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# HUB LOCATION UNDER HUB CONGESTION AND DEMAND UNCERTAINTY: THE BRAZILIAN CASE STUDY

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**ABSTRACT.** In this work, a mixed integer nonlinear programming model combining direct service links, demand uncertainty and congestion effects is proposed. This model is efficiently solved by Generalized Benders Decomposition, for instances of moderate sizes and reasonable number of scenarios. The deployed algorithms are further used for re-designing the Brazilian air transportation network, enabling the analysis of future demand scenarios and providing decision support about the optimal investment policy for Brazil.

**Keywords**: hub-and-spoke networks, Benders Decomposition, large scale optimization, mixed integer non-linear programming, demand uncertainty.

## 1 INTRODUCTION

In the last few years, the Brazilian air transportation network has gone through several operational breakdowns. The apparent reason is the lack of capacity or infra-structure on the main airports of the system. But a closer look of the network reveals that the total installed capacity is much larger than the total traffic volume to be handled, indicating that the network seems to operate in a sub-optimal way. As a consequence, the network performance has been severely compromised, reducing the opportunity of the involved logistic operators for attaining the economies of scale that lead to sustainable revenues.

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In order to provide the necessary operational efficiency to this system, as well as to define optimal investment policies considering the system expansion and future demand sceneries, a mixed integer non-linear programming formulation is proposed. This model aims to redesign the current network, optimizing the use of the installed capacity and balancing the flows through the network. Here, the underlying structure adopted for the Brazilian air transportation network is a customized hub-and-spoke topology. The desired effect is eventually to deactivate some of the current network hubs while activating several others to handle the additional third party traffic.

Hub-and-spoke systems have been largely employed in the telecommunication and transportation areas (see [18, 43, 4, 42, 14]). This class of systems arises when commodities (passenger, cargo parcels, telecommunication packets) of a set of origins must be sent to a set of destinations. Instead of establishing a direct connection for each origin-destination pair, facilities that serve as switching, transshipment, sorting and distribution nodes, designated as hubs, are used as the only valid intermediate points in a path from an origin to a destination. Flows from different origins are gathered at these hub facilities prior to be routed to an intermediate hub or to be delivered to their final destinations.

The employment of these hub facilities and the routing of consolidated flows through inter-hub links allow the centralization of commodity handling and the transportation cost per unit of flow to be less expensive than directly shipping via a non-hub network structure. In other words, the hub networks take advantage of scale economies on inter-hub connections [58, 59].

Usually, the main concerns of hub network design are the location of hub facilities and the allocation of non-hub nodes to hubs at the least expensive network structure, accounting for that the installation and the transportation costs. This is done by selecting node candidates as hubs and assigning origin and destination nodes to these hubs, given the flow volumes between each origin-destination node pair, the installation costs of hubs and the transportation costs between nodes of the network. While hubs are in general fully connected among them, non-hub nodes can be single allocated, meaning that a non-hub can be assigned to exactly one hub only; and multiple allocated, meaning that a non-hub can be connected to more than one hub of the network. Alternative configurations can be considered in order to ensure the application of a specific protocol [62]. For further basic notions, different considerations and formulations of hub networks see the comprehensive surveys of Campbell [12, 13] and of Campbell *et al.* [14] and of Alumur & Kara [2].

The minimization of the installation and transportation costs approach generates solutions that tend to overload a small number of hubs, forcing the congestion effects into consideration. The hub literature has been dealing with the congestion cost effects on hub networks implicitly, when these effects are represented by constraints, and explicitly, when the congestion costs are expressed on the objective function.

Grove & O'Kelly [36] are among the first authors to study the effects of congestion on hub networks. They have demonstrated how the schedule delays of airline systems are influenced by the amount of flow at the hubs by simulating a single assignment hub network with fixed hub locations.

A great variety of researchers has tackled the congestion effects restricting the amount of flow transiting through a hub by means of capacity constraints. Aykin [3] has devised a lagrangian relaxation approach, while Ernst & Krishnamoorthy [27] have used simulated annealing and random descent methods. Ebery *et al.* [23] have imposed capacities only on incoming flows. They have developed a branch-and-bound based on a shortest path heuristic. Solutions obtained by their method may have flows from a hub to itself routed via another hub as explained by Campbell *et al.* [14]. Campbell *et al.* [17, 15, 16] have proposed the hub arc location problem, in which the hub network is seen as a three-layered network. The bottom one has the origin nodes of the demand, the middle layer has the hub nodes, and the top one has the destination nodes of the demand.

Yaman & Carello [72] have addressed the problem of solving single allocation hub location problems with modular link capacities. They have proposed a non-linear formulation where the size of inter-hub connections are treated as stepwise, instead of the linear approach proposed by Labbé *et al.* [44] to the quadratic capacitated hub location problem with single assignment. In order to accomplish that, capacity restrictions have been imposed on the amount of traffic flowing through the hubs rather than only considering the incoming traffic. The problem has been solved by means of an branch-and-cut algorithm based on a linearization with an exponential number of constraints and a two-level local search method. Both methods have been then combined in a heuristic concentration (see [68, 67]) scheme. Marín [54] also has addressed the capacitated multiple allocation hub location problems, but assumed that the flow between a given origin-destination pair can be split into several routes.

Another interesting work is the one presented by Kara & Tansel [41]. They have modeled the transient times at hubs in addition to the travel times, having as objective function the minimization of the latest-arrival times, and consequently, the excessive delays at the hubs. Yaman *et al.* [73] have presented a similar formulation for a Turkish cargo delivery system where transient times have been taken into account as well as journey times.

A work that stands out is the one of Marianov & Serra [53]. They have modeled the hub network as an M/D/c queuing network, proposing capacity constraints based on the probability of waiting customers in the system. Due to the computational complexity of these constraints, they have linearized them and then solved the resulting model by a tabu search algorithm. A similar work has been proposed by Rodriguez *et al.* [65], but a simulated annealing algorithm has been used.

