our constructions must necessarily model either of the three options or any hybrid version of them, then it is hard to

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[17] I am thankful for an anonymous referee for making this point.

[18] As this would require an answer to what makes our declarations coherent in the first place. This is very similar to the point made by Quine in "Truth by Convention": that "if logic is to proceed *mediately* from conventions, logic is needed for inferring logic from the conventions" (QUINE, 1936, p.271). I thank an anonymous referee for this point.

[19] As Putnam (1995) similarly argued against the "truth by convention" thesis of the logical positivists.

believe that any collective agreement is relevant for such constructions. If a group of mathematicians agrees on the existence of a given mathematical domain that models such facets of reality by serving the relevant **RF**s, the correctness of the domain would be answered by the facets of reality themselves. In the case of Arithmetic, it is expected a canonical domain to be constructed, the unique structure that obeys Hume's Principle and other relevant properties[20]. There seems to be no freedom or authority for mathematicians to declare or collectively agree on whether a given construction is the correct one.

At the same time, if we deny that our constructions must model any of the three options above, it is not clear how **PDR** would be able to defend the universal, objective, and necessary character of mathematics. These features seems only possible by appealing to some *a priori* condition that is not derived from declarative acts or collective agreement. Thus, the proper motivation for **PDR** would not be representational acts or collective agreements, but these *a priori* conditions, as the three options above suggests[21].

As it seems, Cole's **PDR** is in the middle of a dilemma, as he is only able to answer the contingency problem with the aid of something extraneous from the practice. If we accept such an answer, he subsequently fails to offer a convincing solution to the tension between mathematical practice and

[20] Assuming that the theory is at least categorical.

[21] A possible objection would be to recall Benacerraf's dilemma in (BENACERRAF, 1965). But even assuming that there are many equally compatible constructions to fulfill the representational functions --- the Von Neumann and Zermelo's ordinals ---, the representational constraints still persist as a severe limitation for the mathematicians' constructions. The focus should be not on the different ways to construct the ordinals, but on the necessary properties that any number structure must satisfy.

its ontology. If, and using Jill Dieterle's wording, it is *unimaginable* to construct arithmetical realms differently and still serve the desired representational functions, then all hope for any relevant freedom and creativity is lost.

4. The Declarative Issue

Cole's account has an important upshot: it depicts very closely what actually takes place in mathematical practice, as the history of the field suggests. But the task of reconciling such substantial account about the practice with a realist ontology, even a weakened one, is hard to achieve. As mentioned above, if his answer is effective in answering the contingency problem, then it's hard to argue for any relevant role for collective agreements. The other horn of the dilemma is that any weight put on collective agreements undermines such an answer, leaving the contingency problem wide open.

Since **PDR** takes collective agreements as necessary for the existence of mathematical domains, it is reasonable to ask how such agreements take place, and what can be said about it. The immediate Searlian answer, which Cole relies on, is that collective agreements are understood in terms of *declarative speech acts* (COLE, 2013, p.14), precisely the broader conception of declarations offered in (SEARLE, 2010). First, note that one can declare something as being the case by stating it. By saying, "I'll be the goalkeeper," I make it the case that I will play as the goalkeeper. But in many other cases, a declaration can be made simply by representing the reality as having the intended declarative effects, even without a corresponding linguistic utterance. For instance, I can also make the case that I will play as the goalkeeper by simply positioning myself in the right place and start playing it as one. For Cole, these are representational acts, and declarations are to be understood as such broad cases.

This picture is well motivated for some standard facets of reality[22], but it's questionable whether it can be a relevant analysis of mathematical entities. Of course, in one sense, collective agreements are necessary for declarative speech acts, but at others, they seem not to be. Agreements and declarative acts are different kinds of social phenomenon[23] and thus the relation between both must be better explained.

Notice first that a declarative speech act, if successful and non-defective, has an immediate effect on reality. If I have the relevant deontic powers, and if I state that "Your paper got an *A*," it is the case that you received a grade *A* on your paper. Similarly, if a Priest utters, "I pronounce you husband and wife," then it is the case that you are married. A priest does not demand any additional authorization to successfully marry a couple. Neither the couple has to wait to consider themselves as married, provided that the Priest has the relevant deontic powers[24]. Similarly, I do not need any

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[22] As Searle (2010) argues, this single principle is responsible for the whole of our social reality. But since he is not considering Mathematics as a part of such reality, Cole's project can be read as an attempt to complete Searle's picture.

