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# SYNTHETIC CONTROL CHARTS WITH TWO-STAGE SAMPLING FOR MONITORING BIVARIATE PROCESSES

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## Abstract

In this article, we consider the synthetic control chart with two-stage sampling (*SyTS* chart) to control bivariate processes. During the first stage, one item of the sample is inspected and two correlated quality characteristics $(x; y)$ are measured. If the Hotelling statistic $T_1^2$ for these individual observations of $(x; y)$ is lower than a specified value $UCL_1$ the sampling is interrupted. Otherwise, the sampling goes on to the second stage, where the remaining items are inspected and the Hotelling statistic $T_2^2$ for the sample means of $(x; y)$ is computed. When the statistic $T_2^2$ is larger than a specified value $UCL_2$, the sample is classified as nonconforming. According to the synthetic control chart procedure, the signal is based on the number of conforming samples between two neighbor nonconforming samples. The proposed chart detects process disturbances faster than the bivariate charts with variable sample size and it is from the practical viewpoint more convenient to administer.

**Keywords:** bivariate processes; synthetic control chart; two-stage sampling.

## Resumo

Este artigo apresenta um gráfico de controle com regra especial de decisão e amostragens em dois estágios para o monitoramento de processos bivariados. No primeiro estágio, um item da amostra é inspecionado e duas características de qualidade correlacionadas $(x; y)$ são medidas. Se a estatística de Hotelling $T_1^2$ para as observações individuais de $(x; y)$ for menor que um valor especificado $UCL_1$ a amostragem é interrompida. Caso contrário, a amostragem segue para o segundo estágio, onde os demais itens da amostra são inspecionados e a estatística de Hotelling $T_2^2$ para as médias de $(x; y)$ é calculada. Quando a estatística $T_2^2$ é maior que um valor especificado $UCL_2$, a amostra é classificada como não conforme. De acordo com a regra especial de decisão, o alarme é baseado no número de amostras entre duas não conformes. O gráfico proposto é mais ágil e mais simples do ponto de vista operacional que o gráfico de controle bivariado com tamanho de amostras variável.

**Palavras-chave:** processos bivariados; regra especial de decisão; amostragens em dois estágios.

## 1. Introduction

The control charts have been widely used for process surveillance because of their operational simplicity. However, this operational simplicity, that is, taking samples of fixed size at regular time intervals and searching for an assignable cause when a point falls outside the control limits, makes the control chart slow in detecting small to moderate shifts in the process parameter being controlled. Many innovations have been proposed to improve the charts' performance, such as the adaptive schemes, the supplementary run rules, or the two-stage sampling procedure. When the standard Shewhart control charts are in use, the rate of sampling is kept fixed, once samples of equal size are taken from the process at fixed-length sampling intervals. A more elaborated scheme, called adaptive scheme, has been proposed, in which the size of the samples and/or the length of the sampling intervals and/or the false alarm risk are allowed to vary in an adaptive manner, that is, in real time, based on current sample information. The adaptive scheme enhances the ability of the control chart to signal process disturbances (Reynolds *et al.* (1988); Prabhu *et al.* (1993); Costa (1994, 1997, 1998, 1998a, 1999, 1999a); Costa & De Magalhães (2007); De Magalhães *et al.* (2001, 2002, 2006); De Magalhães & Moura Neto (2005); Epprecht & Costa (2001); Epprecht *et al.* (2003, 2005); Michel & Fogliatto (2002)). In the multivariate SPC, the adaptive schemes have been applied for Hotelling's $T^2$ control charts (Aparasi (1996); Aparasi & Haro (2001, 2003); Chou *et al.* (2006); Yeh & Lin (2002)).

Champ & Woodall (1987) determined the properties of control charts with supplementary runs rules. More recently, Wu & Spedding (2000) presented the synthetic control chart, where the signal is given after the occurrence of a second point on the action region. According to Davis & Woodall (2002), the synthetic control chart is a runs rule chart with a head start feature. The growing interest in the synthetic charts (see Wu & Spedding (2000, 2000a), Wu & Yeo (2001), Wu *et al.* (2001), Calzada & Scariano (2001), Davis & Woodall (2002), Machado & Costa (2005), Costa & Rahim (2006)) may be explained by the fact that many practitioners prefer waiting until the occurrence of a second point beyond the control limits before looking for an assignable cause.

In this paper, we consider a synthetic control chart with two-stage sampling (*SyTS* chart) to control bivariate processes. As in the case of Shewhart control charts, samples of fixed size are taken from the process at regular time intervals; however, the sampling is performed in two stages. During the first stage, one item of the sample is inspected and two correlated quality characteristics $(x; y)$ are measured. If the Hotelling statistic $T_1^2$ for these individual observations of $(x; y)$ is lower than a specified value $UCL_1$ the sampling is interrupted. Otherwise, the sampling goes on to the second stage, where the remaining items are inspected and the Hotelling statistic $T_2^2$ for the sample means of $(x; y)$ is computed and compared with $UCL_2$, the upper control limit. When the synthetic control chart is in use, the signal is given by the occurrence of a second point on the action region and under the condition they can not be far from each other.