Costa *et al.* [22] have presented a multi-criteria formulation to the capacitated single allocation hub location problem. Besides the traditional cost minimizing function, they have considered alternatively the minimization of the time the hubs take to process the flow or the minimization of the maximum service time of the hubs. For instance, Rodriguez-Martin and Salazar-Gonzalez [66] have proposed a Benders decomposition algorithm [6] to tackle a capacitated hub location problem based on a multi-commodity formulation where the arcs connecting the hubs are not assumed to create a complete graph. Their model is similar to the one presented by Sridhar and Park [71] for the fixed-charge capacitated network design problem.

Modeling congestion through capacity constraint on the flows does not mimic the exponential nature of the congestion effects. Elhedhli & Hu [24] have been the first ones to consider the costs of the congestion effects explicitly on the objective function. Using a convex cost function that increases exponentially as more flows go through the hubs, they have proposed a non-linear model to a single assignment hub location problem. The congestion convex cost function is a power-law function of airport usage relative to its capacity and it has been widely used to estimate delay costs [35]. They have linearized their model and then solved it using lagrangian relaxation.

On the other hand, stochastic location problems where demand uncertainties are taken into account have been addressed by G. Albareda-Sambola *et al.* [1], Hochreiter & Pflug [40], Snyder *et al.* [70], Gabor & Van Ommeren [31], Snyder & Daskin [69], Laporte *et al.* [46], Louveaux & Peeters [48], Laporte *et al.* [45] and França & Luna [28]. In all these works, it is established that stochastic modeling provides a superior design by considering uncertainty in important parameters. When dealing with HS network design, it is interesting to consider the demand uncertainty, since, according to Camargo *et al.* [10], the impact of demand components in the optimal HS network under congestion is considerable. Another interesting conclusion of Camargo *et al.* [10] is that under huge congestion, the marginal utility of installing a new hub in the HS network is reduced. This counter-intuitive effect is due to the HS classical protocol – if a given demand is considerably high, it is not interesting to have this link on the HS network, since it will always imply in high congestion. It is clear that if a given node is overloaded with his own demand, it is unable to handle third party traffic. It is also uninteresting to route such high flows trough a hub, under penalty of severely impact the hub performance. A further step could be the allowance of direct routes between non hub nodes, where no economy of scale is observed but no congestion either.

In this paper, we explore further the congestion effects written as a convex cost function similarly to [24], but addressing the multiple allocation hub location problem, considering also effects of demand uncertainty combined with direct service links. We propose a non-linear mixed integer programming problem based on a classical formulation [37] due to its linear programming bound quality when compared to others [12, 25, 26, 61]. In our formulation the number of hubs on a route is limited to two, even if there is a route with a lower cost using more than two hubs.

Although, one of the main strategies to handle non-linear problems is to linearize them, we have developed a Generalized Benders Decomposition (GBD) [33] to cope with our non-linear mixed integer program. The problem is decomposed in two smaller problems: at a higher level, named as master problem (MP), the location decisions are made; while at an inferior level, known as subproblem (SP), the flow balance and the congestion are handled. The MP is an integer programming problem, while the SP is a non linear convex transportation problem. Using our approach, we have been able to solve some large problems of the CAB and AP standard data sets to optimality. We have also used the deployed algorithm to deal with a real hub location case, studying the Brazilian air transportation network.

This paper is organized as follows: in Section 2, the model formulation and the used notation are introduced. In Section 3, the Generalized Benders Decomposition is developed to solve the pro-

blem and it is also demonstrated how the subproblems can be optimally solved. Computational experiments and conclusion remarks are presented in Section 4 and 5, respectively.

## 2   MODEL FORMULATION

In this section the our model formulation efforts are presented. The basic components of the model are the following sets: let $N$ be the set of node locations that exchange traffic and let $H$ be the set of node candidates to become hubs, $H \subseteq N$. For any pair of nodes $i$ and $j$ $(i, j \in N)$, we have $w_{ij}$, the flow from origin $i$ to destination $j$ that is routed through either one or two installed hubs. Usually, we have $w_{ij} \neq w_{ji}$.

Further, let $f_k$ be the fixed cost of installing a hub at node $k \in H$ and let $c_{ijkm}$ be the transportation cost per unit of flow from node $i$ to $j$ routed via hubs $k$ and $m$ $(i, j \in N$ and $k, m \in H)$. This transportation cost is the composition of three cost segments: $c_{ijkm} = c_{ik} + \alpha \, c_{km} + c_{mj}$, where $c_{ik}$ and $c_{mj}$ are the standard transportation cost per unit from location $i$ $(j)$ to hub $k$ $(m)$, and $\alpha c_{km}$ is the discounted transportation cost between hubs $k$ and $m$. The discount factor $0 \leq \alpha \leq 1$ represents the scale economies on the inter-hub connection. If only one hub is used in a given route, we have $k = m$ and no discount factor is applied.

The following decision variables are defined:

$$y_k = \begin{cases} 1, & \text{if hub } k \in H \text{ is installed;} \\ 0, & \text{otherwise.} \end{cases}$$

$x_{ijkm} \geq 0$ is the flow from origin $i$ to destination $j$ $(i, j \in N)$ that is routed through hubs $k$ and $m$ $(k, m \in H)$ in that order.

In order to start the development of the congested version of the uncapacitated multiple allocation hub location problem (UMAHLP), the formulation due to Hamacher $et\ al.$ [37] is stated as:

$$\min \sum_k f_k \, y_k + \sum_i \sum_j \sum_k \sum_m c_{ijkm} \, x_{ijkm} \tag{1}$$

s. t.:

$$\sum_k \sum_m x_{ijkm} = w_{ij} \qquad\qquad \forall i, j \in N \tag{2}$$

$$\sum_m x_{ijkm} + \sum_{m \neq k} x_{ijmk} \leq w_{ij} \, y_k \qquad\qquad \forall i, j \in N, \ k \in H \tag{3}$$

$$x_{ijkm} \geq 0 \qquad\qquad \forall i, j \in N, \ k, m \in H \tag{4}$$

$$y_k \in \{0, 1\} \qquad\qquad \forall k \in K \tag{5}$$

The model (1)-(5) is reputed to have a tight linear programming bound [37], being one of the strongest formulations for the UMAHLP available. One of its major drawbacks is its size. However, as demonstrated by the pioneering work of Geoffrion & Graves [32], structure is more important than size when dealing with large scale mathematical programs.

