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USE OF RADIAL BASIS FUNCTIONS FOR MESHLESS NUMERICAL SOLUTIONS APPLIED TO FINANCIAL ENGINEERING BARRIER OPTIONS

Gisele Tessari Santos
Maurício Cardoso de Souza*
Universidade Federal de Minas Gerais (UFMG)
Belo Horizonte – MG
mauricio.souza@pq.cnpq.br

Mauri Fortes
Centro Universitário UNA
Belo Horizonte – MG

* *Corresponding author* / autor para quem as correspondências devem ser encaminhadas

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Abstract

A large number of financial engineering problems involve non-linear equations with non-linear or time-dependent boundary conditions. Despite available analytical solutions, many classical and modified forms of the well-known Black-Scholes (BS) equation require fast and accurate numerical solutions. This work introduces the radial basis function (RBF) method as applied to the solution of the BS equation with non-linear boundary conditions, related to path-dependent barrier options. Furthermore, the diffusional method for solving advective-diffusive equations is explored as to its effectiveness to solve BS equations. Cubic and Thin-Plate Spline (TPS) radial basis functions were employed and evaluated as to their effectiveness to solve barrier option problems. The numerical results, when compared against analytical solutions, allow affirming that the RBF method is very accurate and easy to be implemented. When the RBF method is applied, the diffusional method leads to the same results as those obtained from the classical formulation of Black-Scholes equation.

Keywords: financial engineering; radial basis functions; diffusional method; barrier options.

Resumo

Muitos problemas de engenharia financeira envolvem equações não-lineares com condições de contorno não-lineares ou dependentes do tempo. Apesar de soluções analíticas disponíveis, várias formas clássicas e modificadas da conhecida equação de Black-Scholes (BS) requerem soluções numéricas rápidas e acuradas. Este trabalho introduz o método de função de base radial (RBF) aplicado à solução da equação BS com condições de contorno não-lineares relacionadas a opções de barreira dependentes da trajetória. Além disso, explora-se o método difusional para solucionar equações advectivo-difusivas quanto à sua efetividade para solucionar equações BS. Utilizam-se funções de base radial Cúbica e Thin-Plate Spline (TPS), aplicadas à solução de problemas de opções de barreiras. Os resultados numéricos, quando comparados com as soluções analíticas, permitem afirmar que o método RBF é muito acurado e fácil de ser implementado. O método difusional associado ao método RBF leva aos mesmos resultados obtidos pela formulação clássica da equação de Black-Scholes.

Palavras-chave: engenharia financeira; funções de base radial; método difusional; opções de barreira.

1. Introduction

Determining the value of an option is a major concern in financial engineering. An option contract in the financial world is the right, but not the obligation, to buy (for a call option) or sell (for a put option) an asset (e.g. a share of stock in a corporation, a commodity like grain, foreign currency, etc.) at a fixed price (the strike price) by a certain date in the future (Meyer, 1998; Hull, 2005; Wilmott, 2007). The most important models of financial engineering are based on Black-Scholes equations, and are used to predict the outcome of financial options and derivative securities and, thus, help in decision-making processes (Cox & Rubinstein, 1985; Hull, 2005; Siegel *et al.*, 1992). Only simple contracts in stock markets can be handled in a semi-quantitative way. Black-Scholes (BS) basic equation is a linear parabolic hyperbolic equation, with stochastic and deterministic variables and parameters. Despite its significance, the BS equation has suffered numerous modifications in order to adapt it to the ever increasing available financial options (Wilmott, 2007; Amster *et al.*, 2003); under these modifications, the BS equation may become highly non-linear. This work will address barrier-type non-linear boundary conditions associated to the classical BS equation.

The advection–diffusion equation is the basis of many physical phenomena, and its use has also spread into economics, financial forecasting and other fields, including Black and Scholes equation. Many numerical methods have been introduced to model accurately the interaction between advective and diffusive processes. The most common methods are the finite difference, finite element and boundary element methods; they use local interpolation schemes and require a mesh to support the application. Finite difference and finite element solutions of the advection–diffusion equation present numerical problems of oscillations and damping (Murphy & Prenter, 1985; Lee *et al.*, 1987; Zienkiewicz & Taylor, 1991; Hoffman, 1992; Wilmott *et al.*, 1995; Wilmott, 1998; Tomas III & Yalamanchili, 2001; Boztosun & Charafi, 2002; Amster *et al.*, 2003). More recently, the Radial Basis Function Method, RBF, is claimed to be relatively free from these problems (Boztosun & Charafi, 2002). Tomas III & Yalamanchili (2001) applied the finite elements method (FEM) to the familiar Black-Scholes differential equation, to price European put options and discrete barrier options. The authors argue that the FEM allows non-uniform mesh construction and direct derivative valuation.

In recent years, considerable effort has been dedicated to the development of mesh-free methods, due to the complexity of mesh-generation (Boztosun & Charafi, 2002; Brown *et al.*, 2005). The method of Radial Basis Functions does not require meshes, is independent of spatial dimension and can be easily extended to solve high dimensional problems (Zhang, 2006). RBFs make use of linear translate combinations of a basis function $\phi(\mathbf{r})$ of one variable ($\mathbf{r}$), expanded about given scattered 'data centers' $S_i \in \Re^d,\ i = 1,...,N$ to approximate an unknown function $V(S,t)$. The RBF method generates, thus, a system of linear equations, which can be solved to obtain the updated solution in time plane n+1 from known previous solutions. RBFs have been used for interpolations problems as well as for numerically solving partial differential equations (Brown *et al.*, 2005). Koc *et al.* (2003) were among the first authors to present RBF methods for numerical solution of the Black-Scholes equation. They evaluated the predictive capability of four radial basis functions: Cubic, Thin-Plate Spline (TPS), Multiquadrics (MQ) and Gaussian. However, they have not analyzed the predictive capability of the RBF method to problems involving non-linear boundary conditions. The accuracy of MQ and Gaussian radial basis functions depends on a radial basis function shape parameter *c*, which can only be optimized by means of empirical approaches (Rippa, 1999). A recent paper (Goto *et al.*, 2007) shows that empirical approaches for obtaining the shape parameter may be misleading or require previous knowledge of analytical solutions. Thus, this paper aims at analyzing Radial Basis Functions that do not depend on empirical shape functions evaluations while leading to acceptable numerical errors.

Fortes (1997) and Fortes & Ferreira (1998; 1999), proposed the diffusional method to numerically solve convection-diffusion equations. In a nutshell, the method acts by transforming the original hyperbolic- parabolic partial differential equation into a parabolic partial differential equation. The method is simple to apply and was claimed to perform much better when solving benchmark and practical problems, than the commonly employed finite difference techniques (such as implicit finite-difference methods; see Hoffman, 1992). In recent papers the diffusional finite difference method was applied to analyze derivatives in financial engineering, with special attention to Black-Scholes call option equation (Fortes *et al.*, 2000-a; Fortes *et al.*, 2000-b; Fortes *et al.*, 2005).