Like the adaptive schemes, the two-stage sampling enhances the ability of the control chart in detecting process disturbances. According to Costa & Rahim (2004, 2006a), when the process is stable, the monitoring becomes monotonous because the sample values rarely fall outside the control limits. The natural consequence is that the user pays less and less attention to the steps required to obtain these values, such as the sample mean, the sample variance, or the sample range. When the two-stage sampling scheme is in use, the second stage is frequently discarded, consequently most of the time the user will be dealing with the direct information from the parameters under surveillance, specifically, with their $(x; y)$ values. The go-no-go gauge devices are appropriate to decide if the sampling should be interrupted or not.

## 2. The properties of the synthetic control chart with two-stage sampling

A synthetic control chart with two-stage sampling (*SyTS* chart) is proposed for simultaneously monitoring of two correlated quality characteristics $(x; y)$, described by a bivariate normal distribution with mean vector $\boldsymbol{\mu}' = (\mu_x; \mu_y)$ and a known covariance matrix $\boldsymbol{\Sigma} = \begin{bmatrix} \sigma_x^2 & \sigma_{xy} \\ \sigma_{yx} & \sigma_y^2 \end{bmatrix}$, where $\sigma_{xy} = \sigma_{yx} = \rho\sigma_x\sigma_y$ is the covariance between *x* and *y*. The process is considered to start with the mean vector on target, $\boldsymbol{\mu}_0' = (\mu_{0x}; \mu_{0y})$, but at some random time in the future an assignable cause shifts the mean vector from $\boldsymbol{\mu}_0$ to $\boldsymbol{\mu}_1$, where $(\boldsymbol{\mu}_1 - \boldsymbol{\mu}_0)' = (\delta_x\sigma_x; \delta_y\sigma_y)$, with $\delta_x \neq 0$ and/or $\delta_y \neq 0$. During the in-control period $\boldsymbol{\mu} = \boldsymbol{\mu}_0$ and during the out-of-control period $\boldsymbol{\mu} = \boldsymbol{\mu}_1$. The objective of process monitoring is the detection of any assignable cause that shifts $\boldsymbol{\mu}'$.

Similar to the Shewhart control charts, samples of size $n_0 + 1$ are taken from the process at regular time intervals. The sampling is performed in two stages. At the first stage, one item of the sample is inspected and two correlated quality characteristics $(x; y)$ are measured; if the statistical distance between $\mathbf{X}' = (x, y)$ and $\boldsymbol{\mu}_0' = (\mu_{0x}; \mu_{0y})$, given by $T_1^2 = (\mathbf{X} - \boldsymbol{\mu}_0)' \boldsymbol{\Sigma}^{-1} (\mathbf{X} - \boldsymbol{\mu}_0)$, is lower than a specified value $UCL_1$ the sampling is interrupted. Otherwise, the sampling goes on to the second stage, where the remaining $n_0$ items are inspected and the statistical distance between $\overline{\mathbf{X}}' = (\bar{x}; \bar{y})$ and $\boldsymbol{\mu}_0' = (\mu_{0x}; \mu_{0y})$, given by $T_2^2 = n(\overline{\mathbf{X}} - \boldsymbol{\mu}_0)' \boldsymbol{\Sigma}^{-1} (\overline{\mathbf{X}} - \boldsymbol{\mu}_0)$ is computed; being $\bar{x}$ and $\bar{y}$ the sample mean of the two correlated quality characteristics $(x; y)$ taking into account the entire sample of size $n_0 + 1$. When the statistic $T_2^2$ is larger than a specified value $UCL_2$, the sample is classified as nonconforming. According to the synthetic procedure, the signal is based on the Conforming Run Length (*CRL*). The *CRL* is the number of samples taken from the process since the previous nonconforming sample until the occurrence of the next nonconforming sample or, in case of the absence of previous nonconforming sample, the *CRL* is the number of samples taken from the beginning of the process monitoring until the occurrence of a nonconforming sample. The signal is given when the *CRL* is smaller than or equal to *L*, where *L* is a specified positive integer.

In order to obtain the properties of the *SyTS* control chart we consider the following theorem (Wexler, 1962): If a point P has rectangular coordinates $(x, y)$ then the polar coordinates of P is $(r, \theta)$ where

$$x^2 + y^2 = r^2, \tag{1}$$

$$\theta = \arctan(y/x), \tag{2}$$

hence, for any disk $D(0,a)$ centered at the origin, $(x,y) \in D(0,a)$ if and only if $r^2 < a^2$. Based on that we define $T^2 = n(\overline{\mathbf{X}} - \boldsymbol{\mu})' \boldsymbol{\Sigma}^{-1}(\overline{\mathbf{X}} - \boldsymbol{\mu}) = g_1^2(n,\boldsymbol{\mu}) + g_2^2(n,\boldsymbol{\mu})$, where

$$g_1(n,\boldsymbol{\mu}) = h_1(\bar{x})\cos\varphi - h_2(\bar{y})\sin\varphi, \tag{3}$$

$$g_2(n,\boldsymbol{\mu}) = -h_1(\bar{x})\sin\varphi + h_2(\bar{y})\cos\varphi, \tag{4}$$