In Camargo *et al.* [11] the above formulation has been used to solve instances up to 200 nodes with the aid of Benders decomposition. In this sense, formulation (1)-(5) is adopted as a basis to develop a congested approach to hub location due to its nice theoretical properties and good structure. Hamacher *et al.* [37] have demonstrated that constraints (3) are facet defining. In addition, for a fixed hub structure, the remaining formulation is fully decomposable for each $i - j$ pair. In spite of this, O'Kelly & Brian [60] have shown that having the same discount factor $\alpha$ for all the inter-hub connections may be unrealistic. Optimal solutions where large discounts are given on links having small flows can be generated in this way. Overcoming this drawback, a hub location formulation with flow-dependant economies of scale can be addressed, as proposed by Camargo *et al.* [9].

In order to allow the use of direct service links, is necessary to add a new set of variables:

$\theta_{ij} \geq 0$ is the amount of flow transported through a direct connection for pair $i - j$.

It is also necessary to modify constraints (2) to include the direct paths, producing:

$$\theta_{ij} + \sum_k \sum_m x_{ijkm} = w_{ij} \qquad \forall i, j \in N \qquad (6)$$

Implying to rewrite the objective function as:

$$\min \sum_k f_k \, y_k + \sum_i \sum_j \sum_k \sum_m c_{ijkm} \, x_{ijkm} + \sum_i \sum_i c_{ij} \, \theta_{ij} \qquad (7)$$

Moreover, Marín *et al.* [55] have shown how formulation (1)-(5) can allow optimal solutions with routes having more than two hubs when the triangular inequalities are not satisfied. Instead of using the constraints derived by Marín to deal with those possible unfeasible paths, the following additional constraints are preferable in a decomposition framework, since they preserve the $i - j$ pair decomposition:

$$x_{ijij} \geq w_{ij} \, (y_i + y_j - 1) \qquad \forall i, j \in N \qquad (8)$$

$$\sum_{k \neq j} x_{ijik} \geq w_{ij} \, (y_i - y_j) \qquad \forall i, j \in N \qquad (9)$$

$$\sum_{k \neq i} x_{ijkj} \geq w_{ij} \, (y_j - y_i) \qquad \forall i, j \in N \qquad (10)$$

Constraints (8) make sure that the flow $w_{ij}$ goes through route $i - j - i - j$ when $i$ and $j$ are hubs. Constraints (9) and (10) limit the number of hubs to two when $i$ or $j$ are hubs. Furthermore, the already deployed subproblem solution algorithm of Camargo *et al.* [11] can be partially reused. Remark that the above cited origin-destination special structure is a consequence of the interchangeability of indexes $i, j, k$ and $m$, *i.e.* $H \equiv N$. It is also straight-forward to modify the model and the deployed algorithms if $H \subseteq N$ is required. Since we are considering the congestion effects as a function of the total hub flow, we need an extra decision variable for each hub to account it:

$g_k \geq 0$ is the total flow which goes through hub $k \in H$.

Further, a set of additional coupling constraints have to be added in order to amount the total flow in a given hub:

$$\sum_i \sum_j \sum_m x_{ijkm} + \sum_i \sum_j \sum_{m \neq k} x_{ijmk} = g_k \qquad \forall k \in H \qquad (11)$$

A congestion cost function $\tau_k(g_k)$ is assumed as given for each installed hub $k \in H$. This function is considered increasing on $[0, +\infty)$, proper convex and smooth. It is responsible for distributing the flow load along each installed hub, thus reducing the overall congestion expressed in monetary values. The congestion cost function used is a power law where $\tau_k = e\, g_k^b$, and $e$ and $b$ are positive constants, $b \geq 1$. This class of functions has been already used by [24] to the single allocation case and by [10] in his recent work. The congested multiple allocation hub location problem with direct service can now be stated as follows:

$$\min \sum_k [f_k\, y_k + \tau_k(g_k)] + \sum_i \sum_j \sum_k \sum_m c_{ijkm}\, x_{ijkm} + \sum_i \sum_j c_{ij}\, \theta_{ij} \qquad (12)$$

s. t.:

$$\sum_i \sum_j \sum_m x_{ijkm} + \sum_i \sum_j \sum_{m \neq k} x_{ijmk} = g_k \qquad \forall k \in H \qquad (13)$$

$$\theta_{ij} + \sum_k \sum_m x_{ijkm} = w_{ij} \qquad \forall i, j \in N \qquad (14)$$

$$\sum_m x_{ijkm} + \sum_{m \neq k} x_{ijmk} \leq w_{ij}\, y_k \qquad \forall i, j \in N,\ k \in H \qquad (15)$$

$$x_{ijkm} \geq 0 \qquad \forall i, j \in N, k, m \in H \qquad (16)$$

$$g_k \geq 0 \qquad \forall k \in H \qquad (17)$$

$$y_k \in \{0, 1\} \qquad \forall k \in H \qquad (18)$$

$$x_{ijij} \geq w_{ij}\, (y_i + y_j - 1) \qquad \forall i, j \in N \qquad (19)$$

$$\sum_{k \neq j} x_{ijik} \geq w_{ij}\, (y_i - y_j) \qquad \forall i, j \in N \qquad (20)$$

$$\sum_{k \neq i} x_{ijkj} \geq w_{ij}\, (y_j - y_i) \qquad \forall i, j \in N \qquad (21)$$

$$\theta_{ij} \geq 0 \qquad \forall i, j \in N \qquad (22)$$

In the above model the objective function (12) minimizes the total cost, *e.g.*, the sum of the installation, congestion and linear transportation costs. Constraints (13) determine the total amount of flow which goes through hub $k$. Constraints (14) assure that the flow for every pair

$i$-$j$ is routed via some hub pair or use a direct service link. Once again, if only one hub is used, we have $k = m$. Constraints (15) guarantee that the flow going through a hub only happens if that hub is installed. Constraints (19), (20) and (21) avoid problematic routes as discussed above. Constraints (16), (17) and (22) are the non-negativity constraints of the continuous variables $x_{ijkm}$, $r_{ij}$ and $g_k$, while the constraints (18) restrict the integer variables $y_k$ to be 0 or 1.