Since the late 1980s, Barrier options have been extensively used for hedging and investment in over-the-counter (OTC) foreign exchange, equity and commodity markets (Hui, 1997). They are commonly traded types of exotic derivatives. Barrier options are activated (knock-ins) or terminated (knock-outs) if a specific trigger is reached before the expiry date (Fusai *et al.*, 2006). European barrier options are path-dependent options in which the existence of the European options depends on whether the underlying asset price has touched a barrier level during the option life (Hui, 1997). Several papers have been dealt with the pricing of barrier options and a great number of associated valuation techniques have been proposed (Fusai *et al.*, 2006). The non-linearity associated to barrier options is treated in this work.

This paper aims at presenting a radial basis functions approach, with focus on Cubic and TPS functions, to solve time dependent jump barrier-type boundary condition associated to Black-Scholes equation. More specifically, this paper presents:

- A comparative analysis between the diffusional and the classical approach to numerical solutions of Black-Scholes equation.
- Full modeling of call options and barrier options and respective numerical solutions by means of Cubic and TPS radial basis functions.
- Validated results of the numerical solutions by means of analytical solutions.
- A sensitivity (parametric) analysis including the effects of stock value mesh size, time step, and maximum stock value required for precision numerical analysis and integrations method.
- Criteria for optimizing numerical solutions of BS equation via the RBF method.

This paper is structured as follows. In section 2, the classical and diffusional forms of Black-Scholes equation are presented. The following section describes the main aspects of barrier options. Section 4 introduces the radial basis function method for both forms of BS equation. Results are reported in section 5. The paper ends with concluding remarks.

Further justification for utilizing the proposed method to solve Black-Scholes basic equation and Barrier options benchmark problems with available analytical solutions are:

- As Wilmott (1998), a worldwide recognized financial engineering expert and professor points out; "*When I describe the numerical methods, I often use the Black-Scholes equation as the example. But the methods are all applicable to other problems, such as stochastic interest rates*". He continues to point out that the finite difference methods are readily appropriate to real world problems.

- Available analytical solutions of benchmark problems in finance and in other different fields allow evaluating the effectiveness of numerical methods (Goto *et al.*, 2007; Tomas III & Yalamanchili, 2001; Fortes & Ferreira, 1999; Zienkiewicz & Taylor, 1991).
- Solutions to barrier options are highly demanded by the options markets and recent works keep on showing analytical and numerical solutions (Fusai *et al.*, 2006) due to highly complex, non-linear temporal and spatial characteristics of these types of problems. Thus, this paper also relates to solving these problems.

2. Classical and diffusional forms of Black-Scholes equation

The 1997 Nobel Prize was granted to Robert C. Merton and Myron S. Scholes, who, in collaboration with the late Fischer Black, had developed a pioneering formula for the valuation of stock options. Their methodology has paved the way for economic valuations in many areas (Sveriges Riskbank, 1997). Their widely accepted mathematical model for evaluating V (S, t), the time changing value of an option, is the so-called Black-Scholes equation. It is based on a stochastic model for the behavior of the price S of the underlying asset, whose solution leads to current price V (S, T) of an option which expires at T (Meyer, 1998). The **classical** form of Black-Scholes or BS basic equation is (Wilmott, 1998):

$$\frac{\partial V}{\partial \tau} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \tag{1}$$

where V, τ, σ, S and r stand, respectively, for option value, time, volatility, asset (underlying security) price (a stochastic variable) and *riskless* interest rate.

In this work, part of the numerical calculations used to solve BS equation for a **call option** is performed with the following payoff function, that is, the value of the call option at expiry (τ = T), in a neutral-risk world:

$$M(S, T) = V(S, T) = \text{Payoff}\,(S, T) = \max\,(S-E, 0) \tag{2}$$

where E is the option exercise or strike value (price), that is, its value at τ = T. The respective boundary conditions are:

$$V(0,\tau) = 0\ ;\ \ V(\infty,\tau) = S - Ee^{-r(T-\tau)} \tag{3}$$

For a simple **call** option, Wilmott (2007) gives the analytical solution

$$V(S, T-t) = S \cdot N(d_1) - E \cdot N(d_2) \cdot e^{-r(T-t)} \tag{4}$$

where

$$d_1(S, T-t) = \frac{\ln\left(\frac{S}{E}\right) + \left(r + \frac{1}{2}\sigma^2\right)\cdot(T-t)}{\sigma\cdot\sqrt{T-t}} \tag{5}$$

$$d_2(S, T-t) = d_1(S, T-t) - \sigma\cdot\sqrt{T-t} \tag{6}$$

and N is the cumulative normal probability density function

$$N(x) = (2\pi)^{-\frac{1}{2}} \int_0^x e^{-\frac{1}{2}u^2}\,du + \frac{1}{2} \tag{7}$$

One should note that equation (1) is not an initial value problem, since the payoff function is given at t = T. In order to make it an initial boundary value problem let us make t = T – τ, so that the above **classical** equation becomes

$$\frac{\partial V}{\partial t} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rS\frac{\partial V}{\partial S} + rV = 0 \qquad (8)$$

The BS **diffusional** form of the above equation is a form that eliminates the convective term. The convective term usually generates spurious solutions in numerical solutions of partial differential equations (Zienkiewicz & Taylor, 1991; Hoffman, 1992; Fortes, 1997). Thus, Fortes (1997) showed that, by eliminating the convective term, the resulting diffusional equation could be easily calculated by any numerical scheme and, thus, led to highly accurate solutions to the convective-diffusional equation. Here, the motivation is to analyze the usefulness of the diffusional method while using the RBF method. Thus, in order to put this last equation into the **diffusional** form, the following identity is used

$$-\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rS\frac{\partial V}{\partial S} = A\frac{\partial}{\partial S}\left(B\frac{\partial V}{\partial S}\right) \qquad (9)$$

By comparing the right hand side of equation (9) with its left hand side, after algebraic manipulations, one arrives at:

$$B = \frac{B_0}{(S_0)^{\frac{2r}{\sigma^2}}}(S)^{\frac{2r}{\sigma^2}} = C_0\,(S)^{\frac{2r}{\sigma^2}} \qquad (10)$$

and

$$A = -\frac{1}{2}\frac{\sigma^2 S^2}{C_0 S^{\frac{2r}{\sigma^2}}} \qquad (11)$$

where B_0 and C_0 are integration constants. By substituting the values of A and B into equations (9 and 8), one obtains the **diffusional** form of Black-Scholes equation:

$$\frac{2S^{\frac{2r}{\sigma^2}-2}}{\sigma^2}\frac{\partial V}{\partial t} - \frac{\partial}{\partial S}\left(S^{\frac{2r}{\sigma^2}}\frac{\partial V}{\partial S}\right) + \frac{2S^{\frac{2r}{\sigma^2}-2}\,rV}{\sigma^2} = 0 \qquad (12)$$

The non-linear initial and boundary conditions are:

$$V(S,0) = \text{Payoff}(S,0) = \max(S\text{-}E, 0); \quad V(0,t) = 0\ ;\ V(\infty,t) = S\text{-}Ee^{-rt} \qquad (13)$$

3. Main Barrier Options

There are many types of options, ranging from the general classification of European, American and Asian, to specialized forms known as exotic options. Barrier options are highly used options that allow a purchaser to reduce his risks. Barrier options have a payoff that is contingent on the underlying asset, S, reaching some specified level before expiry. There are two main types of barrier option (Wilmott, 1998):

- The **out option** that only pays off if a level, S_u, is not reached. If the barrier is reached then the option is said to have **knocked out**.
- The **in option** that pays off as long as a level, S_d, is reached before expiry. If the barrier is reached then the option is said to have **knocked in**.

