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COMPUTATIONAL COMPLEXITY OF CLASSICAL PROBLEMS FOR HEREDITARY CLIQUE-HELLY GRAPHS

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Abstract

A graph is clique-Helly when its cliques satisfy the Helly property. A graph is hereditary clique-Helly when every induced subgraph of it is clique-Helly. The decision problems associated to the stability, chromatic, clique and clique-covering numbers are NP-complete for clique-Helly graphs. In this note, we analyze the complexity of these problems for hereditary clique-Helly graphs. Some of them can be deduced easily by known results. We prove that the clique-covering problem remains NP-complete for hereditary clique-Helly graphs. Furthermore, the decision problems associated to the clique-transversal and the clique-independence numbers are analyzed too. We prove that they remain NP-complete for a smaller class: hereditary clique-Helly split graphs.

Keywords: computational complexity; hereditary clique-Helly graphs; split graphs.

1. Introduction

All graphs in this paper are finite, without loops or multiple edges. For a graph *G* we denote by *V*(*G*) and *E*(*G*) the vertex set and the edge set of *G*, respectively.

A graph is complete if every pair of vertices is connected by an edge. A complete in a graph *G* is a subset of pairwise adjacent vertices of *G*. A clique in a graph is a complete maximal under inclusion. The clique number of a graph *G* is the cardinality of a maximum clique of *G* and is denoted by $\omega(G)$.

The chromatic number $\chi(G)$ of a graph *G* is the smallest number of colours that can be assigned to the vertices of *G* in such a way that no two adjacent vertices receive the same colour.

A clique cover of a graph *G* is a subset of cliques covering all the vertices of *G*. A clique-transversal is a set of vertices intersecting all the cliques of *G*. The clique-covering number *k*(*G*) and the clique-transversal number $\tau_C(G)$ are the cardinalities of a minimum clique cover and a minimum clique-transversal of *G*, respectively.

A stable set in a graph *G* is a subset of pairwise non-adjacent vertices of *G*. A clique-independent set is a subset of pairwise disjoint cliques of *G*. The stability number $\alpha(G)$ and the clique-independence number $\alpha_C(G)$ are the cardinalities of a maximum stable set and a maximum clique-independent set of *G*, respectively.

Consider a finite family of non-empty sets. The intersection graph of this family is obtained by representing each set by a vertex, two vertices being connected by an edge if and only if the corresponding sets intersect. The clique graph *K*(*G*) of *G* is the intersection graph of the cliques of *G*.

A family *S* of subsets satisfies the Helly property when every subfamily of *S* consisting of pairwise intersecting subsets has a common element. A graph is clique-Helly (*CH*) when its cliques satisfy the Helly property. A graph *G* is hereditary clique-Helly (*HCH*) when *H* is clique-Helly for every induced subgraph *H* of *G*. These graphs have been characterized in [Pr93] as the graphs which contains none of the four graphs in Figure 1 as an induced subgraph. This characterization leads to a polynomial time recognition algorithm for hereditary clique-Helly graphs.

An interesting survey on clique-Helly and hereditary clique-Helly graphs appears in [Fa02].

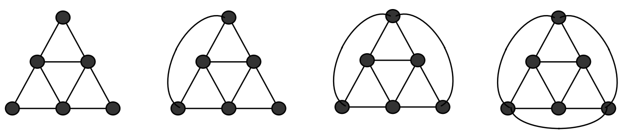

Figure 1 – Hajös graphs

A graph is split if its vertices can be partitioned into a clique and a stable set.

The neighborhood of a vertex *v* in a graph *G* is the set *N*(*v*) consisting of all the vertices that are adjacent to *v*. The closed neighborhood of *v* is $N[v] = N(v) \cup \{v\}$. A vertex *v* of *G* is called simplicial when *N*[*v*] is a complete of *G*, and universal when *N*[*v*]=*V*(*G*).