In order to consider effects of demand uncertainty it is possible to adopt a discrete set of states of the world $S$, associating each one to a probability $p^s, s \in S$, rewriting the demand parameter as $w_{ij}^s$, the flow from origin $i$ to destination $j$ in the state of the world $s \in S$. Clearly, the formulation (12)-(22) must be generalized to deal with the new stochastic parameters. We have chosen to replicate the sets of decision variables $x_{ijkm}$, $\theta_{ij}$, and $g_k$ and associated constraints for each state of the world, in order to minimize the sum of fixed (installation) costs plus the expected variable (linear transportation and congestion) cost components. The first step is to redefine the above cited decision variables:

$x_{ijkm}^s \geq 0$ is the flow from origin $i$ to destination $j$ $(i, j \in N)$ that is routed through hubs $k$ and $m$ $(k, m \in H)$ in that order, in the state of the world $s \in S$.

$\theta_{ij}^s \geq 0$ is the amount of flow transported through a direct connection for pair $i - j$ in the state of the world $s \in S$.

$g_k^s \geq 0$ is the total flow which goes through hub $k \in H$ in the state of the world $s \in S$.

Rewriting formulation (12)-(22) by introducing the new sets of decision variables leads to the following stochastic programming problem, that we are calling hub location problem under hub congestion and demand uncertainty:

$$
\min \sum_k \left[ f_k\, y_k + \sum_s p^s\, \tau_k(g_k^s) \right] + \sum_{(i,j)} \sum_{(k,m)} \sum_s p^s\, c_{ijkm}\, x_{ijkm}^s
$$

$$
+ \sum_{(i,j)} \sum_s p^s\, c_{ij}\, \theta_{ij}^s \tag{23}
$$

s. t.:

$$
\sum_{(i,j)} \sum_m x_{ijkm}^s + \sum_{(i,j)} \sum_{m \neq k} x_{ijmk}^s = g_k^s \qquad \forall k \in H,\ \forall s \in S \tag{24}
$$

$$
\theta_{ij}^s + \sum_{(k,m)} x_{ijkm}^s = w_{ij}^s \qquad \forall i, j \in N,\ \forall s \in S \tag{25}
$$

$$
\sum_m x_{ijkm}^s + \sum_{m \neq k} x_{ijmk}^s \leq w_{ij}^s\, y_k \qquad \forall i, j \in N,\ k \in H,\ \forall s \in S \tag{26}
$$

$$
x_{ijkm}^s \geq 0 \qquad \forall i, j \in N, k, m \in H,\ \forall s \in S \tag{27}
$$

$$
g_k^s \geq 0 \qquad \forall k \in H,\ \forall s \in S \tag{28}
$$

$$
y_k \in \{0, 1\} \qquad \forall k \in H \tag{29}
$$

$$x_{ijij}^s \geq w_{ij}^s \ (y_i + y_j - 1) \qquad\qquad \forall i, j \in N, \ \forall s \in S \qquad\qquad (30)$$

$$\sum_{k \neq j} x_{ijik}^s \geq w_{ij}^s \ (y_i - y_j) \qquad\qquad \forall i, j \in N, \ \forall s \in S \qquad\qquad (31)$$

$$\sum_{k \neq i} x_{ijkj}^s \geq w_{ij}^s \ (y_j - y_i) \qquad\qquad \forall i, j \in N, \ \forall s \in S \qquad\qquad (32)$$

$$\theta_{ij}^s \geq 0 \qquad\qquad \forall i, j \in N, \ \forall s \in S \qquad\qquad (33)$$

As the above model has not capacity constraints, there is always a feasible solution for (23)-(33). Furthermore, for any given fixed structure of hubs, *i.e.* a feasible vector $y$, the model transforms into a non-linear convex transportation model. This non-linear transportation problem can then be solved by a feasible directions search method such as [29] or by the proximal decomposition algorithm of Mahey *et al.* [52]. Gathering all of these features, we have been motivated in deploying the Generalized Benders decomposition method (GBD) [6, 33] to tackle the problem.

Concerning GBD, the algorithm performance is very sensitive to the overall contribution of the non-linear terms, since it is a first order approach. This is particularly true if the associated mixed integer linear program has a poor linear programming bound (see [51, 28, 63]). For this reason, formulations having less variables but weaker lower bounds, as those proposed by Ernst & Krishnamoorthy [26], may be disregarded for this kind of application.

## 3  APPLYING THE GENERALIZED BENDERS DECOMPOSITION (GBD) METHOD

In 1962, Benders [6] proposed a partitioning method for solving mixed linear and nonlinear integer programming problems. Benders has defined a relaxation algorithm which partitions the original problem into two simpler problems: the relaxed master problem (RMP) and the subproblem (SP). The first one is a relaxed version of the original problem with the set of integer variables and its associated constraints; while the last one is the original problem with the values of the integer variables fixed by the RMP.

In general terms, Benders decomposition relies on a *projection* problem manipulation, followed by solution strategies of dualization, outer linearization and relaxation. The complicating variables (integer variables) of the original problem are projected out, resulting into an equivalent model with fewer variables, but many more constraints. However, when attaining optimality, a large number of these constraints will not be binding. Hence a strategy of relaxation can be devised to ignore all but a few of these constraints.