A barrier option can also be characterized by the position of the barrier relative to the initial value of the underlying:

- If the barrier is above the initial asset value, one has an **up** option.
- If the barrier is below the initial asset value, one has a **down** option.

The main boundary conditions for the most common barrier options are presented in Table 1.

Table 1 – Characterization of barrier options.

Barrier option	Barrier level	Solution involves	Boundary condition	If not triggered
Up-and-Out	$S = S_u$	$0 \le S \le S_u$	$V(S_u, t) = 0 \quad \text{for} \quad t < T$	$V(S,T) = \max(S-E,0)$
Up-and-In	$S = S_u$	$S_u \le S \le \infty$	$V(S_u, t) = \max(S-E,0)$	$V(S,T) = 0$
Down-and-Out	$S = S_d$	$S_d \le S < \infty$	$V(S_d, t) = 0$	$V(S,T) = \max(S-E,0)$
Down-and-In	$S = S_d$	$0 \le S \le S_d$	$V(S_d, t) = \max(S-E,0)$	$V(S,T) = 0$

Thus, for an Up-and-Out barrier option, the option is terminated, that is, its value is zero, if the underlying asset price (S) reached the superior barrier (S_u) before expiry, ($V(S_u, t) = 0$ for $t < T$). Otherwise, the option is activated, that is, its value is equal to an ordinary European call option, ($V(S,T) = \max(S-E,0)$).

It is important to notice that for simple call options, the boundary conditions are $V(S,T) = \max(S-E,0)$ for $0 \le S < \infty$. Analytical solutions for call and barrier options can be found in Wilmott (1998, 2007); as an example, only the solution for the up and out call options problems is given below.

The barrier S_u for an up-and-out call option must be above the strike price E (otherwise the option would be worthless). This makes the solution for the price more complicated. The exact value of an up-and-out call option is (Wilmot, 1998; Fusai *et al.*, 2006):

$$V(S, T-t) = S\left(N(d_1) - N(d_3) - b\left(N(d_6) - N(d_8)\right)\right) - Ee^{-r(T-t)}\left(N(d_2) - N(d_4) - a\left(N(d_5) - N(d_7)\right)\right) \tag{14}$$

where d_1 and d_2 are given by equations (5 and 6) and

$$a = \left(\frac{S_b}{S}\right)^{-1+\frac{2r}{\sigma^2}} \quad \text{and} \quad b = \left(\frac{S_b}{S}\right)^{1+\frac{2r}{\sigma^2}} \tag{15}$$

$$d_3(S,T-t)=\frac{\ln\left(\frac{S}{S_b}\right)+\left(r-\frac{1}{2}\sigma^2\right)\cdot(T-t)}{\sigma\cdot\sqrt{T-t}};$$

$$d_4(S,T-t)=\frac{\ln\left(\frac{S}{S_b}\right)+\left(r+\frac{1}{2}\sigma^2\right)\cdot(T-t)}{\sigma\cdot\sqrt{T-t}};$$

$$d_5(S,T-t)=\frac{\ln\left(\frac{S}{S_b}\right)-\left(r+\frac{1}{2}\sigma^2\right)\cdot(T-t)}{\sigma\cdot\sqrt{T-t}};$$

$$d_6(S,T-t)=\frac{\ln\left(\frac{S}{S_b}\right)-\left(r-\frac{1}{2}\sigma^2\right)\cdot(T-t)}{\sigma\cdot\sqrt{T-t}}; \tag{16}$$

$$d_7(S,T-t)=\frac{\ln\left(\frac{SE}{S_b^2}\right)-\left(r-\frac{1}{2}\sigma^2\right)\cdot(T-t)}{\sigma\cdot\sqrt{T-t}};$$

$$d_8(S,T-t)=\frac{\ln\left(\frac{SE}{S_b^2}\right)-\left(r+\frac{1}{2}\sigma^2\right)\cdot(T-t)}{\sigma\cdot\sqrt{T-t}}$$

4. Radial Basis Functions

The idea behind RBFs is to use linear translate combinations of a basis function $\phi(\mathbf{r})$ of one variable, expanded about given scattered 'data centers' $S_i \in \Re^d$, $i = 1,...,N$ to approximate an unknown function $V(S,t)$ by

$$V(S,t)=\sum_{j=1}^{N}\lambda_j(t)\,\phi\left(r_j\right)=\sum_{j=1}^{N}\lambda_j\phi\left(\left\|S-S_j\right\|\right) \tag{17}$$

where $r_j = \left\|S - S_j\right\|$ is the Euclidean norm and λ_j are the coefficients to be determined. Usual radial basis functions are (Koc *et al.*, 2003):

$$\text{Thin-Plate Spline, TPS}: \phi\left(r_j\right)=r_j^4\log\left(r_j\right) \tag{18}$$

$$\text{Multiquadrics, MQ}: \phi\left(r_j\right)=\sqrt{c^2+r_j^2} \tag{19}$$

$$\text{Cubic}: \phi\left(r_j\right)=r_j^3 \tag{20}$$

$$\text{Gaussian}: \phi\left(r_j\right)=e^{-c^2 r_j^2} \tag{21}$$

One should note that mesh points, here established by r_j, do not require any formal rule, except for defining what the boundaries are. The method is, thus, meshless. Furthermore, besides obvious extension to higher dimensions, the method only requires setting points at important boundaries. No rules for meshing are necessary and here lies the importance of the collocation method of radial basis functions as opposed to other numerical methods.

In this work, only Cubic and TPS RBF will be used, due to their simplicity and proven accuracy for other types of problems and because of the difficulty associated for choosing good values for the shape parameter *c*, which depends on the problem type (Boztosun & Charafi, 2002; Goto *et al.*, 2007).

4.1 Application of Radial Basis Functions to the original Black-Scholes equation

As shown below, the RBF methodology for obtaining the numerical solution of the BS equation requires discretizing the original equation (17). Thus, the original Black-Scholes equation shown above in equation (1) can be discretized using the θ-weighted method (for details, applied to finite difference schemes, see Hoffman, 1992):

$$\frac{\partial V(S,t)}{\partial t} = f(S,t) \approx (1-\theta)\cdot f(S_t,t) + \theta\cdot f(S_{t+\Delta t}, t+\Delta t) \qquad \text{for } 0 \le \theta \le 1 \tag{22}$$

In the above equation (22), setting θ equal to 0, 0.5 or 1 leads to Euler's explicit, Crank-Nicolson or Euler's implicit scheme, respectively.

