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Contrast sensitivity to radial frequencies modulated by J_n and j_n Bessel profiles

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Abstract

We measured human contrast sensitivity to radial frequencies modulated by cylindrical (J_o) and spherical (j_o) Bessel profiles. We also measured responses to profiles of j_o, j_1, j_2, j_4, j_8, and j_{16}. Functions were measured three times by at least three of eight observers using a forced-choice method. The results conform to our expectations that sensitivity would be higher for cylindrical profiles. We also observed that contrast sensitivity is increased with the j_n order for *n* greater than zero, having distinct orderly effects at the low and high frequency ends. For n = 0, 1, 2, and 4 sensitivity tended to occur around 0.8-1.0 cpd while for n = 8 and 16 it seemed to shift gradually to 0.8-3.0 cpd. We interpret these results as being consistent with the possibility that spatial frequency processing by the human visual system can be defined *a priori* in terms of polar coordinates and discuss its application to study face perception.

Key words
- Spatial frequency
- Angular frequency
- Polar gratings
- Face perception
- Contrast sensitivity

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Introduction

We have been trying to evaluate the human visual system through its characteristic frequency responses to contrast of radial and angular frequencies defined as elementary stimuli in a polar coordinate system. Some of our measurements of angular and radial frequency filters have been reported elsewhere (1-6; Simas MLB, Frutuoso JT and Santos NA, unpublished results). We have been calling radial frequency the stimuli whose modulation (of any sort, including Bessel functions) is made across the radius of a circle, while angular frequency refers to sine wave modulation across the angle and is independent of the observer's distance.

Radial frequencies as elementary stimuli

There have been few attempts in the literature to conceptualize the human visual system as a processor of spatial information in terms that could be better described in a polar system of coordinates.

A first proposition in that direction was made by Kelly in 1960 (7), who suggested the use of circularly symmetric "J_o targets", i.e., stimuli modulated by cylindrical Bessel functions of zero order. Since this study attracted little attention in the literature, Kelly and Magnuski (8) proceeded to compare contrast sensitivity to sine wave gratings to sensitivity to J_o targets. Perhaps due to the fact that their findings showed sensitivity to J_o

targets to be much lower than to gratings, no further work on the subject was done until 1982. In that year, Kelly (9) repeated part of the work of 1975 and went on to use circularly symmetric stimuli in a later study (10). In the latter, he also used concentric cosines to evaluate spatial summation.

Few other investigators came close to applying these concepts. Only recently did Amidor (11) and Wilkinson et al. (12) use the term radial frequency to refer to different meanings of modulations, some of them concerning modulations across the radius. Mainstream research seems to be more directed towards local filtering of spatial characteristics employing Gabor's or DOG functions as elementary stimuli, generally defined in Cartesian coordinates to probe local processing (e.g., 13-19).

Dodwell (20) has proposed a Lie transformation group model based on Hoffman's suggestion (21). These researchers see filtering in polar coordinates as a by-product of their model. Also, along Dodwell's line, Gallant, Van Essen and colleagues (22,23) used sine wave and square radial and angular frequency targets to test the sensitivity of V4 cells in macaques.

The studies cited above contain all or most of the literature on the subject.

In our original work published in 1985 and 1990 (1,2), we conceived the visual system in terms of spherically shaped eyes processing visual information and replaced Kelly's (7) suggestion of cylindrical Bessel function (mathematical notation J_n) with the choice of spherical ones (mathematical notation j_n, see Appendix).

Even if we assume simultaneous separate decompositions by each retina, this substitution of cylindrical by spherical should be made because each eye constitutes spheres upon which spatial information is projected. We could only consider cylindrical Bessel functions for 2-D surfaces or for extremely restricted areas at the center of the eye.

As the main difference between spherical and cylindrical Bessel functions is the intensification of contrast from the center towards the periphery, we would expect small, or almost no difference in trends, but eventual increases in sensitivity for cylindrical Bessel functions. Figure 1 shows profiles for cylindrical functions, J_n (top), compared to spherical functions, j_n (bottom), for *n* values 0, 1 and 8. Figure 2 shows side by side photographs of cylindrical, J_n (right) and spherical, j_n (left) functions for n = 0, 2 and 8.

Figure 3 shows profiles of j_4 (μ) for different radial frequencies, i.e., n-order = 4 (constant) with varying radial frequencies, while Figure 4 shows photos of stimuli at n = 0.4 and 8 at radial frequencies of 0.5 and 3.0 cpd (as originally drawn for our experiment run at 1.5 m distance).