with $\sin 2\varphi = \sigma_{\mathrm{xy}}/(\sigma_{\mathrm{x}}\sigma_{\mathrm{y}})$, $h_1(\bar{x}) = \sqrt{n}\sigma_{\mathrm{y}}(\bar{x} - \mu_x)/\sqrt{|\boldsymbol{\Sigma}|}$, $h_2(\bar{y}) = \sqrt{n}\sigma_{\mathrm{x}}(\bar{y} - \mu_y)/\sqrt{|\boldsymbol{\Sigma}|}$, and $|\boldsymbol{\Sigma}| = \det \boldsymbol{\Sigma} = \sigma_{\mathrm{x}}^2\sigma_y^2 - \sigma_{\mathrm{xy}}^2 > 0$. According to Theorem A1, see the appendix, $g(n,\boldsymbol{\mu}) = (g_1^2(n,\boldsymbol{\mu}), g_2^2(n,\boldsymbol{\mu}))$ has normal distribution with zero mean vector and unit covariance matrix $\boldsymbol{\varepsilon} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Consequently,

$$\Pr[T^2 < a^2] = \int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} f_{N(0,\boldsymbol{\varepsilon})}(x,y)\,dy\,dx = \int_0^a r\exp(-r^2)\,dr = 1 - \exp(-a^2/2) \tag{5}$$

where $f_{\mathrm{N}(0,\boldsymbol{\varepsilon})}(x,y) = \dfrac{1}{2\pi}\exp[-0.5(x^2 + y^2)]$. When the process is in control, $p_0 = Pr[T_1^2 < UCL_1]$ is the probability that the two-stage sampling ends at the first stage:

$$p_0 = 1 - \exp(-UCL_1/2) \tag{6}$$

During the in-control period, the rate of inspected items per sampling, $\bar{n}$, is given by

$$\bar{n} = 1 + n_0(1 - p_0) = 1 + n_0[\exp(-UCL_1/2)] \tag{7}$$

If the parameters $n_0$ and $UCL_1$ are designed to make $\bar{n}$ equal to $n$, the size of the samples when the standard bivariate $T^2$ control chart is in use, then both, the *SyTS* and the standard $T^2$ control charts will have the same rate of inspected items per sampling.

When the sample of size $n$ is split in two sub-samples of size $n_1$ and $n_2$, with $n_1 + n_2 = n$, it follows that

$$g(n,\boldsymbol{\mu}) = \left(\sqrt{n_1}\,g(n_1,\boldsymbol{\mu}) + \sqrt{n_2}\,g(n_2,\boldsymbol{\mu})\right) \Big/ \left(\sqrt{n_1 + n_2}\right) \tag{8}$$

In the case of the two-stage sampling, $n_1 = 1$, $n_2 = n_0$, and the sample is classified as nonconforming if $g(n_1,\boldsymbol{\mu}) \notin D(0,\sqrt{UCL_1})$ and $g(n,\boldsymbol{\mu}) \notin D(0,\sqrt{UCL_2})$. According to (8), the false alarm risk is given by

$$\alpha = \iint_{\notin D(0,\sqrt{UCL_1})} f_{N(0,\boldsymbol{\varepsilon})}(x,y)\left[\iint_{\notin D(C,R)} f_{N(0,\boldsymbol{\varepsilon})}\,du\,dv\right]dx\,dy \tag{9}$$

where $C(-x/\sqrt{n_0};-y/\sqrt{n_0})$ and $R=\sqrt{\dfrac{1+n_0}{n_0}UCL_2}$ .

From (3) and (4), follows that

$$g_1(n,\boldsymbol{\mu}_1)-g_1(n,\boldsymbol{\mu}_0)=\frac{\sqrt{n}}{\sqrt{(1-\rho^2)}}\left[-\delta_x\cos\ \varphi+\delta_y\ \sin\ \varphi\right]=\sqrt{n}\,\delta_x^{'},$$

$$g_2(n,\boldsymbol{\mu}_1)-g_2(n,\boldsymbol{\mu}_0)=\frac{\sqrt{n}}{\sqrt{(1-\rho^2)}}\left[-\delta_y\cos\ \varphi+\delta_x\ \sin\ \varphi\right]=\sqrt{n}\,\delta_y^{'}$$

consequently, the power of the control chart will be given by

$$p=\iint_{\notin(a,\sqrt{UCL_1})}f_{N(0,\boldsymbol{\varepsilon})}(x,y)\left[\iint_{\notin D(C,R)}f_{N(0,\boldsymbol{\varepsilon})}dudv\right]dxdy \tag{10}$$

where $a=\left(\delta_x^{'};\delta_y^{'}\right)$ and $C(-x/\sqrt{n_0}+\delta_x^{'}\sqrt{n_0};-y/\sqrt{n_0}+\delta_y^{'}\sqrt{n_0})$ .

## 3. The performance of the synthetic control chart with two-stage sampling

According to the synthetic procedure, the signal is based on the Conforming Run Length (*CRL*). The *CRL* is the number of samples taken from the process since the previous nonconforming sample until the occurrence of the next nonconforming sample or, in case of the absence of previous nonconforming sample, the *CRL* is the number of samples taken from the beginning of the process until the occurrence of a nonconforming sample. The signal is given when the *CRL* is smaller than or equal to *L*, where *L* is a specified positive integer. According to Wu & Spedding (2000),

$$ARL=\frac{1}{Q}\times\frac{1}{1-(1-Q)^L} \tag{11}$$

During the in-control period $Q=\alpha$ and the *ARL* is called $ARL_0$, and during the out-of-control period, $Q=p$. Figure 1 shows a *SyTS* bivariate control chart with the $T_{i1}^2$ values plotted. The $T_{i1}^2$ point on the action region triggers the inspection of the whole sample, in which case the statistic $T_{i2}^2$ is computed and compared with $UCL_2$. If $T_{i2}^2>UCL_2$ then the $T_{i1}^2$ point is surrounded by a circle. In other words, the nonconforming samples have their $T_{i1}^2$ points surrounded by circles, see in Figure 1, 9th and 13th sample points. This simple procedure has a practical appeal, once the signal is based on the Conforming Run Length. According to Figure 1, the previous nonconforming sample is the 9th sample, the next nonconforming sample is the 13th, and the *CRL* is equal to four (13-9). As the *CRL* value is lower than *L*(=5), the *SyTS* bivariate control chart signals an out-of-control condition.