Benders [6] has suggested a procedure to add these constraints on demand, adopting two levels of coordination. At a higher level, the RMP is responsible for fixing the values of the integer variables and for providing a lower bound for the problem. At a lower level, the subproblem is solved in order to obtain an upper bound and a violated cut on the master problem space. This valid inequality is known as Benders cut and will enrich the RMP in the next iteration. The procedure stops when the upper bound and lower bound converge towards the optimal value.

The Benders decomposition has been successfully applied on solving large scale problems since the pioneering paper of Geoffrion and Graves [32], the uncapacitated network design problem with undirected arcs of Magnanti *et al.* [49], the locomotive and car assignment problem of Cordeau *et al.* [20, 21], the cellular manufacturing system design of Heragu & Chen [38], the multicasting network design problems of Miranda [57] and Randazzo & Luna [64] and the hub location problem with economies of scale of Camargo *et al.* [9]. The original procedure has been specialized to deal with stochastic programming problems, receiving sometimes the name of L-Shaped Decomposition Method on the literature of stochastic programming, due to the form of the constraint matrices generated, amenable to dual decomposition techniques.

Several authors have addressed other extensions and improvement of the Benders decomposition method. Magnanti & Wong [50] have proposed an approach for selecting the dual variables and thus improving the overall performance of the algorithm. McDaniel & Devine [56] have developed an algorithm for accelerating the decomposition by removing temporarily the integrality constraints of the RMP. Balas & Bergthaller [5] have devised a nice approach for generating strong cuts prior of the beginning of the Benders iterations. Geoffrion [33] has extended the basic algorithm to non-linear problems, formulating the Generalized Benders Decomposition (GBD). The GBD has been employed on different problems. Cai *et al.* [8] have used it on a nonconvex water resource management problem, Hoang [39] has applied it on a nonlinear model for network design, while Mahey *et al.* [51] have solved capacity and flow assignments in telecommunication networks by means of GBD.

When GBD is used to tackle the congestioned hub design problem, the RMP chooses a hub structure configuration represented by the decision variables $y$, and the SP solves the non-linear transportation problem with price sensitive demands on the resulting network. The dual solution of the transportation problem is then used to generate a Benders cut that will redefine the hub structure configuration in the next iteration. Adjustment of the non-linear objective function is necessary at the subproblem. This is accomplished by dualizing the coupling constraints associated to the non-linearity. After dualization, the resulting subproblems are relatively easy to solve.

In the next Sections 3.1 and 3.2, we present the relaxed master problem and the subproblem, respectively, of the congestioned hub-and-spoke network design problem.

## 3.1   Relaxed Master Problem (RMP)

From the viewpoint of mathematical programming, a projection of the problem (24)-(33) onto the space of the *structural* variables $y$ can be done, resulting on the following problem to be solved:

$$\min_{y \in Y} \sum_k f_k \, y_k + \sum_s p^s \, z(y)^s \tag{34}$$

where $Y = \{y \in \{0, 1\} \mid$ for $y$ fixed, there are feasible flows satisfying (24)-(28) and (30)-(33)$\}$ and $z(y)^s$ is the following subproblem to be solved for each state of the world $s$:

$$z(y)^s = \min_{(x,g,\theta) \in G} \sum_k \tau_k(g_k^s) + \sum_{(i,j)} \sum_{(k,m)} c_{ijkm} \, x_{ijkm}^s + \sum_{(i,j)} c_{ij} \, \theta_{ij}^s \tag{35}$$

subject to (26),(30)-(32) for $y$ fixed and the appropriate $s$, and where

$$G = \left\{ (x, g, \theta) \mid x \geq 0 \text{ and } g \geq 0 \text{ and } \theta \geq 0, \text{ satisfying (24), (25), (27), (28), (33)} \right\}.$$

For any hub structure configuration there is a feasible path connecting the pair of demands $i$-$j$, hence, there is no need for further feasibility constraints on the domain of the projected problem (34). Further, since the subproblem has a convex differentiable objective function and linear constraints, its Karush-Kuhn-Tucker conditions are necessary and sufficient for optimality. This makes the problem consequently amenable to dualization techniques (see [33, 47]).

After associating the vectors of dual variables $u \geq 0$, $v \geq 0$, $t \geq 0$ and $r \geq 0$ to the coupling constraints (26), (30)-(32) and since there is no duality gap, the optimal value of the subproblem (35) for any $y \in Y$ can be given by:

$$
\begin{aligned}
z(y)^s \;=\; \max_{u,v,r,t \geq 0} \Bigg\{ & \min_{(x,g,\theta) \in G} \sum_k \tau_k(g_k^s) + \sum_{(i,j)} \sum_{(k,m)} c_{ijkm}\, x_{ijkm}^s + \sum_{(i,j)} c_{ij}\, \theta_{ij}^s \\
& + \sum_{(i,j)} \sum_k u_{ijk}^s \left( \sum_m x_{ijkm}^s + \sum_{m \neq k} x_{ijmk}^s - w_{ij}^s\, y_k \right) \\
& + \sum_{(i,j)} v_{ij}^s \left( w_{ij}^s\, (y_i + y_j - 1) - x_{ijij}^s \right) + \sum_{(i,j)} r_{ij}^s \left( w_{ij}^s\, (y_i - y_j) - \sum_{k \neq j} x_{ijik}^s \right) \\
& + \sum_{(i,j)} t_{ij}^s \left( w_{ij}^s\, (y_j - y_i) - \sum_{k \neq i} x_{ijkj}^s \right) \Bigg\}
\end{aligned}
$$