So, the discretized form of equation (1) becomes:

$$\begin{aligned} & V(S,t) - V(S,t+\Delta t) + \Delta t(1-\theta)\cdot\left[\frac{1}{2}\sigma^2 S^2 \nabla^2 V(S,t) + rS\nabla V(S,t) - rV(S,t)\right] + \\ & +\Delta t\theta\cdot\left[\frac{1}{2}\sigma^2 S^2 \nabla^2 V(S,t+\Delta t) + rS\nabla V(S,t+\Delta t) - rV(S,t+\Delta t)\right] = 0 \end{aligned} \tag{23}$$

Or

$$\begin{aligned} & V(S,t^n)\cdot\left[1 + \Delta t(1-\theta)\cdot\left(\frac{1}{2}\sigma^2 S^2 \nabla^2 + rS\nabla - r\right)\right] + \\ & +V(S,t^n+\Delta t)\cdot\left[-1 + \Delta t\theta\cdot\left(\frac{1}{2}\sigma^2 S^2 \nabla^2 + rS\nabla - r\right)\right] = 0 \end{aligned} \tag{24}$$

In this equation, n indicates the nth time plane. By defining $V(\mathbf{S},t^n) = V^n$ and $V(\mathbf{S},t^n+\Delta t) = V^{n+1}$, the previous equation can be written in the form:

$$\left[1 - \alpha\cdot\left(\frac{1}{2}\sigma^2 \mathbf{S}^2 \nabla^2 + r\mathbf{S}\nabla - r\right)\right]\cdot V^{n+1} = \left[1 + \beta\cdot\left(\frac{1}{2}\sigma^2 \mathbf{S}^2 \nabla^2 + r\mathbf{S}\nabla - r\right)\right]\cdot V^n \tag{25}$$

where α = θΔt, β = (1 – θ)Δt, $\nabla = \frac{\partial}{\partial S}$ and $\nabla^2 = \frac{\partial^2}{\partial S^2}$. Now, by defining two new operators, H_L and H_R by:

$$H_L = 1 - \alpha\cdot\left(\frac{1}{2}\sigma^2 S^2 \nabla^2 + rS\nabla - r\right), \quad H_R = 1 + \beta\cdot\left(\frac{1}{2}\sigma^2 S^2 \nabla^2 + rS\nabla - r\right) \tag{26}$$

equation (24) becomes:

$$\sum_{j=1}^{N} \lambda_j^{n+1} H_L \phi(S_{ij}) = \sum_{j=1}^{N} \lambda_j^{n} H_R \phi(S_{ij}) \qquad \text{for } i = 1...N \tag{27}$$

Equation (27) generates a system of linear equations, which can be solved to obtain the unknowns, λ_j^{n+1}, from the known values of λ_j^{n} at a previous time step. Then they give rise to V(**S**, t) by means of equation (17).

4.2 Application of Radial Basis Functions to the diffusional form of Black-Scholes equation

Algebraic manipulations of equation (12), following the θ-weighted method, lead to:

$$\left[\frac{2S^{\frac{2r}{\sigma^2}-2}}{\sigma^2} - \alpha\left(\nabla S^{\frac{2r}{\sigma^2}}\nabla - \frac{2S^{\frac{2r}{\sigma^2}-2}}{\sigma^2} r\right)\right] V^{n+1} = \left[\frac{2S^{\frac{2r}{\sigma^2}-2}}{\sigma^2} + \beta\left(\nabla S^{\frac{2r}{\sigma^2}}\nabla - \frac{2S^{\frac{2r}{\sigma^2}-2}}{\sigma^2} r\right)\right] V^{n} \tag{28}$$

The operators H_L and H_R are, now, defined by:

$$H_L = \left[\frac{2S^{\frac{2r}{\sigma^2}-2}}{\sigma^2} - \alpha\left(\nabla S^{\frac{2r}{\sigma^2}}\nabla - \frac{2S^{\frac{2r}{\sigma^2}-2}}{\sigma^2} r\right)\right], \quad H_R = \left[\frac{2S^{\frac{2r}{\sigma^2}-2}}{\sigma^2} + \beta\left(\nabla S^{\frac{2r}{\sigma^2}}\nabla - \frac{2S^{\frac{2r}{\sigma^2}-2}}{\sigma^2} r\right)\right] \tag{29}$$

Finally, the new operators H_L and H_R, are applied in equation (27), generating another system of RBF linear equations equivalent to the discretized form of equation (12).

5. Results and Discussion

5.1 Simulation parameters and results

This section shows results obtained via Mathcad, a symbolic mathematical programming language and solver. The fixed data used in the simulation studies were:

- Exercise value: E = 50
- Volatility: σ = 20%
- Riskless interest rate: r = 5%
- Expiry time: T = 1
- Present exact analytical call option exercise value = 5.225

This work considers only Up-and-Out and Down-and-Out barriers, since their numerical solutions can be manipulated and lead to the solutions of the other two barrier options problems (see Wilmott, 1998, for details). For these options:

- Present exact analytical Up-and-Out barrier exercise option value = 4.9869
- Present exact analytical Down-and-Out barrier exercise option value = 5.176

The total number of stock value meshes is N; the mesh size, ΔS, is defined by $\Delta S = S/N$. Analogously, the total number of time steps is Nt, while the time step, Δt, is defined by $\Delta t = T/Nt$.

In this work, numerical option value relative errors refer to option prediction values at the strike (exercise) price value ($S = E = 50$) and are defined as:

$$\varepsilon = \text{Relative error of numerical option price (\%)} = \\ = \frac{\text{Numerical option value} - \text{Analytical solution value}}{\text{Analytical solution value}} \times 100\% \quad (30)$$

One of the boundary conditions, typical in BS problems, requires specifying $V(S,t)$ at $S = \infty$; practical numerical solutions require that this value should be reduced. Additionally, the larger the allowable reduction, the better the efficiency of the numerical solution, due to decreased equation matrix size; thus, the practical maximum value used for S in simulations was called S_{max}.

The accuracy of *finite difference* solutions of BS equations can be heavily improved if the diffusional method substitutes the classical approach (Fortes *et al.*, 2005). However, when RBF are considered, the numerical results were identical, that is, the diffusional and the classical form of Black-Scholes equation led to the same results. Thus, this fact will not be shown in the figures and discussion to come.

Figures 1 to 3 show that Cubic and TPS radial basis functions can be effectively and accurately used to simulate typical vanilla and barrier options.