[23] There are at least two senses in which we can agree on some statement *S*. In the assertive sense, we agree that *S* is the case. In the declarative sense, we agree to make *S* the case. Acts of declaration are not the same as agreements.

[24] Marriages still need the state's legal approval and so the couple still has to wait to consider themselves as legally married. But this is only the case if we extend the scope of the relevant deontic powers. Religiously, at least, Priests do have the power to marry under God's will. And for this case, there are no extra demands. Some declarative acts do demand a form of agreement, as it is the

additional approval for rating your paper an *A*, provided that I have the relevant deontic powers by being your professor. With the right deontology, no additional agreement is necessary as the declarative act has immediate effects. In this scenario, any mathematician could declare a domain to exist, provided the relevant deontology. But this isn't an accurate picture of the practice. A mathematician still requires acceptance to make the declaration successful and non-defective. In a way, no mathematician (or any group of) has enough deontic powers to make a successful declaration alone. Cole agrees. He states that "[...] a single individual cannot be responsible for a mathematical domain existing unless his or her work in characterizing it is — in some sense — legitimized by a broader mathematical community." (COLE, 2009, p.601)

On the other hand, we can still find a connection between agreements and declarative acts. One cannot become a professor or a priest by merely choosing to. To be able to grade papers or marry couples, one must have the relevant deontic powers granted by the community. Being accepted as a member of the mathematical community certainly involves collective agreements, as for any disposition of deontic powers. But this is not an adequate picture of the mathematical practice either: proving theorems, making conjectures or even declaring some domain to exist does not depend solely on being a recognized member of the mathematical community[25].

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case in elections and other collective decisions. But they are different kinds of social phenomenon.

[25] Take the example of the Indian mathematician Ramanujan. His works are now well accepted mathematical results, albeit having been received with much resistance from the community at first,

These two explanations must be rejected as incoherent about the practice. We know that mathematicians do not have powers to single-handedly declare something to exist, and we know that such powers are not dependent upon the relevant community of practitioners. A third option is to take a domain to exist if the declarative act responsible for its existence is collectively recognized and accepted. This is Cole's option. He argues that agreement is necessary for accepting that a given declaration is in place. As he says, "[...] institutional facets of reality come to exist in virtue of *our collectively adopting systems of (standing) Declarations*." (COLE, 2013, p.16)[26]. To agree that some domain *X* exists is to agree on adopting a (standing) declaration (an utterance or a simple representative act) by which *X* exists. Mathematical domains exist, in such case, in virtue of mathematicians collective and simultaneously representing such domains as existing, or, by mathematicians collectively adopting the same acts of declaration.

This is not convincing enough, at least for mathematical purposes. Because of the objectivity and universality of mathematics, agreements should not be constrained by conditions that are sensitive to time, space, or individual preferences. This condition seems hard to be obtained (especially with so rigid criteria), if we do not want to sacrifice the practice dependent component of **PDR**. Nonetheless, collective agreement, or collective recognition, seems to fail for mathematical purposes in the following points:

as he was not recognized as a member of the mathematical community.

[26] The emphasis is mine.

Agreements are not fine-grained enough:

Declarative speech acts have the special property that, if successful and non-defective, they immediately bring about the content being declared as existing, provided that enough deontic powers are in place. But linking this notion with collective agreements in the Searlian sense does not clarify the existence of mathematical domains, given that the truth conditions for agreements are not precisely defined. In no mathematical field mathematicians reach an absolute agreement. For example, despite acting as they know what numbers are for practical matters, mathematicians still haven't agreed on a convincing and unifying solution for whether they exist, what are their metaphysical properties, and so on.

Agreements are also never unanimous. One might suggest that we don't need a definite agreement to declare a domain X to exist, thus appealing to some notion of a consensus. But this is as much coarse-grained as the notion of an agreement is. Perhaps we could appeal to a "simple majority" argument, but this offers more difficulties. Not only mathematicians do not assemble in order to vote for whether a given domain X exists, but even if they do, the contingency problem would still prevail[27].

[27] It should be mentioned that the notion of a collective agreement does not imply any form of approval, but only a recognition of existence.