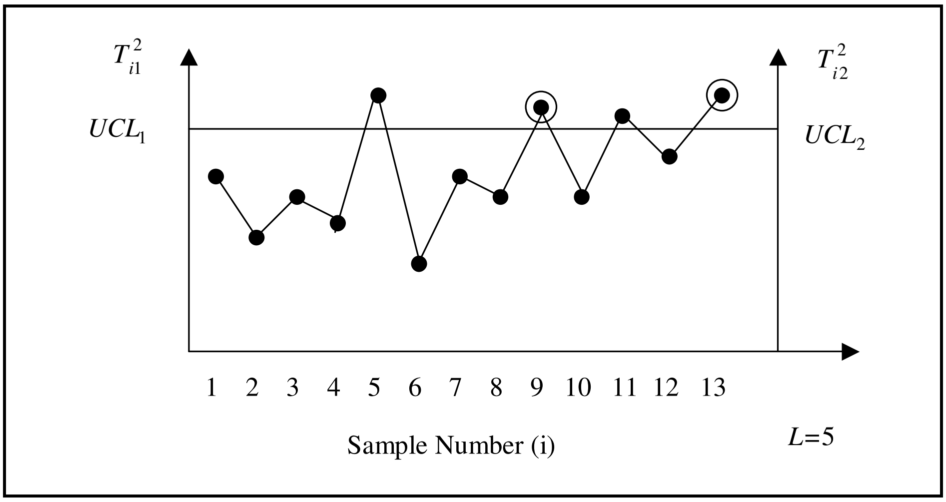


**Figure 1** – *SyTS* bivariate control chart with the $T_{i1}^2$ values plotted.

When a process is in control, it is desirable that the expected number of samples taken since the beginning of the monitoring until a signal ($ARL_0$) be large, to guarantee few false alarms. When a process is out of control, it is desirable that the expected number of samples taken since the occurrence of the assignable cause until a signal (*ARL*) be small, in order to guarantee fast detection of process changes (see Costa *et al.* (2005)).

When the sampling interval is fixed, the *ARL* provides a measure of the time required to detect an assignable cause that is affecting the process since the beginning of the monitoring (zero-state *ARL*) or since the occurrence of the assignable cause (steady-state *ARL*). However, it is more realistic to assume that the assignable cause only occurs after the process has been running for some time. As a result, the assignable cause will occur between two nonconforming samples. The steady-state *ARL* measures how long the synthetic control chart will take to signal in this situation. We used the Markov chain model of the synthetic control chart described in Davis & Woodall (2002) to obtain the steady-state *ARL* for the *SyTS* bivariate chart.

The steady-state *ARL* values are presented in Table 1 for the synthetic bivariate chart with two-stage sampling, $\bar{n}=4$, $n_0=9$, $UCL_1$=2.1973 and $L$=1, 5, 10, 20, 50, 100. As $L$ increases from 1 to 50, all the steady-state *ARL* values decrease. The process disturbance is measured by $\lambda=\sqrt{\bar{n}(\delta_x'^2+\delta_y'^2)}$, with $\delta_x'^2=\delta_y'^2$. For example, if $\lambda=0.5$, the steady-state *ARL* decreases from 84.79 to 59.74. The effect of varying $L$ from 50 to 100 almost not affects the speed with which the chart detects small or large disturbances. For example, if $\lambda=1.5$, the steady-state *ARL* increases from 4.80 to 4.88 as $L$ increases from 50 to 100. And if a larger shift is considered, for example, $\lambda=3.0$, the steady-state *ARL* decreases from 1.62 to 1.59 as $L$ increases from 50 to 100. Table 1 also brings the *ARL* values for the standard bivariate chart ($T^2$ chart). Inspection of Table 1 reveals that the *SyTS* Chart always outperforms the $T^2$ chart, except the case when $L$=1. For example, if $\lambda=1.5$ and $L$=20, the steady-state *ARL* value for the *SyTS* Chart is 4.86 and the equivalent *ARL* value for the $T^2$ chart 5.79.

**Table 1 –** Effect of *L* on the steady-state *ARL* values for the *SyTS* chart.

| $\lambda$ | $T^2$ chart $n$=4 $UCL$=10.597 | $L$=1 $UCL_2$=3.380 | $L$=5 $UCL_2$=4.983 | $L$=10 $UCL_2$=5.635 | $L$=20 $UCL_2$=6.253 | $L$=50 $UCL_2$=7.001 | $L$=100 $UCL_2$=7.503 |
|---|---|---|---|---|---|---|---|
| 0.0 | 200.3 | 200.1 | 200.1 | 200.1 | 200.2 | 200.1 | 200.1 |
| 0.5 | 115.7 | 84.79 | 66.98 | 62.92 | 60.61 | 59.74 | 60.44 |
| 1.0 | 41.97 | 24.12 | 15.33 | 13.93 | 13.32 | 13.30 | 13.64 |
| 1.5 | 15.79 | 9.75 | 5.68 | 5.12 | 4.86 | 4.80 | 4.88 |
| 2.0 | 6.88 | 5.88 | 3.33 | 2.97 | 2.77 | 2.67 | 2.67 |
| 2.5 | 3.55 | 4.18 | 2.45 | 2.18 | 2.02 | 1.92 | 1.90 |
| 3.0 | 2.16 | 3.31 | 2.04 | 1.83 | 1.7 | 1.62 | 1.59 |

$$\lambda = \sqrt{\bar{n}(\delta_x'^2 + \delta_y'^2)}\text{, with } \delta_x'^2 = \delta_y'^2$$