The main results were:

- Figure 1 shows the excellent approximations obtained by the Cubic and TPS radial basis functions, when applied to call option simulation, with Nt = 100 time steps, N = 112 meshes and $\Delta S = 0.714$, an upper value of S equal to 80 and θ equal to 0.5. Cubic and TPS RBF, respectively, led to relative errors at the exercise option value (E=50) of 0.00039% and 0.019%.
- Figure 2 shows that accurate solutions can be obtained by the use of Cubic and TPS RBF when used to simulate Up-and-Out (UAO) barrier options. Figure 2 was obtained with 100 time steps, N = 112 meshes and $\Delta S = 0.893$, with an upper value of S equal to 100 and θ equal to 1. Cubic and TPS RBF, respectively, led to relative errors at the exercise option value (E=50) of 1.15%, and 1.12%. The influence of the highly non-linear time-dependent boundary condition can be felt by noticing the increase of relative errors, as compared to the errors associated to milder non-linear boundary condition of call options (Figure 1).
- Figure 3 shows again that accurate results can be obtained by the Cubic and TPS RBF, when applied to solve the Down-and-Out (DAO) barrier options. Figure 3 was obtained with 100 time steps, N = 112 meshes and $\Delta S = 0.714$, with an upper value of S equal to 80 and θ equal to 1. In the case of Cubic RBF, the relative error at the exercise option value (E=50) was 0.38%, while in the case of TPS RBF, the relative error was 0.36%.

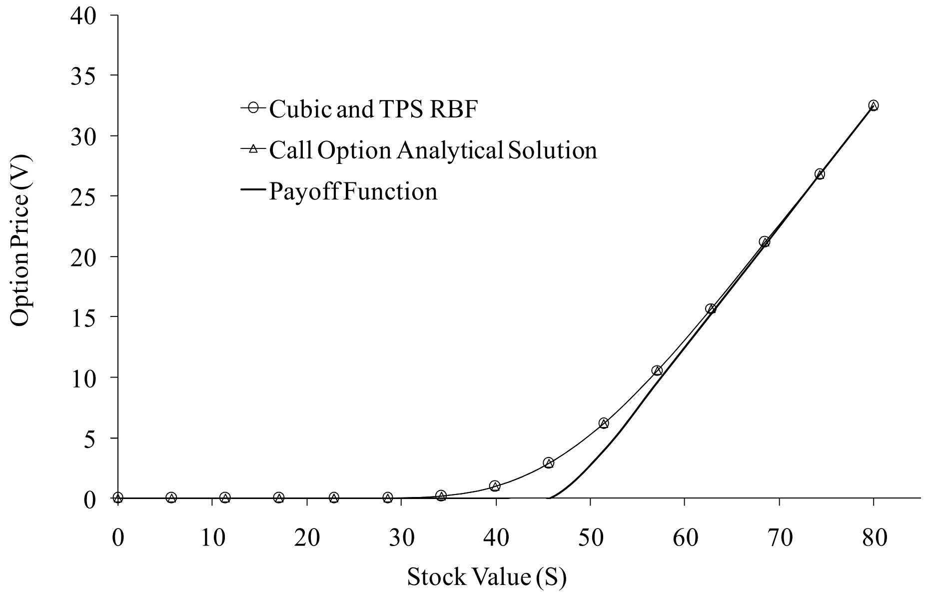

Figure 1 – Cubic and TPS RBF simulated values of a call option compared against the analytical solution and payoff function values.

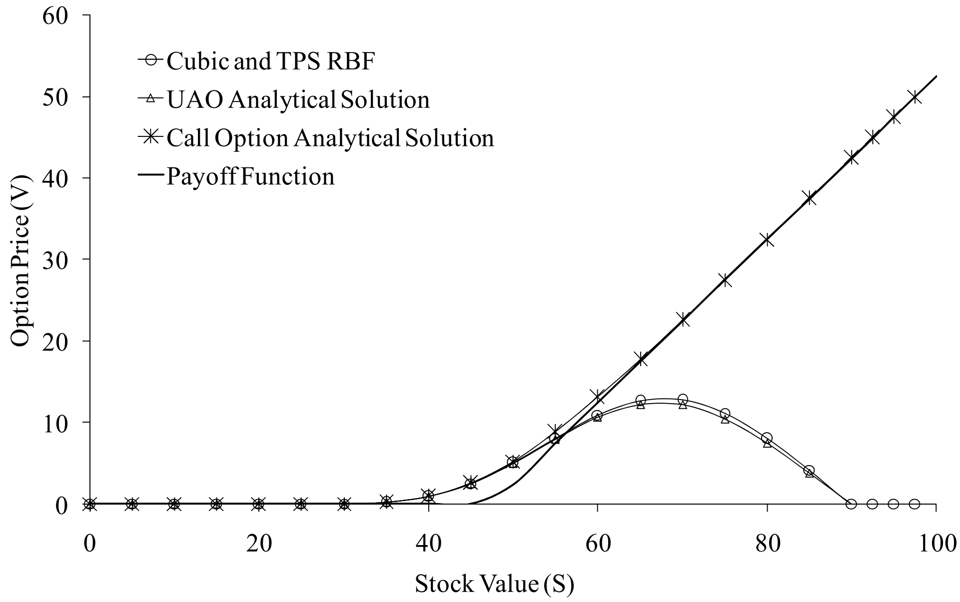

Figure 2 – Cubic and TPS RBF simulated values of an Up-and-Out barrier options compared against the analytical solution and payoff function values.

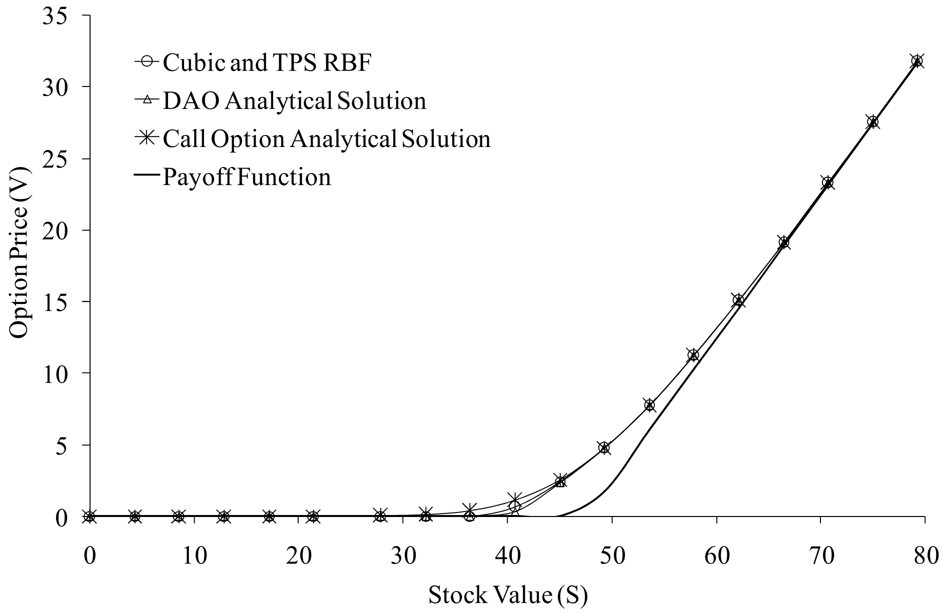

Figure 3 – Cubic and TPS RBF simulated values of a Down-and-Out barrier options compared against the analytical solution and payoff function values.