It is easy to see that the decision problems associated to the stability, chromatic, clique and clique-covering numbers are NP-complete for clique-Helly graphs. The reduction is trivial: we have to add a universal vertex to the general graph *G* in order to generate a clique-Helly graph G^+.

However, $\omega(G)$ can be obtained in polynomial time for *HCH* graphs. The number of cliques is bounded by the number of edges [Pr93] and all the cliques can be generated in $O(nmk)$, where *m* is the number of edges, *n* the number of vertices and *k* the number of cliques of the graph [TIAS77].

The stable set and the colorability problems remain NP-complete for *HCH* graphs. These results are direct corollaries of the NP-completeness of these problems for triangle-free graphs [Pol74], [MP96]. For triangle-free graphs, a subclass of *HCH* graphs, the clique-covering number can be obtained in polynomial time [GJ79].

So, the following question arises naturally: what happens with the complexity of the clique-cover problem for hereditary clique-Helly graphs?

The decision problems associated to the problems of finding the clique-independence number and the clique-transversal number are NP-complete [CFT93] and NP-hard [EGT92], respectively. This last problem is not known to be in NP, in fact the problem of determining if a subset of vertices is a clique-transversal is NP-hard [DLS02].

The clique-transversal problem is NP-complete for *HCH* graphs. Again, this result is a consequence of the NP-completeness of this problem for triangle-free graphs. In this class of graphs, the clique-transversal problem is equivalent to vertex cover, and vertex cover is NP-complete for triangle-free graphs [Pol74]. Remember that in this case the problem is in NP for the property of *HCH* graphs above mentioned. This problem remains NP-complete for split graphs [GP00].

However, the clique-independence number can be obtained in polynomial time for triangle-free graphs, because it is equivalent in this case to maximum matching. This problem is NP-complete for split graphs [GP00] but, to our knowledge, it was not known its complexity for clique-Helly graphs.

Again, the following question appears naturally: what happens with the complexity of the clique-independence problem for hereditary clique-Helly graphs?

In this note, we prove that clique-cover and clique-independence problems remain NP-complete for *HCH* graphs. Additionally, it is proved that clique-transversal and clique-independence problems remain NP-complete for a smaller class: the intersection between *HCH* and split graphs.

2. Preliminaries

There are some relations between the parameters defined in the introduction in a graph *G* and its clique graph *K*(*G*).

Theorem 2.1 *Let G be a graph. Then:*

(i) $\alpha_C(G) = \alpha(K(G))$.

(ii) *If G is a clique-Helly graph then* $\tau_C(G) = k(K(G))$.

Proof: (i) It follows from the fact that independent cliques of *G* correspond to non adjacent vertices in *K*(*G*), and conversely, non adjacent vertices in *K*(*G*) correspond to independent cliques in *G*.

(ii) Let $v_1,\ldots,v_{\tau_C(G)}$ be a clique-transversal set of *G*. For each *i*, analyze the vertices in *K*(*G*) corresponding to the cliques in *G* that contain the vertex v_i. They form a complete of *K*(*G*). This complete must be included in some clique L_i of *K*(*G*). Observe that these cliques L_i ($i=1,\ldots,\tau_C(G)$) are not all necessarily different. Let us see that these at most $\tau_C(G)$ cliques are a clique cover of *K*(*G*). Let *w* be a vertex of *K*(*G*). Then *w* corresponds to some clique M_w of *G*. As the set $v_1,\ldots,v_{\tau_C(G)}$ intersects all the cliques of *G*, there is some vertex v_j that belongs to M_w. This means that the corresponding vertex of M_w in *K*(*G*) belongs to the clique L_j, i.e, $w \in L_j$. Then, the size of the minimum clique cover of *K*(*G*) is at most the size of this clique cover which is at most $\tau_C(G)$.

All we need to prove is that if *G* is clique-Helly, then $\tau_C(G) \le k(K(G))$. By the Helly property, each clique *L* of *K*(*G*) has an associated vertex v_L in *G* such that v_L belongs to all the cliques of *G* corresponding to the vertices of *L* in *K*(*G*).