In the present study, in addition to reporting data for two different experiments, in detail we compare contrast thresholds to spherical Bessel functions with cylindrical ones for n-order 0 (zero), a comparison not previously made in the literature. We compare measurements of contrast thresholds for radial frequency stimuli modulated by j_n of Bessel profiles of j_0, j_1, j_2, j_4, j_8 and j_{16} with those performed for J_0.

Method

Subjects

Eight 19- to 42-year-old naive subjects with normal or corrected vision participated in the experiments. Most of the participants were selected among undergraduate students interested in carrying out research on vision at our laboratory. Both authors participated in some measurements and were naive with respect to the specific experiment as well as to the expected results.

Equipment and stimulus material

Circular stimuli measuring 7.25 degrees of visual angle in diameter were generated

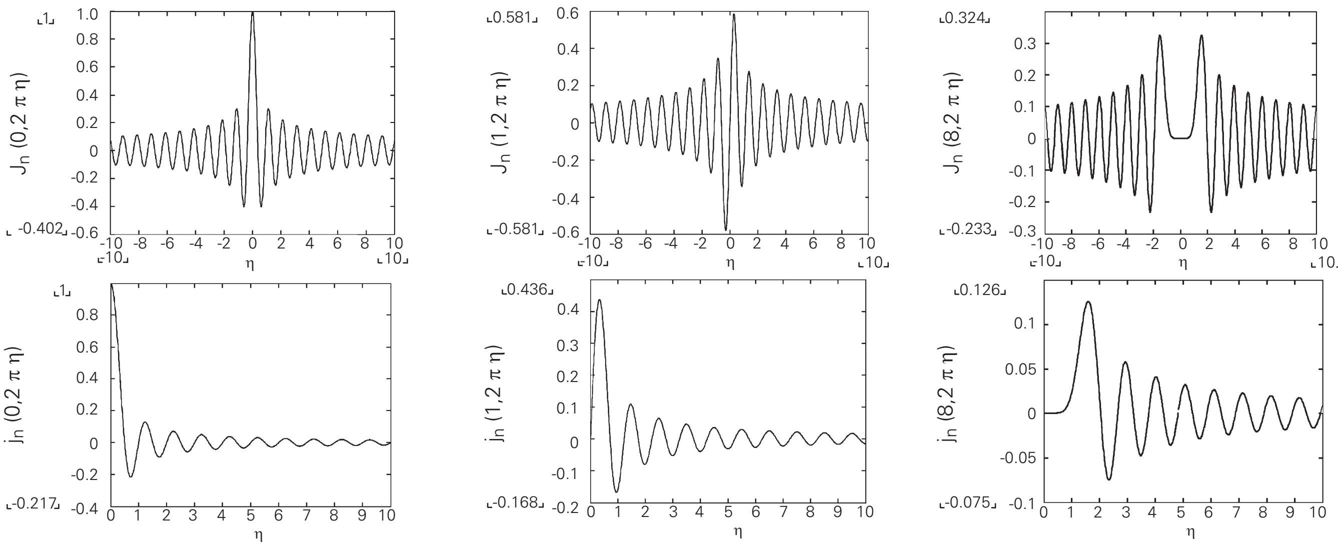

Figure 1. Profiles of cylindrical Bessel functions, J_n (top), and spherical Bessel functions, j_n (bottom), for n = 0 (left), n = 1 (center) and n = 8 (right). Profiles of j_n (bottom) are defined only for n from 0 (zero) to infinity.

on a standard 21" TV display (Telefunken with a standard resolution of 250 lines and medium-low contrast resolution) by a 486 PC through a DT2853 frame grabber with RGBsync output. Measurements were made binocularly at a 150-cm distance with a mean luminance equal to 1.8 fL, or 6.2 cd/m^2. This luminance level is necessary in order to run a forced choice procedure since we have to display mean luminance within the linearity of the TV monitor at human threshold levels. Contrast was defined as usually done in the literature on spatial vision, i.e., Lum_{max} - Lum_{min}/Lum_{max} + Lum_{min}.

In the first experiment we used radial frequencies of 0.2, 0.3, 0.5, 0.8, 1, 2, 3, 4, 5, 6 and 9 cycles per degree of visual angle, cpd, modulated by both J_n (r) and j_n (r) profiles of n-order 0 (zero) to characterize each curve. Figure 2 shows examples of J_n (r) (right) and j_n (r) (left) profiles for n-orders 0, 2 and 8 (from top to bottom, respectively).