Recalling the interchangeability of indexes $i$, $j$, $k$ and $m$, $z(y)^s$ can be rewritten in a more compact form as:

$$
\begin{aligned}
z(y)^s \;=\; \max_{u,v,r,t \geq 0} \Bigg\{ & \sum_k \Bigg[ -\sum_{(i,j)} u_{ijk}^s\, w_{ij}^s + \sum_{m \neq k} w_{km}^s (v_{km}^s + r_{km}^s - t_{km}^s) \\
& + \sum_{m \neq k} w_{mk}^s (v_{mk}^s - r_{mk}^s + t_{mk}^s) \Bigg] y_k - \sum_k \sum_{m \neq k} v_{km}^s w_{km}^s + \min_{(x,g,\theta) \in G} \Bigg\{ \sum_k \tau_k(g_k^s) \\
& + \sum_{(i,j)} \sum_{(k,m)} c_{ijkm}\, x_{ijkm}^s - \sum_{(i,j)} \sum_k u_{ijk}^s \left( \sum_m x_{ijkm}^s + \sum_{m \neq k} x_{ijmk}^s \right) \\
& + \sum_{(i,j)} v_{ij}^s x_{ijij}^s + \sum_{(i,j)} \sum_{k \neq j} r_{ij}^s x_{ijik}^s + \sum_{(i,j)} \sum_{k \neq i} t_{ij}^s x_{ijkj}^s + \sum_{(i,j)} c_{ij}\, \theta_{ij}^s \Bigg\} \Bigg\}
\end{aligned}
\tag{36}
$$

Defining $\Omega_k^s$ as:

$$\Omega_k^s = -\sum_{(i,j)} u_{ijk}^s\, w_{ij}^s + \sum_{m \neq k} w_{km}^s (v_{km}^s + r_{km}^s - t_{km}^s) + \sum_{m \neq k} w_{mk}^s (v_{mk}^s - r_{mk}^s + t_{mk}^s)$$

for all $k \in H$, (36) can now be stated as:

$$
\begin{aligned}
z(y)^s = \max_{u,v,r,t\geq 0} & \left\{ \sum_k \Omega_k^s \, y_k - \sum_k \sum_{m \neq k} v_{km}^s w_{km}^s + \min_{(x,g,\theta)\in G} \left\{ \sum_k \tau_k(g_k^s) \right.\right. \\
& + \sum_{(i,j)} \sum_{(k,m)} c_{ijkm} \, x_{ijkm}^s - \sum_{(i,j)} \sum_k u_{ijk}^s \left( \sum_m x_{ijkm}^s + \sum_{m \neq k} x_{ijmk}^s \right) \\
& \left.\left. + \sum_{(i,j)} v_{ij}^s x_{ijij}^s + \sum_{(i,j)} \sum_{k \neq j} r_{ij}^s x_{ijik}^s + \sum_{(i,j)} \sum_{k \neq i} t_{ij}^s x_{ijkj}^s + \sum_{(i,j)} c_{ij} \, \theta_{ij}^s \right\} \right\}
\end{aligned} \tag{37}
$$

Hence, the whole problem (34) is equivalent to:

$$
\begin{aligned}
\min_{y \in Y} & \left\{ \sum_k f_k \, y_k + \sum_s p^s \max_{u,v,r,t\geq 0} \left\{ \sum_k \Omega_k^s \, y_k - \sum_k \sum_{m \neq k} v_{km}^s w_{km}^s \right.\right. \\
& + \min_{(x,g)\in G} \left\{ \sum_k \tau_k(g_k^s) + \sum_{(i,j)} \sum_{(k,m)} c_{ijkm} \, x_{ijkm}^s - \sum_{(i,j)} \sum_k u_{ijk}^s \left( \sum_m x_{ijkm}^s + \sum_{m \neq k} x_{ijmk}^s \right) \right. \\
& \left.\left.\left. + \sum_{(i,j)} v_{ij}^s x_{ijij}^s + \sum_{(i,j)} \sum_{k \neq j} r_{ij}^s x_{ijik}^s + \sum_{(i,j)} \sum_{k \neq i} t_{ij}^s x_{ijkj}^s + \sum_{(i,j)} c_{ij} \, \theta_{ij}^s \right\} \right\}\right\}
\end{aligned}
$$

As the supremum is the least upper bound, the mathematical program (23)-(32) can be represented as the following master problem (MP) with the help of variable $\eta \geq 0$:

$$
\min_{\eta, \, y \in Y} \sum_k f_k \, y_k + \sum_s p^s \, \eta^s \tag{38}
$$

s. t.:

$$
\begin{aligned}
\eta^s \geq & \sum_k \Omega_k^s y_k - \sum_k \sum_{m \neq k} v_{km}^s w_{km}^s + \min_{(x,g,\theta)\in G} \left\{ \sum_k \tau_k(g_k^s) \right. \\
& + \sum_{(i,j)} \sum_{(k,m)} c_{ijkm} \, x_{ijkm}^s - \sum_{(i,j)} \sum_k u_{ijk}^s \left( \sum_m x_{ijkm}^s + \sum_{m \neq k} x_{ijmk}^s \right) \\
& + \sum_{(i,j)} \left[ v_{ij}^s x_{ijij}^s + \sum_{k \neq j} r_{ij}^s x_{ijik}^s + \sum_{k \neq i} t_{ij}^s x_{ijkj}^s \right] \\
& \left. + \sum_{(i,j)} c_{ij} \, \theta_{ij}^s \right\} \quad \forall \, u,v,r,t \geq 0, \; \forall s \in S \tag{39}
\end{aligned}
$$

$$
\eta^s \geq 0 \; \forall s \in S \tag{40}
$$

Problem (38)-(40) is solved by the GBD through a strategy of relaxation, *i.e.* ignoring all but a few of the constraints (39). This strategy requires a procedure which adds iteratively the constraints to the MP as needed. So, at any iteration $h$, the optimal value $z(y^h)^s$ is found from (37) after solving the non-linear transportation subproblem for a given installed hub structure $y^h$ and recovering the vector of optimal multipliers $(u^s)^h$, $(v^s)^h$, $(r^s)^h$ and $(t^s)^h$ ($u^s = (u^s)^h$, $v^s = (v^s)^h$, $r^s = (r^s)^h$ and $t^s = (t^s)^h$). Thus the optimal value of $z(y^h)^s$ is given by:

$$
z(y^h)^s = \sum_k (\Omega_k^s)^h y_k^h - \sum_k \sum_{m \neq k} (v_{km}^s)^h w_{km}^s + \min_{(x,g,\theta) \in G} \left\{ \sum_k \tau_k(g_k^s) \right.
$$

$$
+ \sum_{(i,j)} \sum_{(k,m)} c_{ijkm} x_{ijkm}^s - \sum_{(i,j)} \sum_k (u_{ijk}^s)^h \left( \sum_m x_{ijkm}^s + \sum_{m \neq k} x_{ijmk}^s \right)
$$

$$
\left. + \sum_{(i,j)} (v_{ij}^s)^h x_{ijij}^s + \sum_{(i,j)} \sum_{k \neq j} (r_{ij}^s)^h x_{ijik}^s + \sum_{(i,j)} \sum_{k \neq i} (t_{ij}^s)^h x_{ijkj}^s \sum_{(i,j)} c_{ij} \theta_{ij}^s \right\} \quad (41)
$$

From (39), we have the following constraint associated to $u^h$, $v^h$, $r^h$ and $t^h$ for the state of the world $s$:

$$
\eta^s \geq \sum_k (\Omega_k^s)^h y_k - \sum_k \sum_{m \neq k} (v_{km}^s)^h w_{km}^s + \min_{(x,g,\theta) \in G} \left\{ \sum_k \tau_k(g_k^s) \right.
$$

$$
+ \sum_{(i,j)} \sum_k \sum_m c_{ijkm} x_{ijkm}^s - \sum_{(i,j)} \sum_k (u_{ijk}^s)^h \left( \sum_m x_{ijkm}^s + \sum_{m \neq k} x_{ijmk}^s \right)
$$

$$
\left. + \sum_{(i,j)} \left[ (v_{ij}^s)^h x_{ijij}^s + \sum_{k \neq j} (r_{ij}^s)^h x_{ijik}^s + \sum_{k \neq i} (t_{ij}^s)^h x_{ijkj}^s \right] + \sum_{(i,j)} c_{ij} \theta_{ij}^s \right\}
$$

Isolating the minimum given by (41), we get the following Benders cut based on the $y^h$ and $(u^s)^h$, $(v^s)^h$, $(r^s)^h$ and $(t^s)^h$ for the state of the world $s$:

$$
\eta^s \geq z(y^h)^s + \sum_k (\Omega_k^s)^h (y_k - y_k^h)
$$

Resulting then on the following relaxed master problem:

$$
\min_{\eta, \ y \in Y} \sum_k f_k \ y_k + \sum_s p^s \ \eta^s \tag{42}
$$

s. t:

$$
\eta^s - \sum_k (\Omega_k^s)^h (y_k - y_k^h) \geq z(y^h)^s \qquad \forall s \in S, \ \forall \ h = 1, \ldots, L \tag{43}
$$

$$
\eta^s \geq 0 \qquad \forall s \in S \tag{44}
$$

where $L$ is the maximum number of Benders cuts to be taken in order to attain the optimal solution. Note that, at each iteration $h$, new Benders cuts of the form (43) can be generated provided that the optimal values of $z(y^h)^s$ and the vector of multipliers $(u^s)^h$, $(v^s)^h$, $(r^s)^h$ and $(t^s)^h$ have been recovered. These values are attained by solving the subproblems, the subject of the next Section 3.2.

## 3.2  Subproblems

For a hub structure $y^h$ fixed by the RMP at iteration $h$, the subproblem $z(y^h)^s$, see equation (35), to be solved at the state of the world $s$ is given by:

$$\min \sum_k \tau_k(g_k^s) + \sum_{(i,j)} \sum_{(k,m)} c_{ijkm}\, x_{ijkm}^s + \sum_{(i,j)} c_{ij}\, \theta_{ij}^s \tag{45}$$

s. t.:

$$\sum_{(i,j)} \sum_m x_{ijmk}^s + \sum_{(i,j)} \sum_{m \neq k} x_{ijkm}^s = g_k^s \qquad \forall k \in H, \quad \forall s \in S \tag{46}$$

$$\theta_{ij}^s + \sum_{(k,m)} x_{ijkm}^s = w_{ij}^s \qquad \forall i, j \in N, \quad \forall s \in S \tag{47}$$

$$\sum_m x_{ijmk}^s + \sum_{m \neq k} x_{ijkm}^s \leq w_{ij}^s\, y_k^h \qquad \forall i, j \in N,\ k \in H, \quad \forall s \in S \tag{48}$$

$$x_{ijij}^s \geq w_{ij}^s\, (y_i^h + y_j^h - 1) \qquad \forall i, j \in N, \quad \forall s \in S \tag{49}$$

$$\sum_{k \neq j} x_{ijik}^s \geq w_{ij}^s\, (y_i^h - y_j^h) \qquad \forall i, j \in N, \quad \forall s \in S \tag{50}$$

$$\sum_{k \neq i} x_{ijkj}^s \geq w_{ij}^s\, (y_j^h - y_i^h) \qquad \forall i, j \in N, \quad \forall s \in S \tag{51}$$

$$x_{ijkm}^s \geq 0 \qquad \forall i, j \in N,\ k, m \in H, \quad \forall s \in S \tag{52}$$

$$g_k^s \geq 0 \qquad \forall k \in H, \quad \forall s \in S \tag{53}$$

Since the function $\tau_k(g_k^s)$ is increasing, the coupling constraints (46) can be dualized by associating the Lagrangian multipliers $\beta^s \geq 0$. Thus the resulting problem is separable into two subproblems: a linear transportation problem, $d_L(\beta^s)$, having only the $x_{ijkm}^s$ and the $\theta_{ij}^s$ variables; and another, a convex flow assignment transportation problem, $d_{NL}(\beta^s)$, having only the $g_k^s$ variables;

$$d_L(\beta^s) \;=\; \min \sum_{(i,j)} \sum_k (c_{ijkk} + \beta_k^s)\, x_{ijkk}^s + \sum_{(i,j)} \sum_{\substack{(k,m) \\ k \neq m}} (c_{ijkm} + \beta_k^s + \beta_m^s)\, x_{ijkm}^s$$

subject to constraints (47)-(52).