Let $L_1,\ldots,L_{k(K(G))}$ be a clique cover of *K*(*G*). Let $v_{L_1},\ldots,v_{L_{k(K(G))}}$ be the vertices in *G* associated to those *k*(*K*(*G*)) cliques. Let us see that they form a clique-transversal set of *G*. Let *M* be a clique of *G* and w_M its corresponding vertex in *K*(*G*). Then there is an index *j* such that w_M belongs to the clique L_j in *K*(*G*). It follows that v_{L_j} belongs to *M* in *G*. □

Let $M_1,\ldots,M_k$ and $v_1,\ldots,v_n$ be the cliques and vertices of a graph *G*, respectively. A clique matrix $A_G \in \mathbb{R}^{k\times n}$ of *G* is a 0-1 matrix whose entry a_{ij} is 1 if $v_j \in M_i$, and 0, otherwise. Another characterization of *HCH* graphs is the following [Pr93]: a graph *G* is *HCH* if and only if A_G does not contain a vertex-edge incidence matrix of a triangle as a submatrix.

Let $M_1,\ldots,M_k$ and $v_1,\ldots,v_n$ be the cliques and vertices of a graph *G*, respectively. Define the graph *H*(*G*) where $V(H(G)) = \{q_1,\ldots,q_k, w_1,\ldots,w_n\}$, each q_i corresponds to the clique M_i of *G*, and each w_j corresponds to the vertex v_j of *G*. The edges of *H*(*G*) are the following: the vertices $q_1,\ldots,q_k$ induce the graph *K*(*G*), the vertices $w_1,\ldots,w_n$ are a stable set and w_j is adjacent to q_i if and only if v_j belongs to the clique M_i in *G*.

Let $A \in \mathbb{R}^{n\times m}$ and $B \in \mathbb{R}^{n\times k}$ be two matrices. We define the matrix $A \mid B \in \mathbb{R}^{n\times(m+k)}$ as $(A \mid B)(i,j) = A(i,j)$ for *i*=1,…,*n*, *j*=1,…,*m* and $(A \mid B)(i, m+j) = B(i,j)$ for *i*=1,…,*n*, *j*=1,…,*k*. Let I_n be the $n\times n$ identity matrix.

Theorem 2.2 [Ham68] *Let G be a clique-Helly graph and H(G) as it is defined above. Then the cliques of H(G) are* $N[w_i]$ *for each i,* w_i *is a simplicial vertex of H(G) for every i, and K(H(G)) = G.*

Corollary 2.1 *Let G be a clique-Helly graph, |V(G)| = n. Then* $A_{H(G)} = A_G^t \mid I_n$.

Proof: It follows directly from the fact that $N[w_i]$ (i=1,…,n) are the cliques of $H(G)$ and each clique contains the vertex w_i and the vertices q_j whose corresponding cliques M_j contain the vertex v_i in G. □

This corollary leads us to prove the following result:

Theorem 2.3 *Let G be an HCH graph. Then H(G) is HCH.*

Proof: Let G be an *HCH* graph, $|V(G)| = n$. By Corollary 2.1, $A_{H(G)} = A_G^t \mid I_n$. Let us suppose that $A_{H(G)}$ contains a vertex-edge incidence matrix of a triangle as a submatrix. Since it has two 1's in each column, it must be a submatrix of A_G^t, but then A_G contains a vertex-edge incidence matrix of a triangle as a submatrix, which is a contradiction. □

3. Clique Cover

The decision problem associated to the problem of finding the clique-covering number of a graph is the following:

CLIQUE COVER
INSTANCE: A graph $G = (V,E)$, a positive integer $K \leq |V|$.
QUESTION: Are there $k \leq K$ cliques of G covering all the vertices of G ?

To prove that CLIQUE COVER is NP-complete for *HCH* graphs, we will use that the following problem is NP-complete [GJ79]:

EXACT COVER BY 3-SETS (X3C)
INSTANCE: A set X such that $|X|=3q$ and a collection C of 3-element subsets of X.
QUESTION: Does C contain an exact cover (by q sets) of X ?