In the second experiment we used the same set of radial frequencies but modulated by j_n (r) profiles of n-orders 0, 1, 2, 4, 8 and 16. The same measurements of the j_n (r) profile, n-order 0 (zero) were used for both experiments. Figure 5 shows profiles originally obtained for n = 0 (top left), 1 (top right), 2 (center left), 4 (center right), 8 (bottom left) and 16 (bottom right) at 1-cpd radial frequency as calibrated under the specific experimental conditions used.

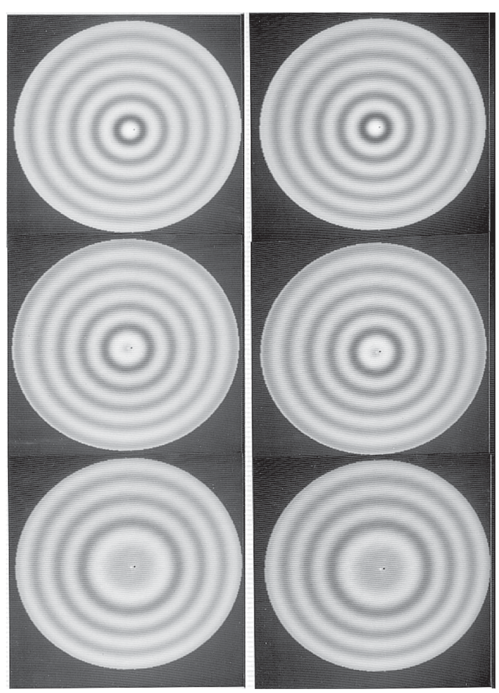

Figure 2. Photographs of spherical Bessel functions, j_n (left), and cylindrical Bessel functions, J_n (right), for n = 0 (top), n = 2 (center) and n = 8 (bottom).

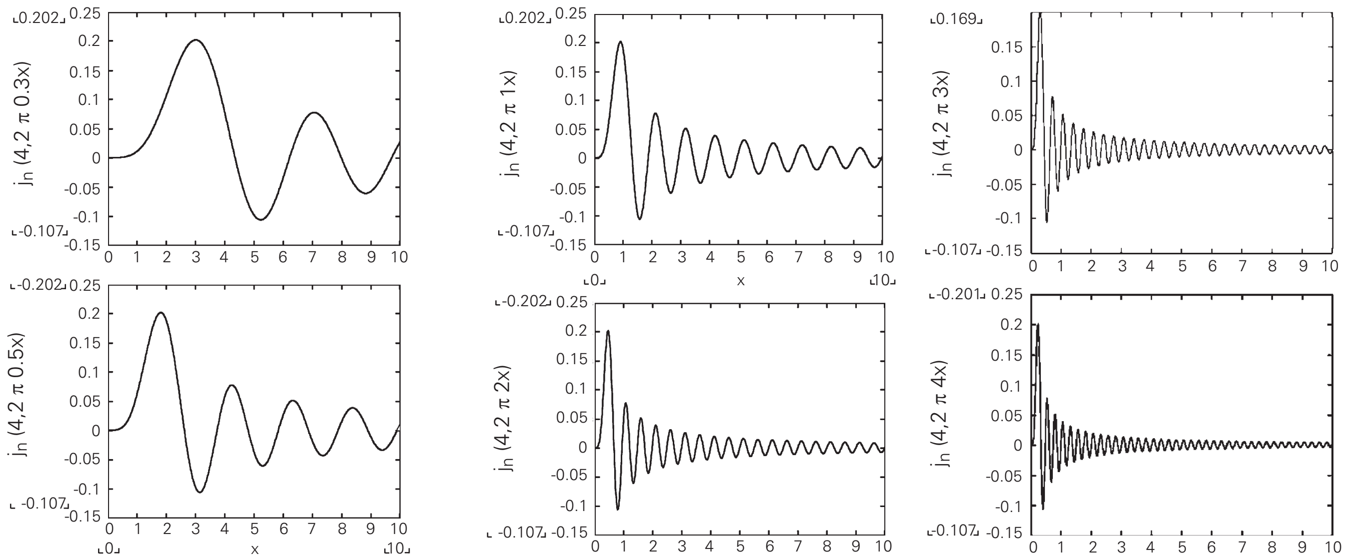

Figure 3. Profiles of spherical Bessel functions, j_n (μ), for n = 4, at varying radial frequencies (μ = 0.3x, 0.5x, left; 1x, 2x, center, and 3x and 4x, right).