Agreements are hard to locate temporally

The following is also a condition for declarative speech acts: any institutional facet of reality that exists in virtue of a declaration is temporally located *after* the declarative act. But, if it is not up to a single mathematician to declare some domain X to exist, and if it is the case that X exists given some collective agreement, what is the actual temporal profile of X? The question begs for another: when can we say that an agreement was finally reached? And if we don't have a precise answer, how can we decide existence matters until then? Surely, not every case of a collective agreement offers such difficulties. We can collectively vote for president and decide *when* the decision is official. The declarative effects, *i.e.*, the election results, are discretely verifiable by counting votes, and thus have a precise temporal profile. But no such tool is available for mathematicians decisions.

Agreements are Contingent

Agreements are clearly a contingent phenomenon, as they could have different outcomes. Even more, they are also reversible. A argued by Dieterle (2010), if mathematical statements are true by virtue of stipulations, then the truth of the statement "$s(0)=1$" would be equally justifiable as "Boston Stage College is an American University"[28]. But at the same time,

[28] The former would depend on the structure of numbers, thus, on the relevant domain declared by mutual agreement. The latter, on the standing declaration in which the Boston Stage College depends to exist as a University.

since both are dependent upon conventions and agreement, both may change: Boston Stage College could cease to exist, and 0 could have another successor if conventions change. In order to block such cases for mathematics, we would need a more robust notion of agreement, one that is immune to contingent features. But a complete and absolute convention, or a complete and absolute agreement, is nowhere to be found.

We saw that Cole's solution states that we can declare some domain to exist in any modal profile required for such domain to fulfill their **RF**s, meaning that we could declare numbers to exist necessarily, even from a contingent standpoint: that X exists necessarily at t if, at t, it is collectively accepted that X exists necessarily. But since adopting a given modal profile is still something dependent on declarations/collective agreement, if some mathematical statement S is necessarily true in virtue of the modal profile of the relevant domain X, then it is not necessary that is necessary that S is true. The ghost of contingency still haunts any domain that exists in virtue of agreements, since it could be the case that X exists necessarily in a given world without existing necessarily at others. Moreover, if we can declare something to exists necessarily, we can declare something to cease to necessarily exists either. A notion of necessary agreement could solve this, but as stated before, a necessary agreement is hardly an agreement at all[29].

[29] Take, as an example, the institution of human rights. We can collective recognize the fundamental values of all humans beings in the past, present and future. But this recognition could still change. This point and a temporal objection to Cole's solution is

Agreements are Historically Rare

The agreements required and presupposed by Cole might be too idealized. His picture of mathematical progress is an idealization that does not faithfully represent the history of many mathematical notions. Contrary to the story of linear progress that seems to underline his description of the evolution of mathematics, mathematical practice is not much different from that of science. If we look closely to the practice, we find a series of trial and errors, together with important disputes on the acceptance of new methods, ideas, and even objects. Without going too much back in history, we can recall the fierce debate that took place around the Axiom of Choice, at the beginning of the twentieth century[30]: How to interpret the existence of choice functions during this debate? As long as a consensus was not reached, was Zermelo wrong in saying that any set could be well-ordered? Were the French analysts right in considering a lighter mathematical ontology? Besides, and more importantly, how to evaluate the outcome of this debate? Has the Axiom of Choice

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well found in (LOGAN, 2015). Cole's solution implies that we could model the sentence "It has always been the case that φ" in an atemporal way, even though the declaration of φ is only temporally located. This means that Cole has to accept that "It has always been the case that φ but it has not always been the case that it has always been the case that φ". In a temporal logic, this is $H\varphi \& {\sim}HH\varphi$, or equivalently, ${\sim}(H\varphi \rightarrow HH\varphi)$, which is only valid in non-transitive frames, a tough condition to ask for any temporal logic.

[30] For more on this debate, the interested reader is referred to (EWALD, 1996).

been accepted by the mathematical community[31]? If anything, the acceptance of the Axiom of Choice was far from being self-evident[32].

Reasons for accepting or rejecting an axiom may vary. Should we require our axioms to be self-evident, as Gödel would claim? Shapiro (2009) argues that such requirement, particularly in Frege and Zermelo, can be read holistically: that reasons for accepting an axiom are ultimately given in the practice[33].