Theorem 3.1 *The problem CLIQUE COVER is NP-complete for HCH graphs.*

Proof: The transformation from X3C to CLIQUE COVER on *HCH* graphs is based on the transformation given in [GJ79] from X3C to PARTITION INTO TRIANGLES and is the following: let the set X with $|X|=3q$ and the collection C of 3-element subsets of X be an arbitrary instance of X3C. We will construct an *HCH* graph $G=(V,E)$, with $|V|=3q'$, such that G has a clique cover of size at most q' if and only if C contains an exact cover of X.

We will replace each subset $c_i = \{x_i, y_i, z_i\}$ in C by the graph of Figure 2. Let E_i be the set of 18 edges of the graph corresponding to $\{x_i, y_i, z_i\}$.

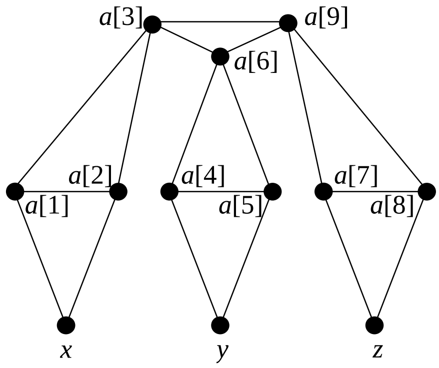

Figure 2 – Local replacement for $c=\{x,y,z\}$ in C for transforming X3C to CLIQUE COVER.

Thus $G=(V,E)$ is defined by

$$V = X \cup \bigcup_{i=1}^{|C|} \{a_i[j] : 1 \leq j \leq 9\}, \;\; E = \bigcup_{i=1}^{|C|} E_i$$

It is easy to see that G does not contain any graph of Figure 1 as an induced subgraph, thus G is an *HCH* graph, $|V| = |X| + 9|C|$ ($q' = q + 3|C|$) and the transformation can be made in polynomial time. Figure 3 shows an example of this transformation from an instance of X3C to an instance of CLIQUE COVER.

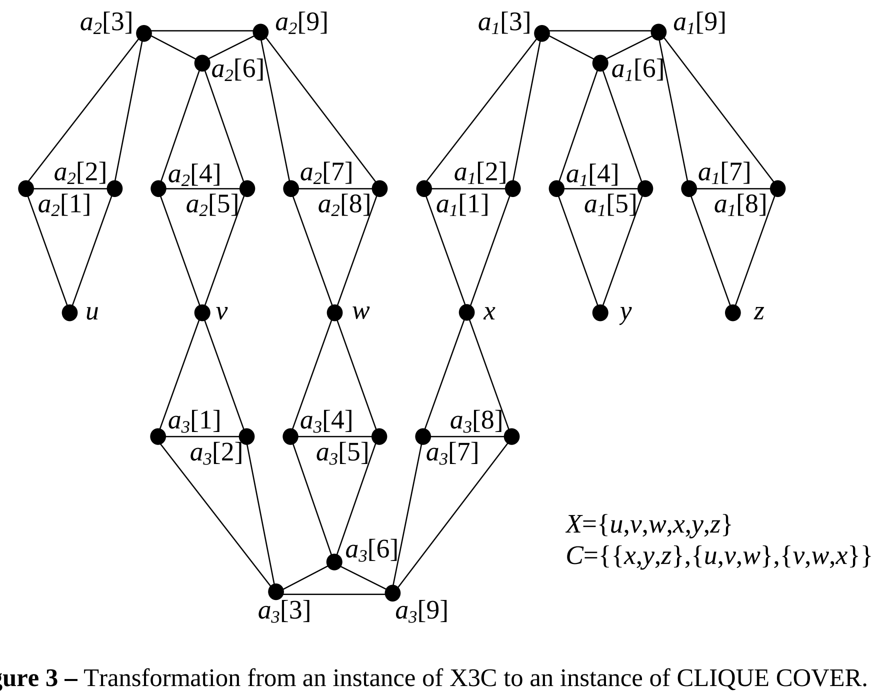

Figure 3 – Transformation from an instance of X3C to an instance of CLIQUE COVER.