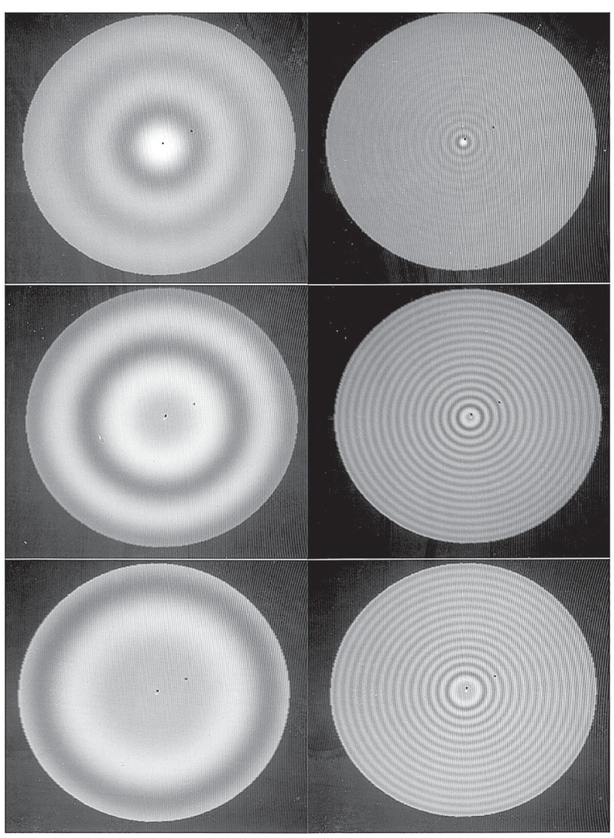

Figure 4. Photographs of spherical Bessel functions, j_n (μ), for n = 0, n = 4 and n = 8 (top, center, bottom) at μ = 0.5 (left) and 3.0 cpd (right) as originally calibrated for a 150-cm viewing distance.

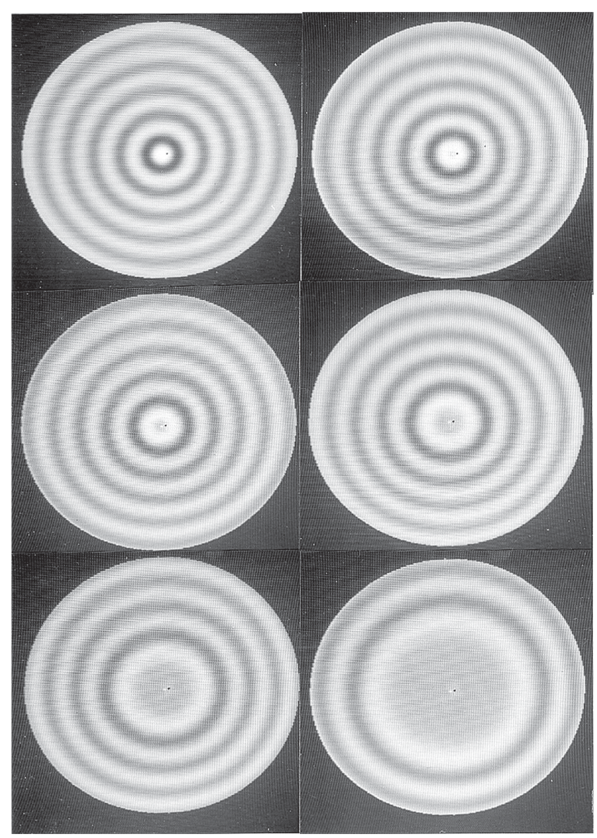

Figure 5. Stimuli defined by spherical Bessel function profiles originally taken for n = 0 (top left), n = 1 (top right), n = 2 (center left), n = 4 (center right), n = 8 (bottom left) and n = 16 (bottom right) at 1-cpd radial frequency as calibrated under the specific experimental conditions.