Table 2 brings the steady-state *ARL* values for the synthetic chart with two-stage sampling, $\bar{n} = 4$, *L*=10, and $n_0 = 6, 9, 12$. The choice of $n_0$ affects the speed with which the *SyTS* chart signals. Larger values of $n_0$ are better for detecting smaller process changes and worse for detecting larger process changes. For example, one can see from Table 4 that, when $\lambda = 1.5$, as $n_0$ increases from 6 to 12, the steady-state *ARL* value decreases from 5.91 to 5.30. On the other hand, when $\lambda = 3.0$, as $n_0$ increases from 6 to 12, the steady-state *ARL* value increases from 1.55 to 2.13. Table 2 also brings the *ARL* values for the standard bivariate chart $T^2$ chart and the *ARL* values for the bivariate control chart with two-stage sampling (*BTS* chart). Inspection of Table 2 reveals the *SyTS* chart, when *L*=10, always outperforms the $T^2$ chart and outperform the *BTS* chart for small disturbances. For example, if $\lambda = 1.0$ and $n_0$=9, the steady-state *ARL* value for the *SyTS* chart is 13.93. The equivalent *ARL* values for the $T^2$ chart and for the *BTS* chart are respectively 41.97 and 17.59. When $\lambda$ increases from 1.5 to 3.0, most of the time the *BTS* chart has a better performance than the proposed chart (the *SyTS* chart). For example, if $\lambda = 2.5$ and $n_0$=9, the *ARL* value for the *BTS* chart is 1.94 and the steady-state *ARL* value for the *SyTS* chart is 2.18

**Table 2 –** Effect of $n_0$ on the steady-state *ARL* values for the *SyTS* chart.

| $\lambda$ | $T^2$ chart $n$=4 $UCL$=10.597 | *BTS* chart $n_0$=6 $UCL_1$=1.39 $UCL_2$=9.92 | *SyTS* chart $n_0$=6 $UCL_1$=1.386 $UCL_2$=6.442 | *BTS* chart $n_0$=9 $UCL_1$=2.20 $UCL_2$=9.20 | *SyTS* chart $n_0$=9 $UCL_1$=2.197 $UCL_2$=5.635 | *BTS* chart $n_0$=12 $UCL_1$=2.77 $UCL_2$=8.56 | *SyTS* chart $n_0$=12 $UCL_1$=2.773 $UCL_2$=4.998 |
|---|---|---|---|---|---|---|---|
| 0.0 | 200.3 | 200.0 | 200.2 | 200.2 | 200.1 | 200.1 | 200.2 |
| 0.5 | 115.7 | 88.09 | 76.38 | 72.14 | 62.92 | 63.04 | 57.25 |
| 1.0 | 41.97 | 24.15 | 17.48 | 17.59 | 13.93 | 14.36 | 12.68 |
| 1.5 | 15.79 | 8.10 | 5.91 | 5.9 | 5.12 | 5.17 | 5.30 |
| 2.0 | 6.88 | 3.58 | 2.94 | 2.93 | 2.97 | 2.88 | 3.40 |
| 2.5 | 3.55 | 2.07 | 1.95 | 1.94 | 2.18 | 2.11 | 2.59 |
| 3.0 | 2.16 | 1.49 | 1.55 | 1.57 | 1.83 | 1.76 | 2.13 |

$$\lambda = \sqrt{\bar{n}(\delta_x'^2 + \delta_y'^2)}\text{, with } \delta_x'^2 = \delta_y'^2$$

## 4. Comparing charts

As the *SyTS* control chart was designed to detect small changes in the process parameters, it seems reasonable to compare its performance with the performance of the Hotelling's $T^2$ control chart with variable sample sizes (*VSS* chart).

The design of univariate Shewhart charts with adaptive sample sizes has been studied by Daudin (1992), Prabhu *et al.* (1993) and Costa (1994). Aparisi (1996) has extended these studies for the Hotelling's $T^2$ control chart. The monitoring procedure is as follows: the statistic $T_i^2 = n(\overline{\mathbf{X}}_i - \boldsymbol{\mu}_0)' \boldsymbol{\Sigma}^{-1}(\overline{\mathbf{X}}_i - \boldsymbol{\mu}_0)$ is charted with upper control limit $UCL_{VSS}$. Two samples sizes are used, $n_1$ and $n_2$, $n_2 > n_1$. A warning limit $w$ is defined, where $0 < w < UCL_{VSS}$ as a threshold limit that identifies when to shift from one sample size to another (Fig. 2). The sample size of subgroup *i* depends on the value of $T_{i-1}^2$. If $T_{i-1}^2 < w$ the *ith* subgroup sample size is $n_1$, and corresponds to the black points in the chart. If $w < T_{i-1}^2 < UCL_{VSS}$ the *ith* subgroup sample size is $n_2$, and correspond to the white points in the chart (see sample points 6, 8, 10 and 12). If $T_i^2 > UCL_{VSS}$ then the control chart signals.