Thus, as a main conclusion, it can be stated that, at a relative error level inferior to 1.3%, or 0.013 (decimal), the RBF methods are very accurate. Further discussion on error optimization procedures are shown below.

5.2 Error analysis through simulation results for RBF methods applied to Up-and-Out, UAO, barriers

A parametric analysis of numerical errors associated to both Cubic and TPS RBF applied to UAO barrier options are shown in Figures 4 to 7.

The main results were:

- Figure 4 shows the effect of the integration scheme. As can be noted, it is advisable to use $\theta \geq 0.25$, as a rule, for Cubic RBF; smaller θ values may lead to divergence. TPS RBF are not subject to divergence, even for the explicit solution condition ($\theta = 0$). No effort was made to optimize the error level, although the results show that the higher the θ-value, the lower the errors.
- Figure 5 allows observing that the maximum simulated stock value (S_{max}) should be larger than 100, or, in other words, larger than 2E. By following this procedure, both RBF methods lead to similar and smaller errors and more stable solutions.
- Figure 6 shows that the time step should be kept equal to or smaller than 0.1, in practical terms, when RBF are used.
- Figure 7 shows that both Cubic RBF and TPS RBF do not depend heavily on mesh size and lead to approximately the same results when mesh sizes are considered.
- Figures 6 and 7 show that the TPS RBF leads to slightly better results than Cubic RBF.

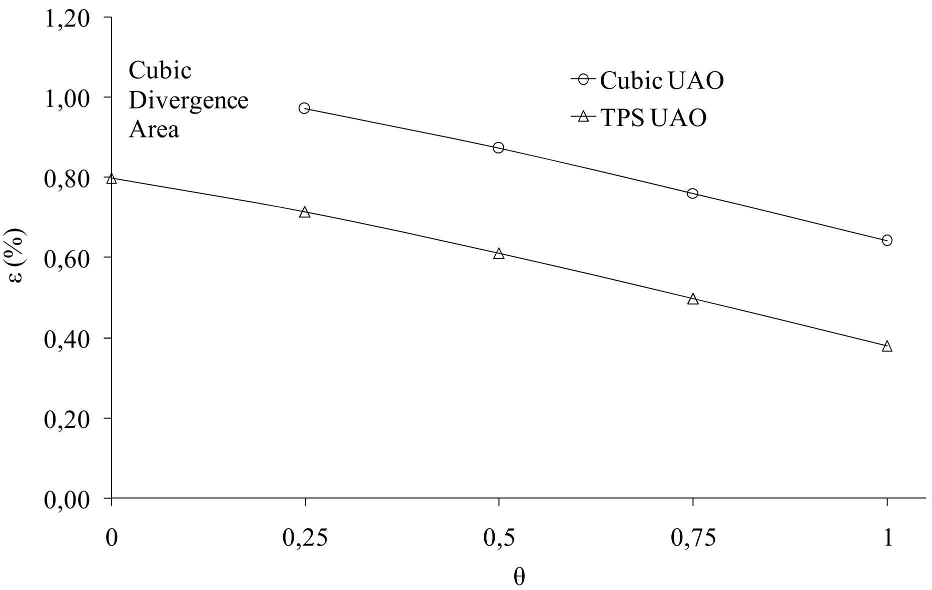

Figure 4 – Relative error between RBF **Up-and-Out** barrier option values at the exercise option value (%) as affected by the integration θ-value; Nt = 100, N = 40, S_{max} = 100.

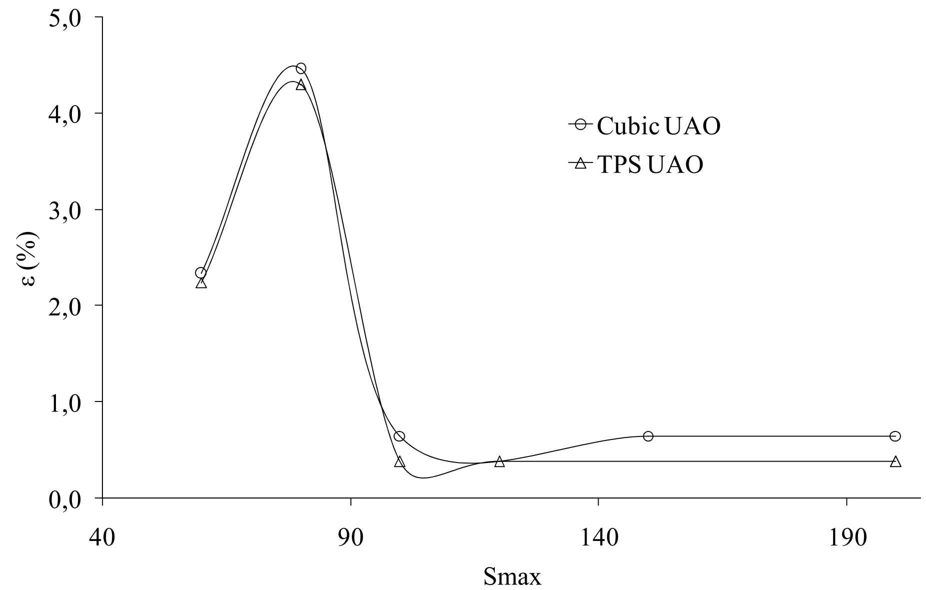

Figure 5 – Relative error between RBF **Up-and-Out** barrier option values at the exercise option value (%) as affected by the maximum simulated stock value; Nt = 100, θ = 1 and ΔS = 2.5.

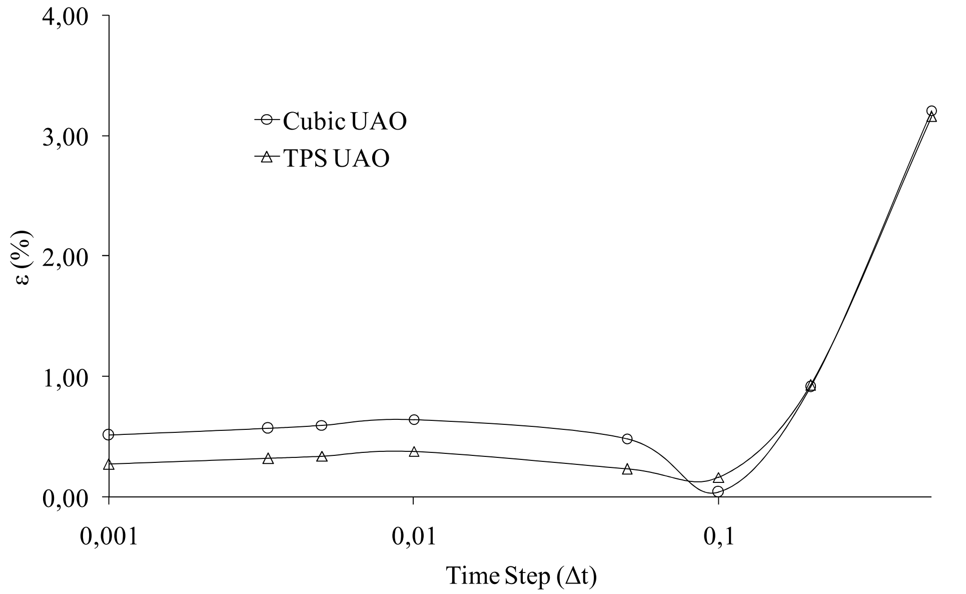

Figure 6 – Relative error between RBF **Up-and-Out** barrier option values at the exercise option value (%) as affected by the time step; N = 40, S_{max} = 100 and θ = 1.