Procedure

In the first experiment we had eight subjects (TPL, ALL, MMM, ERB, SAL, NAS, FMR and MLS) and each of the 11 radial frequencies of j_n (r) (n = 0) was measured three times, on different days, using a forced-choice method (24). For J_n (r) (n = 0) measurements were made under the same conditions but with only five subjects (TPL, ALL, MMM, ERB and SAL). The order of measurement for conditions and curves was random for each observer. A whole curve was measured per day. In every experimental session pairs of stimuli were presented centered on the screen at the same site over two different time intervals: one contained one of the radial frequencies and the other was merely a circle at mean luminance. The same radial frequency was presented throughout an entire session. Each stimulus was presented over a period of 2000 ms. The interstimulus interval lasted 1000 ms, whereas the intertrial interval lasted 3000 ms following the answer of the subject. The task of the observer was to correctly select the stimulus containing the radial frequency. The criterion was to lower contrast by 0.8% after three correct choices and to increase it by the same amount after each subject's error. A session was terminated after obtaining 20 estimates, i.e., 10 pairs of peaks and valleys.

In the second experiment, j_1, j_4, j_8 and j_{16} Bessel profiles characterized by the same set of 11 radial frequencies were measured three times each by each of five subjects (TPL, ALL, NAS, FMR and MLS), for a total of 132 experimental sessions per subject (not including those of the first experiment). The remaining procedure and conditions were exactly as described above. The j_2 Bessel profile was measured three times by only three subjects (NAS, FMR and MLS).

Results

Figure 6 shows contrast thresholds for radial frequencies modulated by J_n (r) and j_n (r) Bessel profiles of n-order 0 (zero). Thresholds were lower for cylindrical profiles for all but one point estimate.

Figure 7 shows the contrast thresholds for radial frequencies modulated by j_n (r) Bessel profiles of n-orders 0 (zero), 1, 2, 4, 8 and 16.

Contrast sensitivity (i.e., the reciprocal of the contrast threshold) is increased with the j_n order for *n* greater than zero. We can observe distinct order effects at the low and high frequency ends for increasing *n*. Stimuli having j_8 and j_{16} profiles are the most difficult to detect at the low frequency end, whereas these tend to be the easiest at the high frequency end.

Also, for Bessel profiles j_0, j_1, j_2 and j_4, maximum sensitivity tends to occur at the 0.8- to 1-cpd interval, while for the j_8 and j_{16} profiles this sensitivity appears to shift gradually to the 0.8- to 3-cpd interval. For the j_{16} profile, maximum sensitivity for all observers was in the 2- to 3-cpd radial frequency interval. We believe this is due to increased iso-luminance at the center of the stimuli which occurs with increasing n-order. That is to say that the task of detection of these stimuli is shifting towards the periphery. We did not study enough n-orders to observe at what radial frequency this shifting stabilizes.

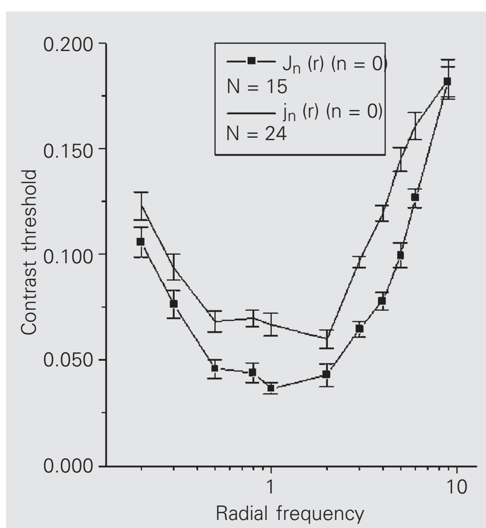

Figure 6. Contrast thresholds for radial frequencies modulated by J_n (r) and j_n (r) Bessel profiles of n-order 0 (zero). N represents the number of measured functions for all points.

Our statistical treatment was to estimate the standard error of the mean for each distribution of 180-300 values measured (within and across subjects) per point and correct for sample size using the *t*-Student statistic to obtain the 99% confidence level.

In previous studies we have demonstrated that this corrected estimate is more rigorous than testing means by comparison for two correlated samples or by ANOVA treatment. Using the confidence level method, when two error bars show a 50% overlap, if we calculate the *t*-statistic obtained by testing difference between two means for correlated samples, we find a significance of at least P = 0.05, and sometimes smaller. In our treatment, most of the error bars that do not overlap would show a significance of P<0.000 if a *t*-test for comparison of means for correlated samples were performed. The use of ANOVAS (very traditional in psychology) in this case tends to show significant effects in all factors and interactions and does not add much information.

Discussion

As expected, sensitivity was higher for cylindrical Bessel profiles. Since the difference in peak at zero and modulation of periphery is higher for spherical profiles (see Figure 1) we did expect higher sensitivities for cylindrical profiles because, as one can see in Figures 1 and 2, contrast at the periphery is perceived as higher than that for spherical profiles. However, to study the expansion effects of increasing the n-order, we used spherical profiles because of the spherical shape of the eyes (and retina).