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[31] I'm inclined to answer yes, and yet there are mathematicians who still have reservations in accepting it.

[32] What is indeed self-evident in the practice is that mathematicians (and philosophers of mathematics) often disagree with each other, contrary to the common belief that mathematics is the cornerstone of indubitability. A good assessment of this point is found in Clarke-Doane (2020, ch.2). The author further argues that disagreements in mathematics are on a par with disagreements in morality. Surely, it is easier to see how disagreements take place on over-specialized fields, such as Set-Theory, involving over-specialized individuals. But these are also the relevant individuals for mathematical practice, thus relevant for Cole's **PDR.**

[33] A similar position was held by Russell and Whitehead (1910, p.62): "The reason for accepting an axiom, as for accepting any other proposition, is always largely inductive, namely that many propositions which are nearly indubitable can be deduced from it, and that no equally plausible way is known by which these propositions could be true if the axiom were false, and nothing which is probably false can be deduced from it." But on this account, the acceptability of an axiom is largely dependent on the acceptability of its consequences, on which propositions we want as theorems in the first place. In the case of mathematics, this is not an easy question to settle. For a more detailed account of the justificatory problem for axioms and possible answers, see Clarke-Doane (Forthcoming, ch.1).

Obviousness may be a good starting point for framing a given proposition as an axiom, but whether we should accept or reject it is a matter of further developments. As Shapiro puts it,

> The web of mathematics is supported, and properly pruned and extended, by discovering deductive connections between propositions, both within a given branch of mathematics and between branches. By deriving a hitherto assumed proposition, or a working proposition, from others, we see what is involved in accepting or rejecting it. (SHAPIRO, 2009, p.204)

The holistic reading is a fair description of the axiomatic practice, and Cole has reasons to accept it too. But it is still not his idealized narrative on how mathematicians actually resolve their disagreements and collectively agree on which axioms to adopt or on which standing declarations to simultaneously perform. And even if they *do* end up agreeing, the pressing question is on what grounds the agreement is reached. As argued in the previous section, the answer would still impose some motivational problems for **PDR**.

In conclusion, the connection between collective agreements and declarative acts is not entirely clear, as far as the application to mathematics is concerned. We considered and rejected two options above. The first take agreements as irrelevant for declarations, while the second takes them as only relevant for granting deontic powers. Both are incoherent representations of the practice and so must be rejected.

Cole, of course, opted for a third option, in which Collective agreements are, in the mathematical case, the mutual acceptance of the same act of declaration. Even assuming this highly unlikely scenario, agreements are not fine-grained, nor are their temporal profiles clear enough for the ontology of mathematics. It is hard to believe that mathematicians are accepting the exact same acts of declaration, or representing the exact same domains as existing without any significant level of confusion. Since agreements are not precisely defined, they could speak of different things --- about different domains --- under the mist of a so-called common agreement, as history shows.

Finally, it is hard to achieve the strong conditions imposed on mathematical domains and truths relying on a contingent factor such as agreements or collective recognition. In a sense, it is quite obvious that the success of mathematical ideas and definitions depend on contingent, sociological, historical, and even idiosyncratic factors. But it is very hard to couple a realist take on ontology with such high standards (necessity and objectivity) relying on collective agreements. If Cole wants to choose this path, he must surrender to the contingency problem.

5. From PDR to Speech Act Theory

Putting the ontological drawbacks aside, Cole's Practice Dependent Realism has its fortunes, as it is the case for other social constructivists. The main benefit is that it draws attention to the *act-based* dimension of mathematics. The complaint that philosophies of mathematics lack such attention goes back to Lakatos (1976), as he claims that mainstream separates too much crudely the philosophical facets of mathematics from the fact that mathematics is a social activity. Lakatos' diagnosis is certainly right, as formalists (and platonists) had ignored much of the practical phenomenon that mathematics involves. But formal mathematics must still include some pragmatic features, given the fact that axiomatic systems --- the ones to be blamed in Lakatos' account --- are languages as well. Moreover, the purely formal devices developed by logicians are not what one founds in mathematical texts.