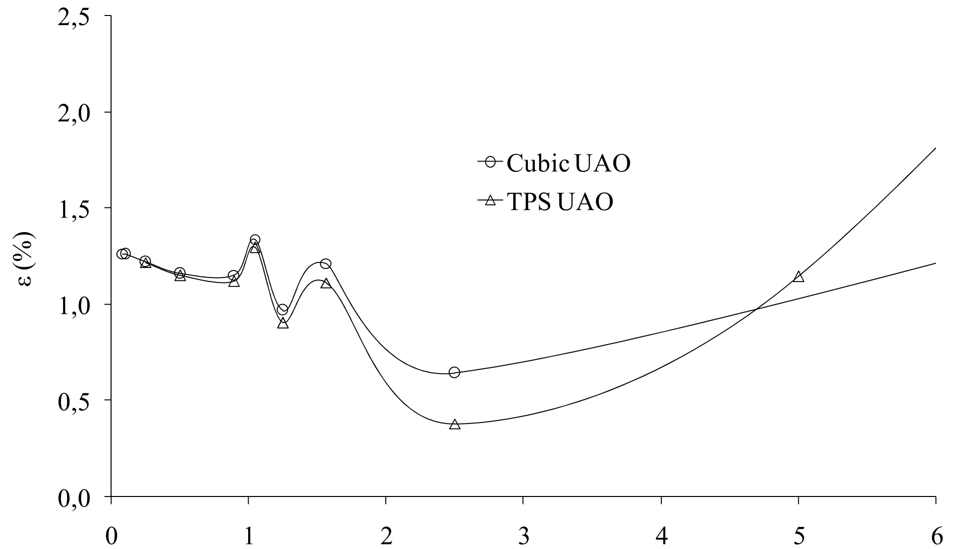

Figure 7 – Relative error between RBF **Up-and-Out** barrier option values at the exercise option value (%) as affected the mesh size; Nt = 100, S_{max}= 100 and θ = 1.

5.3 Error analysis through simulation results for RBF methods applied to Down-and-Out, DAO, barriers

A parametric analysis of numerical errors associated to both Cubic and TPS RBF applied to DAO barrier options are shown in Figures 8 to 11.

The main results were:

- Figure 8 shows the effect of the integration scheme. As can be seen, differently from the UAO problem, it is advisable to use $\theta \geq 0.5$, as a rule, for both Cubic and TPS RBF; smaller θ values lead to divergence. Again, no effort was made to optimize the error level, although the results show that the higher the θ-value, the lower the error. Relative errors associated to DAO (Figure 8) are smaller than the ones associated to UAO (Figure 4).
- Figure 9 allows observing that the maximum simulated stock value (S_{max}) should be larger than 90, or, in other words, larger than 2E for both RBF methods. By following this procedure, both RBF methods lead to similar and smaller errors and more stable solutions. Relative errors of DAO are inferior with respect to those associated to UAO barriers solutions.
- Figure 10 shows, as in the case of UAO barriers, that the time step should, be kept equal to or smaller than 0.1, in practical terms, when RBF are used.
- Figure 11 shows that both Cubic RBF and TPS RBF do not depend heavily on mesh size and lead to approximately the same results when mesh size are considered. It is advisable to use mesh sizes larger than 0.5 for TPS RBF since it diverges for smaller ΔS values. The associated relative errors are smaller than in the case of UAO solutions.

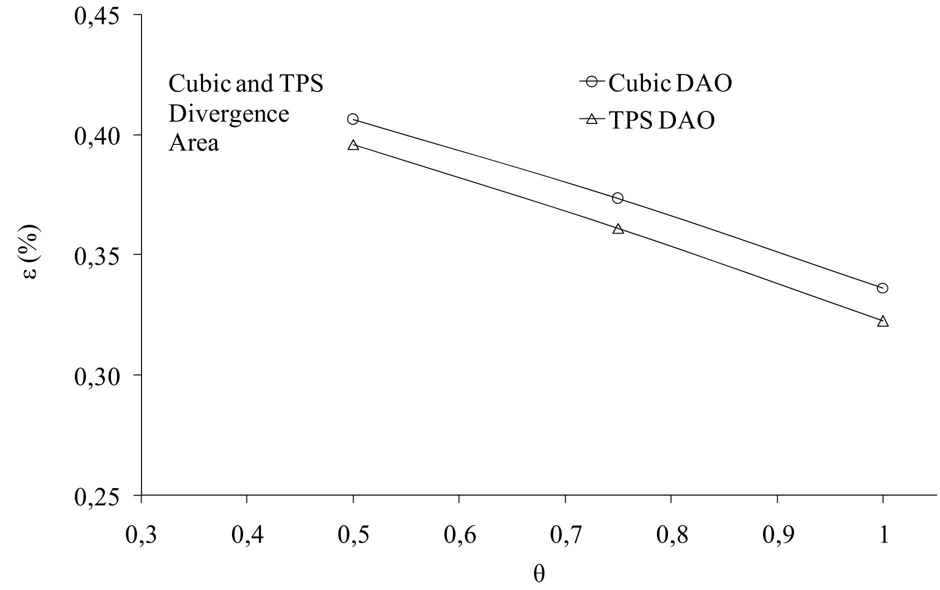

Figure 8 – Relative error between RBF **Down-and-Out** barrier option values at the exercise option value (%) as affected by the integration θ-value; Nt = 100, N = 112, S_{max} = 100.

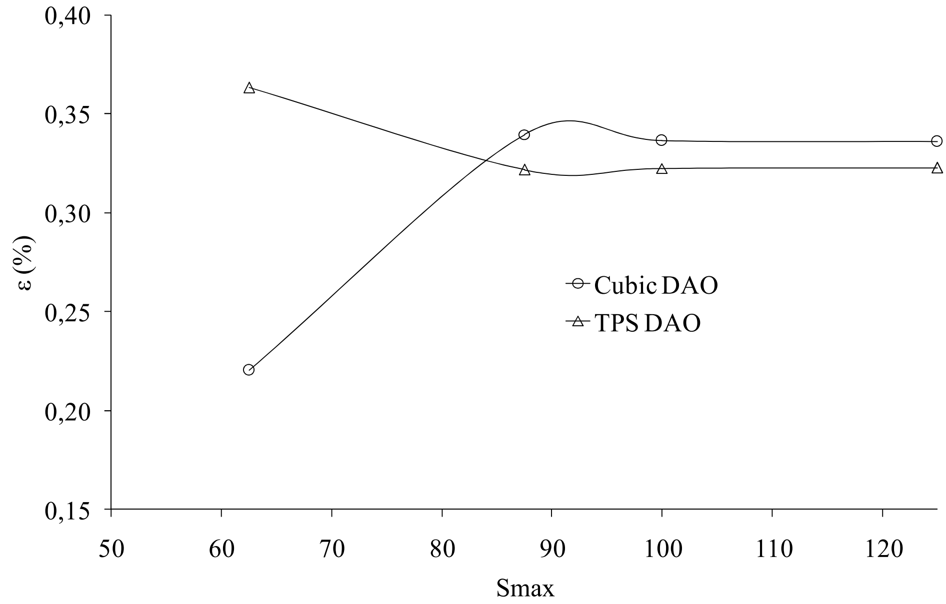

Figure 9 – Relative error between RBF **Down-and-Out** barrier option values at the exercise option value (%) as affected by the maximum simulated stock value; Nt = 100, $\theta = 1$ and $\Delta S = 0.893$.