Figure 8 also shows the comparisons between sensitivity to sine wave gratings, cylindrical Bessel J_0, angular frequencies and coupled radial/angular frequencies of n-order equal to 4. Note that Figure 8 shows contrast thresholds rather than sensitivity.

We also obtained results comparing narrow-band spatial frequency filters to narrow-band radial frequency filters modulated by j_0, j_1, j_4, j_8 and j_{16} Bessel profiles and

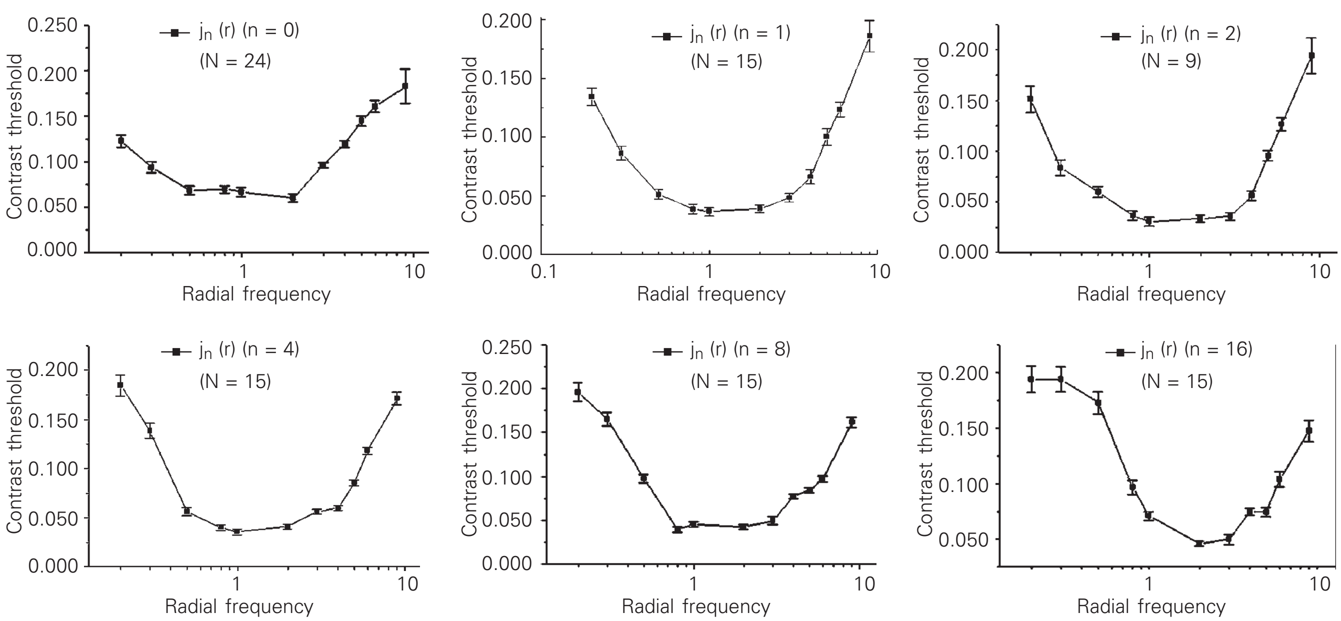

Figure 7. Contrast thresholds for radial frequencies modulated by j_n (r) Bessel profiles of n-orders 0 (zero), 1, 2, 4, 8 and 16. N represents the number of measured functions for all points.

centered at radial frequencies of 1 and 4 cpd (Simas MLB, Frutuoso JT and Santos NA, unpublished results). These results are consistent with the possible existence of narrow-band radial frequencies that seem to operate linearly at higher contrast levels than spatial frequency filters, as shown by a supra-threshold summation paradigm.

Although the results reported here are only for the radial part, because we used n>0, they should be considered together with the orthogonal part, i.e., as angular frequencies because when n>0 the angular component is also present.

Filtering of radial and angular frequencies by the visual system

Our studies of 1985 (1), 1990 (2) and 1992 (3) on angular frequency detection and filtering by the human visual system should be considered in relation to the data reported here. Selective neighboring inhibition around the testing frequency was consistently observed for angular frequency filters of 2, 4, 9, 13, 16, 24 and 47 cycles (3). The previous results were obtained with the use of phases different from those we use today. The study was carried out only with 24 cycles (1,2).