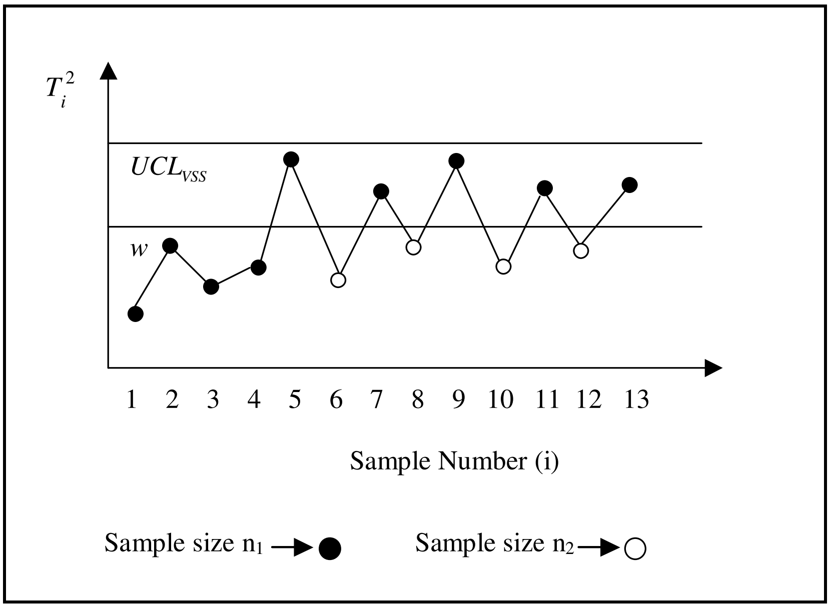


**Figure 2** – Hotelling's $T^2$ control chart with variable sample sizes.

Table 3 provides the *ARL* for the *SyTS* chart and for the Hotelling's $T^2$ control chart with adaptive sample sizes. Similarly to Costa (1994), Aparisi (1996) obtained the *ARL* values for the Hotelling's $T^2$ control chart with adaptive sample sizes using Markov chains.

According to Table 3 the *SyTS* chart has a better overall performance than its competitor, the *VSS* chart.

**Table 3 –** *ARL* values for the *VSS* and for the *SyTS* control charts with $L$=10 and $\bar{n} = 4$ .

| $n_1 = 1$ | $n_2 = n_0 = 6$ | | $n_2 = n_0 = 9$ | | $n_2 = n_0 = 12$ | |
|---|---|---|---|---|---|---|
| $\lambda$ | *VSS*<br>$w$=1.011<br>$UCL$=10.597 | *SyTS*<br>$UCL_1$=1.386<br>$UCL_2$=6.442 | *VSS*<br>$w$=1.961<br>$UCL$=10.597 | *SyTS*<br>$UCL_1$=2.197<br>$UCL_2$=5.635 | *VSS*<br>$w$=2.611<br>$UCL$=10.597 | *SyTS*<br>$UCL_1$=2.773<br>$UCL_2$=4.998 |
| 0.00 | 200.1 | 200.2 | 198.42 | 200.1 | 197.93 | 200.2 |
| 0.50 | 111.64 | 76.38 | 107.02 | 62.92 | 104.79 | 57.25 |
| 1.00 | 33.76 | 17.48 | 26.30 | 13.93 | 21.44 | 12.68 |
| 1.50 | 10.56 | 5.91 | 7.30 | 5.12 | 5.90 | 5.30 |
| 2.00 | 4.42 | 2.94 | 3.43 | 2.97 | 3.21 | 3.40 |
| 2.50 | 2.58 | 1.95 | 2.35 | 2.18 | 2.49 | 2.59 |
| 3.00 | 1.84 | 1.55 | 1.98 | 1.83 | 2.17 | 2.13 |

## 5. Illustrative example

In this section the proposed model is applied to the example considered by Montgomery (2004) to explain the use of the bivariate $T^2$ control chart. The two quality characteristics are the tensile strength (psi) and the diameter of a textile fiber (x$10^{-2}$ inch). He considered a set of twenty samples of size ten to estimate the in-control values of the mean vector and the covariance matrix:

$$\boldsymbol{\mu}_0 = \begin{pmatrix} 115.59 \\ 1.06 \end{pmatrix}, \ \Sigma_0 = \begin{pmatrix} 1.23 & 0.79 \\ 0.79 & 0.83 \end{pmatrix}.$$

Based on these estimations, ten samples of size $n_0 + 1$, with $n_0 = 6$, were generated using the DRNMVN sub routine available on the IMSL FORTRAN library (1995). After fixing $\bar{n} = 3$, the rate of inspected items per sampling, $\alpha = 0.005$, the false alarm risk, and $L = 2$, the expressions (6) and (11) were used to obtain the control chart limits, $UCL_1 = 2.197$ and $UCL_2 = 4.911$. The general expression of the Hotelling $T^2$ statistic for two quality characteristics is:

$$T^2(n,\boldsymbol{\mu}) = \frac{n}{|\boldsymbol{\Sigma}|}\left[\sigma_y^2\left(\bar{x}-\mu_x\right)^2 + \sigma_x^2\left(\bar{y}-\mu_y\right)^2 - 2\sigma_{xy}\left(\bar{y}-\mu_y\right)\left(\bar{x}-\mu_x\right)\right]$$