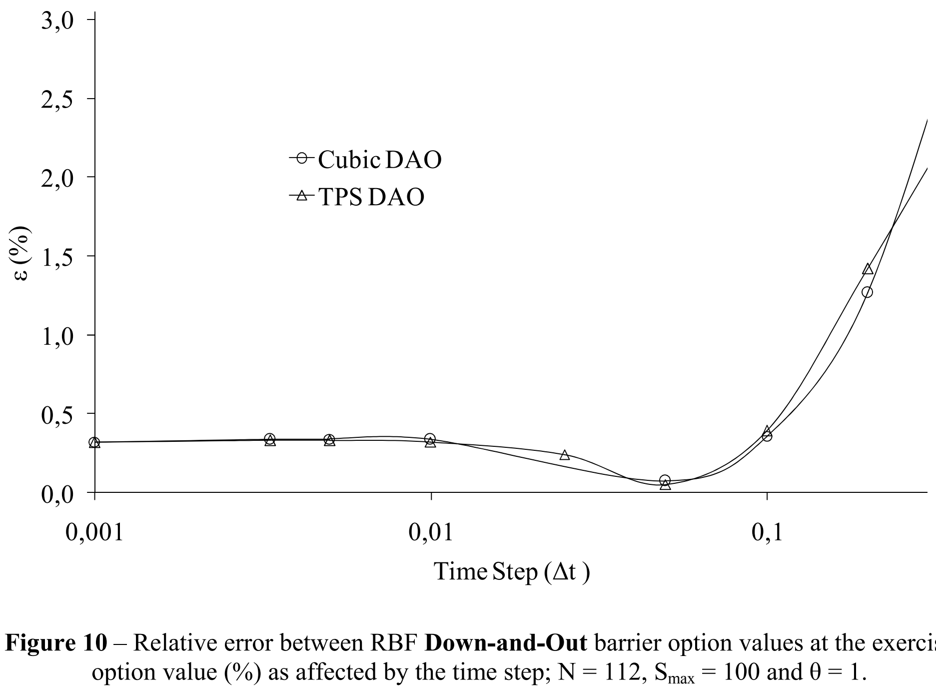

Figure 10 – Relative error between RBF **Down-and-Out** barrier option values at the exercise option value (%) as affected by the time step; N = 112, $S_{max} = 100$ and $\theta = 1$.

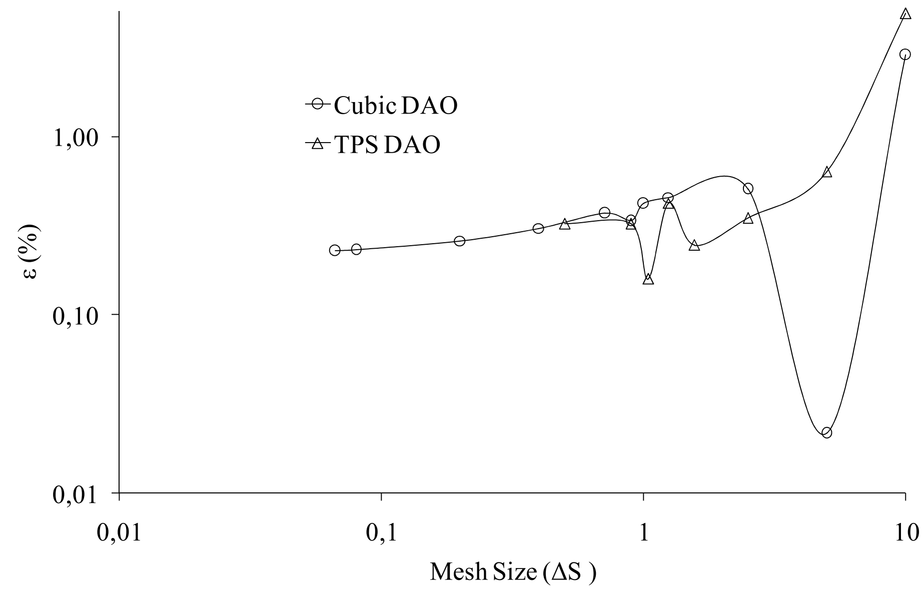

Figure 11 – Relative error between RBF **Down-and-Out** barrier option values at the exercise option value (%) as affected by the mesh size; Nt = 100, S_{max} = 100 and θ = 1.

6. Conclusions

This work presents a detailed analysis and modeling of Black-Scholes equation, both in the classical and the diffusional version, using Radial Basis Functions. In order to assure the range of applicability of the RBF method, numerical solutions were compared against analytical solutions for the classical Vanilla option and the problems of non-linear boundary conditions, as defined by path-dependent barrier options. Furthermore, an analysis taking into account the most important numerical parameters was undertaken in order to establish a convenient way for reaching accurate solutions using the RBF method.

The main conclusions are:

1. There is no noticeable difference between solutions obtained through the classical and the diffusional forms of the BS equation.
2. As general procedures for financial engineering simulation, the problems at hand suggest to decrease the mesh size from a starting coarse mesh, to obtain reasonably close solutions. Non-linearity, as in the case of barrier options, precludes affirming that smaller stock value mesh size will necessarily lead to smaller relative errors; however, if the mesh size is heavily decreased, divergence will occur and the mesh size should be increased. Under these conditions, this work shows the occurrence of stable and accurate results.
3. Implicit simple time-integration methods with $\theta \geq 0.5$ lead to stable solutions.
4. In order to warrantee a proper choice for the boundary condition of option value at infinity, that is, $V(t, S_{max}) \approx V(t, \infty)$, it seems to be safer to take $S_{max} \geq 2E$.

5. Time step sizes should be decreased until reaching a desired convergence of numerical results. Again, as in conclusion 2, above, small fluctuations are to be expected.
6. Call option simulations are simple to implement and, due to the very accurate results obtained, were not subject to detailed analysis. Barrier options, on the other hand, required a more detailed analysis.

Thus, the results allow concluding that the Cubic and the TPS RBF method are well suited for modeling and analyzing Black and Scholes equation under non-linear time-dependent boundary conditions.

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