Those results, together with the findings of Gallant and colleagues (22,23) about groups of cells selective for angular and radial frequencies strongly encouraged us to proceed with our psychophysical studies in that direction. To our knowledge, except for Kelly's early study (7), no investigations were carried out with targets modulated by Bessel functions, or perhaps circular cosine functions. These are not equivalent and Kelly and Magnuski (8) measured them both.

To better evaluate filtering of radial and angular spatial frequencies by the human visual system we should point out that when n = 0 in Equation 4, only the radial part exists, precisely the J_o (we preferred j_0) part pointed out by Kelly (7). This part is well suited to describe retinal acuity and light intensity centered in fovea and near fovea regions. The angular part would be zero, i.e., mean luminance. When this happens, higher luminance levels are required as compared to both sensitivity to gratings and to angular frequencies.

The opposite situation, i.e., only the angular component, would occur when the stimulus has radial frequencies in the lowest range of the radial spectrum so that a large portion of a cycle is present. In this case, light intensity would not be concentrated at the fovea and would spread evenly across large portions of the visual field except for the modulation of the angular part present. This could occur for any n>0, but may occur more commonly for n>8. The angular part would be always present for n>0 and the

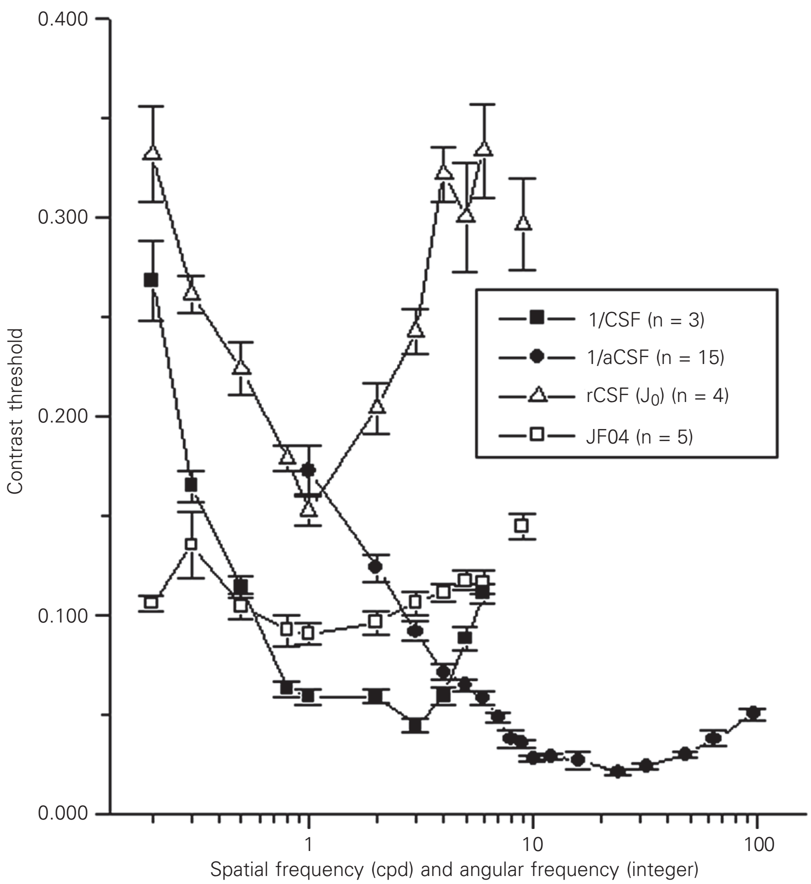

Figure 8. Comparisons between contrast thresholds for sine wave gratings (i.e., reciprocal of contrast sensitivity function, 1/CSF, dark square), cylindrical Bessel J_0 (i.e., 1/radial CSF, rCSF, triangle), angular frequencies (i.e., 1/angular CSF, 1/aCSF, circle) and coupled radial/angular frequencies of n-order equal to 4 (i.e., JF04, light square).

exact number and values of *n* would depend on the particular visual system in question. For the human visual system, the angular modulation transfer function ranges from 1 to 100 cycles (we made n = integer, to fit exactly in 360 degrees) (1,2,4).

Since we were also interested in learning how this filtering process could be involved in face detection and/or recognition, in a related study on narrow-band angular frequency filters we presented the face of the night monkey aotus as an example showing that the angular part appears to be somewhat dominant over the radial one, particularly at 1, 2, 3 and 4 cycles (25). We also find it interesting that the aotus has nocturnal habits because if we consider very low angular frequencies such as these, very large portions of the retina would be involved in visually processing information and thus would require (work best under) lower luminance levels.