According to the proposed sampling scheme, the sampling is performed in two stages. At the first stage, one item of the sample is inspected and the two correlated quality characteristics $(x; y)$ are measured. The statistical distance between $\mathbf{X}' = (x, y)$ and $\boldsymbol{\mu}_0' = (\mu_{0x}; \mu_{0y})$ is given by

$$T_1^2 = \frac{1}{(1.23)(0.83) - (0.79)^2}[0.83(x_1 - 115.59)^2 + 1.23(y_1 - 1.06)^2 \\ - 2(0.79)(x_1 - 115.59)(y_1 - 1.06)]$$

If the statistic $T_1^2$ is lower than $UCL_1$ the sampling is interrupted. Otherwise, the sampling goes on to the second stage, where the remaining $n_0$ items are inspected and the statistical distance between $\bar{\mathbf{X}}' = (\bar{x}; \bar{y})$ and $\boldsymbol{\mu}_0' = (\mu_{0x}; \mu_{0y})$, given by

$$T_2^2 = \frac{7}{(1.23)(0.83)-(0.79)^2}[0.83\left(\bar{x}-115.59\right)^2 + 1.23\left(\bar{y}-1.06\right)^2$$
$$-2(0.79)\left(\bar{x}-115.59\right)\left(\bar{y}-1.06\right)]$$

is computed; being $\bar{x}$ and $\bar{y}$ the sample mean of the two correlated quality characteristics $(x;y)$ taking into account the entire sample of size $n_0 + 1$. When the statistic $T_2^2$ is larger than $UCL_2 = 4.911$, the sample is classified as nonconforming. According to the synthetic procedure, the signal is based on the Conforming Run Length (*CRL*). The *CRL* is the number of samples taken from the process since the previous nonconforming sample until the occurrence of the next nonconforming sample or, in case of the absence of previous nonconforming sample, the *CRL* is the number of samples taken from the beginning of the process monitoring until the occurrence of a nonconforming sample. The signal is given when the *CRL* is smaller than or equal to *L*.

Table 4 presents the data of $(x;y)$, the sample means, the statistics $T_1^2$ and $T_2^2$ and the sampling status.

**Table 4 –** Data for the illustrative example.

| # | | Observations | | | | | | | Sample Mean | $T_1^2$ | $T_2^2$ | * |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | 1 | 2 | 3 | 4 | 5 | 6 | 7 | | | | |
| 1 | x | 114.37 | 116.30 | 117.37 | 114.78 | 114.51 | 115.72 | 115.06 | 115.44 | 3.55 | 2.18 | 2 |
| | y | 1.14 | 1.46 | 2.17 | 1.24 | 0.54 | 1.47 | 0.89 | 1.27 | | | |
| 2 | | 116.39 | 116.02 | 115.30 | 114.20 | 115.28 | 116.62 | 115.98 | 115.68 | 2.86 | 0.05 | 2 |
| | | 2.44 | 1.65 | 0.12 | 0.21 | 1.20 | 1.44 | 0.83 | 1.13 | | | |
| 3 | | 114.34 | - | - | - | - | - | - | 114.34 | 1.36 | - | 1 |
| | | 0.42 | - | - | - | - | - | - | 0.42 | | | |
| 4 | | 114.27 | 116.37 | 115.27 | 115.74 | 116.11 | 116.75 | 115.19 | 115.67 | 3.64 | 0.04 | 2 |
| | | 1.06 | 1.13 | 0.54 | 1.28 | 0.82 | 1.75 | 1.21 | 1.11 | | | |
| 5 | | 114.49 | 114.37 | 116.11 | 113.81 | 116.58 | 115.36 | 115.09 | 115.11 | 4.42 | 7.38 | 2 |
| | | 1.41 | 1.18 | 1.55 | 1.04 | 1.29 | 1.06 | 1.46 | 1.28 | | | |
| 6 | | 116.49 | 114.86 | 115.77 | 116.51 | 116.01 | 115.85 | 115.37 | 115.84 | 1.54 | 0.83 | 2 |
| | | 1.11 | 0.99 | 0.34 | 1.45 | 1.51 | 1.67 | 0.41 | 1.07 | | | |
| 7 | | 115.52 | - | - | - | - | - | - | 115.52 | 0.24 | - | 1 |
| | | 1.29 | - | - | - | - | - | - | 1.29 | | | |
| 8 | | 115.84 | - | - | - | - | - | - | 115.84 | 0.15 | - | 1 |
| | | 1.40 | - | - | - | - | - | - | 1.40 | | | |
| 9 | | 115.49 | 114.97 | 114.95 | 116.02 | 115.30 | 115.05 | 114.50 | 115.18 | 2.30 | 5.92 | 2 |
| | | 1.85 | 1.64 | 1.07 | 1.35 | 1.55 | 0.61 | 0.86 | 1.28 | | | |
| 10 | | 115.52 | - | - | - | - | - | - | 115.52 | 0.79 | - | 1 |
| | | 0.51 | - | - | - | - | - | - | 0.51 | | | |

#: Sample Number; * (1): Sampling is interrupted, (2): Sampling goes on to the second stage

Figure 3 presents the *SyTS* control chart for the illustrative example. According to this figure, the first nonconforming sample is the 5th sample, the second nonconforming sample is the 9th, and the *CRL* is equal four (9-5). As the *CRL* value is larger than $L$ (= 2), the *SyTS* bivariate control chart does not signal an out-of-control condition.