Cole is offering an important step towards this point. He correctly notes that the acts on which mathematics is grounded also involve linguistic acts. Even though the representational acts considered are not necessarily linguistic, I contend that, at least for mathematics, these are the interesting cases. For instance, we might make a declaration, and thus put some institutional facet of reality into existence, by simply *acting* in accordance with the existence of such facet of reality. Still, as far as mathematics is concerned, some level of communication is required. Sometimes, a mathematician (or a group of) can act in accordance with the existence of some mathematical domain or objects, but this never goes out without some level of communication. A representational act without a corresponding communication might be possible but is useless as far as any social practice is concerned. Even though mathematics involves a number of different interactions, its official vehicle of communication is the mathematical texts.

In Searle's (2010) account of social reality, freestanding social institutions, *i.e.* those not reducible to some physical entity, still require some physical realization of the **SF**s that they carry. Money can exist without its physical realization --- as cryptocurrencies make it clear --- but some level of physicality is needed for money to properly fulfill its functions. The data recorded in a computer trace the money in your account and properly represent the deontic powers that they carry, even though they are not actually money. Generally, all such cases can be reduced to some form of written language. According to Searle (2010, p.115), "All of these are made possible by the existence of writing, for a written record provides an enduring representation of the status functions in question." Even granting Cole's assumption that mathematical domains are freestanding institutional entities, we still have to consider the fact that mathematical ideas are officially transmitted through the written language of mathematics. In this context, declarations are always presented as written definitions.

But even if we take declarations as definitions within mathematical texts, in the context of mathematical proofs, **PDR** still only refers, at best, to just a small fragment of the practice. Every attempt to extend his analysis to cover the whole of mathematics can be made only at the cost of seriously deforming the way in which mathematics is commonly understood. This is because contemporary mathematics is permeated by notions that are *not* objectual: *e.g.*, functions, morphisms, algorithms, transformations, etc. All of these notions can undoubtedly be represented within ZFC as sets (and thus as objects). But this is not how mathematicians typically conceive them. For instance, functions obviously incorporate a *directive* component, *i.e.*, they express how to go *from* some x *to* some y. By restricting the focus to representative or declarative acts (as Cole does), one can take account only of how functions are introduced without being able to represent how mathematicians *use* functions.

We can find this richness of the mathematical practice also embedded within mathematical texts. Even if **PDR** is restricted to an ontological thesis, adopting such a limited perspective would conflict with the fact that mathematics consists of much more than only definitions. To portrait it properly, one shall do justice to all of its components, including the non-declarative ones. Moreover, sometimes the nature of mathematical notions can be grasped only by acknowledging the (often delicate) way in which different linguistic acts combine. Functions offer again a paradigmatic example. The modern set-theoretic definition of a function as the subset of the Cartesian product hide the fact that functions were historically regarded as *rules* (rather than objects); in fact, some modern areas of mathematics are build upon a more intensional understanding of functions where the emphasis is on how the functions are computed and thus, linguistically, on their directive core.

Therefore, Cole's insight should be extended, since there are more aspects of the mathematical phenomenon other than its objectual side. And this can be advanced from a linguistic perspective. Mathematics is a complex activity involving many different kinds of linguistic acts. Declarative acts, even if crucial, are only one of them.

5.1. Speech Acts and Mathematics

It might sound innocent in saying that mathematics is grounded in acts of communication, but this has not been thoroughly discussed in the literature, at least not from the perspective opened up by Cole's work. To be fair, the intuition of adopting a Searlian perspective to tackle the ontology of mathematics is consistent with the facts that (1) mathematical objects are indeed introduced by certain linguistic acts (*i.e.*, mathematical definitions); and (2) in this respect, they resemble many social institutions that, similarly, come to existence via declarative acts. That said, the analogy between mathematical objects and social institutions, although appealing, remains problematic. But the idea of taking philosophical problems in mathematics from a linguistic perspective might be fruitful, as far as the practice is concerned.