If we assume that the visual system of an organism must have the necessary characteristics to recognize its own species, as shown in the literature (26,27), we could try to predict maximum sensitivity to radial frequencies for certain species. In this case, we could take into consideration base focal distance together with the typical size of their faces in their habitat. It would be just a matter of calculating for a given distance which first minimum of radial frequency would coincide with their black/white regions around the eyes and head. If the angular component is present, this would show on the face of a given species, particularly for those species using few low angular frequency filters. Note that the tuning for spatial frequencies is not necessarily equivalent to that for radial frequencies. For example, the human visual system is tuned to 3-4 cpd for spatial frequencies and 0.8-2.0 cpd for radial frequencies of n = 0 profiles.

Since it is very unlikely that all organisms having circular pupils will have a complete set of all radial and angular frequency components, a few animals possessing less developed visual systems could show their most basic angular/radial filters on their own patterned faces. The aotus would be a good example for low *n*, i.e., 2, 3 and 4.

If we bear in mind that, as we proceed to select phases for the angular frequency stimuli, we find that for an even *n* there could be symmetry among quadrants, whereas for an odd *n* there could be symmetry between hemispheres, this would have strong implications in visual projections from one area to another and among different species. Animals having homogeneously colored faces, less protruding noses, etc., would need more complete sets to process spatial information. In this case, they could develop coupled radial and angular frequencies presenting n = even and n = odd-even series. An example of n = even series is 2, 4, 8, 16 cycles, etc. An n = odd-even series would be 1, 2, 3, 4, 6, 8, 12, 16, etc. If we consider that n = even harmonic series may suffer biological "up-grades" more easily, n = even series would be more frequent in lower species. Since n = odd series may require hemispheric symmetry, angular frequencies of 1, 2, 3, 5, 7 cycles, etc., would be originators of sub-series.

Thus, in the present study we are just starting to explore how the human visual system responds as we manipulate the elementary frequency stimulus sets of the radial component. We observed that as *n* is increased (n>0), maximum sensitivity tends to be displaced to higher frequencies even though the absolute value of the contrast threshold may remain in the same range as for low *n*. Since we used n>0, the angular component is also relevant. We already have some initial results from studies coupling angular and radial components which show that adding the angular part to j_n increases overall sensitivity.

Finally, we should mention a very recent study (28) that used Bessel function stimulus modulation, but only the cylindrical kind at n = 0. This study deals with issues quite distinct from those addressed in the present study.

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Appendix

Angular frequencies as elementary stimuli

Despite the fact that if we adopt a polar coordinate system it is necessary to divide elementary stimuli into two components, i.e., radial and angular, in his original proposal, Kelly (7) defined only the radial part as cylindrical Bessel functions of zero order and left out the angular part.

In 1985, Simas (1), using an equation from Sneddon (29) defining a Hankel series, introduced the idea of angular frequency stimuli as the orthogonal component of radial frequencies, i.e. defined by Kelly as J_n targets.

From Sneddon's equation describing a series defined in polar coordinates:

$$\exp(i\boldsymbol{x}\sin\theta) = \sum_{n=-\infty}^{\infty} J_n(\boldsymbol{x})e^{in\theta} \qquad \text{(Equation 1)}$$

where $J_n(\boldsymbol{x})$ is the radial part and $e^{in\theta}$ the angular one, we substituted $\boldsymbol{x}$ for $\boldsymbol{2}\pi\boldsymbol{\rho r}$:

$$\exp(i\boldsymbol{2\pi\rho r}\sin\theta) = \sum_{n=-\infty}^{\infty} J_n(\boldsymbol{2\pi\rho r})e^{in\theta} \qquad \text{(Equation 2)}$$

and made:

$$e^{in\theta} = \cos(n\theta) + i\,\sin(n\theta). \qquad \text{(Equation 3)}$$

Thus,

$$\exp(i\boldsymbol{2\pi\rho r}\sin\theta) = \sum_{n=-\infty}^{\infty} J_n(\boldsymbol{2\pi\rho r})\ \{\cos(n\theta) + i\,\sin(n\theta)\} \qquad \text{(Equation 4)}$$

Equation 4 defines a complete set of elementary stimuli in polar coordinates. Using this equation we can treat any arbitrary stimulus containing both radial and angular elements, describing it by coupled radial and angular frequencies that have the same *n* value.