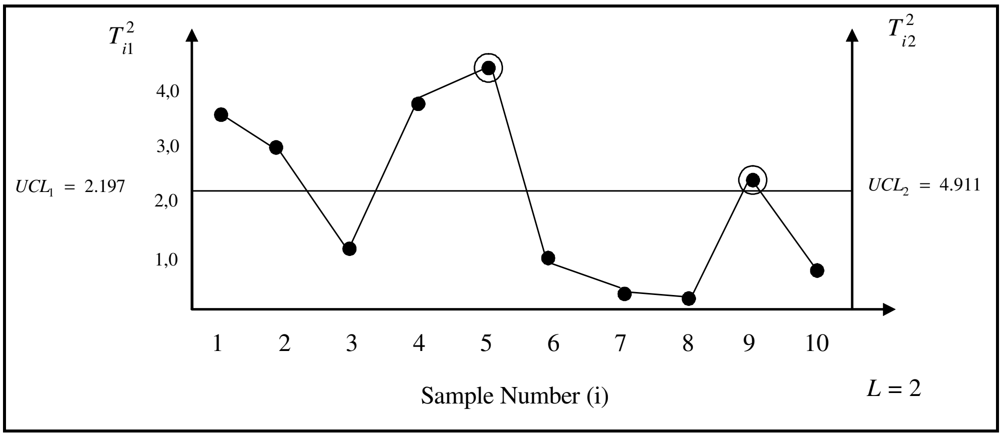


**Figure 3 –** The *SyTS* control chart for tensile strength and diameter.

## 6. Conclusions

In this article, a synthetic control chart with two-stage sampling (*SyTS* chart) to control bivariate processes was proposed for detecting assignable causes. An important advantage of the proposed chart is that the practitioner can control the process by looking at only one chart. The synthetic control chart with two stage sampling (*SyTS*), the standard control chart, the control chart with two-stage sampling and the *VSS* control charts for bivariate processes were compared, in terms of the speed they detect changes in the process parameters. The numerical results we obtained show a better overall performance of the *SyTS* chart if compared with the other control charts.

In order to obtain the properties of the synthetic control chart with two-stage sampling, the IMSL FORTRAN library (1995) was used, where the subroutine DCSNDF is available to evaluate the non-central chi-square distribution function. The *SyTS* chart was conceived to attend the user because frequently they will be dealing with the situation where the inspection is reduced to the first item of the sample. Moreover, they feel more secure in stopping the process only after the occurrence of a second point beyond the control limits.

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**Appendix: Theorem's proof**

**Theorem A1:** The random functions $g_1(n,\boldsymbol{\mu})$ and $g_2(n,\boldsymbol{\mu})$ have normal distribution with zero mean vector and unit covariance matrix $\boldsymbol{\varepsilon} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

**Proof:** The random functions $g_1(n,\boldsymbol{\mu})$ and $g_2(n,\boldsymbol{\mu})$ are normal, because they are linear combinations of normal variables $x_i$ and $y_i$. We know that

$$E(\bar{x}-\mu_x)^2 = \frac{\sigma_x^2}{n},\ E(\bar{y}-\mu_y)^2 = \frac{\sigma_y^2}{n},\ E(\bar{x}-\mu_x)(\bar{y}-\mu_{xy}) = \frac{\sigma_{xy}}{n}$$

After some calculation we obtain

$$E[g_1(n,\boldsymbol{\mu})]^2 = \frac{n}{|\boldsymbol{\Sigma}|}\left[\frac{\sigma_x^2\sigma_y^2}{n} - \frac{2}{n}\sigma_x\sigma_y\sigma_{xy}\sin\varphi\cos\varphi\right].$$

By symmetry with respect to indices *x* and *y*, we have

$$E[g_1(n,\boldsymbol{\mu})]^2 = E[g_2(n,\boldsymbol{\mu})]^2 = \frac{n}{|\boldsymbol{\Sigma}|}\left[\frac{\sigma_x^2\sigma_y^2}{n} - \frac{1}{n}\sigma_x\sigma_y\sigma_{xy}\sin 2\varphi\right]$$

As $\sin 2\varphi = \sigma_{\mathrm{xy}}/(\sigma_{\mathrm{x}}\sigma_{\mathrm{y}})$ and $|\boldsymbol{\Sigma}| = \sigma_x^2\sigma_y^2 - \sigma_{xy}^2$ follows

$$E[g_1(n,\boldsymbol{\mu})]^2 = E[g_2(n,\boldsymbol{\mu})]^2 = \frac{1}{|\boldsymbol{\Sigma}|}\left[\sigma_x^2\sigma_y^2 - \sigma_{xy}^2\right] = 1$$

Similarly

$$E[g_1(n,\boldsymbol{\mu})g_2(n,\boldsymbol{\mu})] = \frac{n}{|\boldsymbol{\Sigma}|}\left[\frac{\sigma_x\sigma_y\sigma_{xy}}{n} - \frac{1}{n}\sigma_x^2\sigma_y^2\sin 2\varphi\right] = \frac{\sigma_x\sigma_y}{|\boldsymbol{\Sigma}|}\left[\sigma_{xy} - \sigma_x\sigma_y\sin 2\varphi\right] = 0$$

Theorem A1 is proved.