Avoiding the discussion about the ontological consequences of such perspective, we can extend the analysis by considering Searle's Speech Act theory in general[34]. This can be done for both the informal mathematical language used daily by mathematicians for communicative purposes and also for the highly formalized language part of mathematical texts. Recently, Ruffino, San Mauro and Venturi (2020) had initially developed such analysis. They argue that the mathematical language is embedded with pragmatic phenomenon. This is trivially true in the case of mathematical communication in general, where we often find different pragmatic features --- *e.g.* metaphors, rhetorical figures, irony, etc. --- that are employed in lectures, conversations, the explanation of ideas, and many other mathematical activities. But the authors also argue that even written mathematical texts often contain expressions that are hard to account without any appeal to pragmatics. This goes for simple auxiliary phrases that encode propositional attitudes (*e.g.* "we believe that"), directions (*e.g.* "suppose that"), assertions (*e.g.* "it is the case that"), among others, but also for formal devices. Even quantifiers, as they argue, seem to be context-sensitive. There is substantial evidence that "[...] literal meaning is simply not sufficiently fine-grained to encode all possible shades of meaning provided by different mathematical context" (RUFFINO *et al.,* 2020, p.5), they conclude.

A more substantial analysis is then offered in (RUFFINO *et al*, 2021). The core idea is to build from the previous claim that, even in formal mathematical language, there are hidden illocutionary force indicating devices in play. This idea is not new. Frege already recognized and implemented them within the *Begriffsschrift* in the late nineteenth century. But since

[34] Following the taxonomy in (SEARLE, 1979).

Speech Act Theory had developed itself from the works of Austin and Searle, a proper connection between speech acts and mathematics was still missing. The idea then is to link Searle's taxonomy of Speech Acts with the inferential structure presented in mathematical texts, given that

- they are commonly built upon affirmative statements (theorems, lemmas, corollaries) that can be analyzed from a variety of assertive illocutionary forces;
- they also made extensive use of definitions that are uttered with a declarative illocutionary force, and
- reasoning is performed using inferential rules, which can be read as having a directive illocutionary force in play.

Their work fell short of just analyzing these rather initial cases, but the perspective is certainly productive enough to be continued.

The linguistic analysis offers a thoughtful addition to discussions on the philosophy of mathematical practice, one that is missing in the works of philosophers of the *practical turn,* such as (MANCOSU, 2008). Moreover, Ruffino, San Mauro and Venturi, have not endorsed any specific ontological perspective as far as both works are concerned. But it is also not clear why should they offer one. As it seems, we can highlight the pragmatical features of the mathematical language, thus giving an account about the practice, without an explicit commitment to any ontology of mathematical objects. Frege, for example, is both a paradigmatic case for platonism and the first to include illocutionary force indicating devices within formal logic. Therefore, it is not clear how a given perspective of mathematical language forces one in adopting a specific account about mathematical objects and domains. Mathematical language seems to be invariant concerning the multiplicity of philosophical perspectives.

To be fair, **PDR** is not an account of the mathematical language, but it is still an important step towards such an analysis. Even granting that his starting point is broader than a linguistic one, declarative acts (or even collective agreements in his case) are presented in the context of mathematical texts as definitions. Following this, perhaps a more precise analysis of such linguistic devices can shed some light on Cole's ontological goals. It could make the practice consistent with the realist ontology, even if it isn't the practice-dependent type.

6. Concluding remarks

Even though **PDR** is motivated by a picture of mathematics that is clearly more faithful to its practice than a bare form of platonism, it fell short in offering a convincing case for the main problem that it proposes itself: that of reconciling mathematical practice with a realist ontology. Further explanation is missing in order to sufficiently reconcile **PDR** with some of the intuitions we have about mathematical epistemology and mathematical ontology.

Even though Cole is right in considering Declaratives as a constitutive element of mathematical practice, the full story is yet to be told. Of course, declarations are essential aspects of mathematics, but there is still a lot to be considered about the richness of mathematical practice. Even if considering the practice from a linguistic perspective seems like an important step toward a better understanding of mathematics, I believe that we should include in this analysis the whole spectrum of actions that we can perform by means of mathematical language, *viz.*, we assert, we give instructions, we convince, and so forth. Moreover, even though adopting a relevant role for declarations is justified from a linguistic perspective, it is still unclear whether it is ontologically relevant. Until this point is clarified, it remains unclear if mathematical objects can be equated to social institutions.

Regardless of the difficulties, **PDR** offers some fresh air in the philosophy of mathematics and surely points to the right direction: to account for mathematical practice in a way compatible with the objectivity of mathematics. Even if this turns out to be a difficult goal to achieve, the philosophy of mathematical practice might benefit from the linguistic perspective that Cole leaves open to be explored. This, I believe, is an important challenge that cannot anymore be ignored by philosophers.

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