Let us suppose that *C* contains an exact cover of *X*, then we construct a clique cover of *G* of size *q*', by taking for each $1 \leq i \leq |C|$

$$\{a_i[1], a_i[2], x_i\}, \{a_i[4], a_i[5], y_i\}, \{a_i[7], a_i[8], z_i\}, \{a_i[3], a_i[6], a_i[9]\},$$

whenever $c_i = \{x_i, y_i, z_i\}$ is in the exact cover and

$$\{a_i[1], a_i[2], a_i[3]\}, \{a_i[4], a_i[5], a_i[6]\}, \{a_i[7], a_i[8], a_i[9]\},$$

otherwise.

Let us now suppose that *G* has a clique cover of size at most *q*'. Since the cliques of *G* are triangles, the number of cliques in the clique cover must be *q*' and each vertex of *G* must be covered exactly once.

In the graph of Figure 2, the only two ways of covering by triangles each vertex $a_i[j]$ (*j*=1,…,9) exactly once are the above mentioned, covering or not x_i, y_i and z_i, respectively. Then the exact cover of *X* is given by choosing those $c_i \in C$ such that $\{a_i[3], a_i[6], a_i[9]\}$ belongs to the clique cover of *G*.

Finally, the membership in NP for the restricted problem follows from that for the general problem. □

4. Clique Transversal and Clique-Independent Set

The decision problems associated to the problems of finding the clique-independence number and the clique-transversal number of a graph, respectively, are the following:

CLIQUE-INDEPENDENT SET

INSTANCE: A graph *G* = (*V*,*E*), a positive integer $K \leq |V|$.

QUESTION: Is there a set of *K* or more pairwise disjoint cliques of *G* ?

CLIQUE-TRANSVERSAL

INSTANCE: *G* = (*V*,*E*), a positive integer $K \leq |V|$.

QUESTION: Is there a set of *K* or fewer vertices of *G* intersecting all the cliques of *G* ?

Theorem 4.1 *The problems CLIQUE-TRANSVERSAL and CLIQUE-INDEPENDENT SET are NP-complete for HCH split graphs.*

Proof: We will show a polynomial time transformation from CLIQUE COVER on *HCH* graphs (by Theorem 3.1 it is NP-complete) to CLIQUE-TRANSVERSAL on *HCH* split graphs.

Define the graph G^+ where $V(G^+) = V(G) \cup \{u\}$, *V*(*G*) induces the graph *G* and *u* is a universal vertex. Since for any graph *G* all the cliques of G^+ share the vertex *u*, the graph $K(G^+)$ is complete and thus the graph $H(G^+)$ is a split graph.

Let G be an *HCH* graph. As the set of cliques of an *HCH* graph has polynomial size and can be computed in polynomial time, $H(G^+)$ can be built in polynomial time. By Theorem 2.3, since G^+ is an *HCH* graph, $H(G^+)$ is an *HCH* graph. By Theorem 2.2 $K(H(G^+)) = G^+$, and by Theorem 2.1 $k(G) = k(G^+) = \tau_C(H(G^+))$. Finally, the problem of determining if a subset of vertices is a clique-transversal is solvable in polynomial time for *HCH* graphs, and therefore CLIQUE-TRANSVERSAL is NP-complete for *HCH* split graphs.

In a similar way, using the equality $\alpha(G) = \alpha(G^+) = \alpha_C(H(G^+))$ instead of $k(G) = k(G^+) = \tau_C(H(G^+))$, and the NP-completeness of the STABLE SET problem for *HCH* graphs, CLIQUE-INDEPENDENT SET is NP-complete for *HCH* split graphs. □

Corollary 4.1 *The problem CLIQUE-INDEPENDENT SET is NP-complete for HCH graphs.*

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