$$d_{NL}(\beta^s) = \min_{g \geq 0} \sum_k (\tau_k(g_k^s) - \beta_k^s\, g_k^s)$$

With the correspondent dual variables $\beta^s \in \mathbb{R}$, the optimal value $z(y^h)^s$ can then be computed as:

$$z(y^h)^s = \max_{\beta \geq 0} \left\{ d_L(\beta^s) + d_{NL}(\beta^s) \right\}$$

Recalling that $d_{NL}(\beta^s)$ has a convex differentiable objective function and linear constraints, its Karush-Kuhn-Tucker conditions are necessary and sufficient for optimality. So, for each $k \in H$ and a fixed hub structure $y_k^h$, the optimal solution of $(\beta_k^s)^h$ should minimize its correspondent parcel of the $d_{NL}(\beta)$ objective function, what implies, for iteration $h$:

$$(\beta_k^s)^h = \tau_k'(g_k^s)^h \qquad\qquad \forall\, k \in H \qquad\qquad (54)$$

The optimal values of $(g_k^s)^h$ can be obtained by employing an algorithm based on the Flow Deviation algorithm (FD) [34], described in the Section 3.2.1.

### 3.2.1  The Non-linear Transportation Problem

The subproblem $z(y^h)^s$, given by (45)-(53), is a non-linear transportation problem that can be solved by means of a feasible directions search method such as [29] or the proximal decomposition algorithm of Mahey *et al.* [52]. In the present paper, a specialized Flow Deviation [34] algorithm has been deployed.

The Flow Deviation algorithm is an optimization method designed to find the minimum of a convex function subject to linear constraints. The algorithm has been shown to be efficient since the early works of Fratta *et al.* [30] and Cantor & Gerla [19]. In order to apply the method to the subproblem (45)-(53) and thus compute the optimal values of $g_k^s$, a more adequate formulation has to be applied. An arc-node formulation with side constraints has been used. The role of the side constraints is to restrain the set of feasible paths from an origin to a destination.

The specialized Flow Deviation algorithm takes advantage of the fact that each commodity may only follow a reduced set of routes. Because of this, when evaluating the shortest path for a given $i - j$ pair, it is not necessary to deploy an algorithm like Dijkstra, but only enumerate this reduced set. Furthermore, as it is not necessary to assess the flow of each commodity in each arc during the running of the algorithm, but just the total flow in each arc, specially for each hub, it is possible to save computer memory and processing time.

Once computed the values of $(g^s)^h$ and hence the values of $(\beta^s)^h$ (see (54)), the subproblem $d_L(\beta^s)$ can now be solved, subject of the next Section 3.2.2.

### 3.2.2  The Linear Transportation Problem

After fixing the optimal values of $(\beta^s)^h$ and $(g^s)^h$, the subproblem $d_L(\beta^s)$ can now be solved and thus the values of $(u^s)^h$, $(v^s)^h$, $(r^s)^h$ and $(t^s)^h$ are now available. Instead of solving the $d_L(\beta^s)$ by means of simplex or interior point methods, it can be solved by specializing an algorithm which relies on the complementary slackness condition and on the linear programming duality theory.

As proved in Camargo *et al.* [11] the primal problem $d_L(\beta^s)$ is always feasible and bounded for any fixed feasible $y_k$ satisfying constraints (47)-(52) as well as its dual. The linear programming dual is also decomposable for each $i - j$ pair and state of the world $s$:

$$\max w_{ij}^s \ \rho_{ij}^s - \sum_k w_{ij}^s \ y_k^h \ u_{ijk}^s + w_{ij}^s \ (y_i^h + y_j^h - 1) \ v_{ij}^s$$

$$+ w_{ij}^s \ (y_i^h - y_j^h) \ r_{ij}^s + w_{ij}^s (y_j^h - y_i^h) \ t_{ij}^s \tag{55}$$

s. t.:

$$\rho_{ij}^s - u_{ijk}^s \le c_{ijkk} + \beta_k^s \qquad\qquad \forall \ k \neq i, k \neq j \tag{56}$$

$$\rho_{ij}^s - u_{ijk}^s - u_{ijm}^s \le c_{ijkm} + \beta_k^s + \beta_m^s \qquad \forall \ k \neq i, k \neq m, m \neq j \tag{57}$$

$$\rho_{ij}^s - u_{iji}^s + r_{ij}^s \le c_{ijii} + \beta_i^s \tag{58}$$

$$\rho_{ij}^s - u_{ijj}^s + t_{ij}^s \le c_{ijjj} + \beta_j^s \tag{59}$$

$$\rho_{ij}^s - u_{iji}^s - u_{ijj}^s + v_{ij}^s \le c_{ijij} + \beta_i^s + \beta_j^s \tag{60}$$

$$\rho_{ij}^s - u_{iji}^s - u_{ijk}^s + r_{ij}^s \le c_{ijik} + \beta_i^s + \beta_k^s \qquad \forall \ k \neq i, k \neq j \tag{61}$$

$$\rho_{ij}^s - u_{ijj}^s - u_{ijk}^s + t_{ij}^s \le c_{ijkj} + \beta_k^s + \beta_j^s \qquad \forall \ k \neq i, k \neq j \tag{62}$$

$$\rho_{ij}^s \le c_{ij} \tag{63}$$

$$\rho_{ij}^s \in \mathbb{R} \tag{64}$$

$$u_{ijk}^s \ge 0 \qquad\qquad\qquad \forall \ k \tag{65}$$

$$v_{ij}^s \ge 0 \tag{66}$$

$$r_{ij}^s \ge 0 \tag{67}$$

$$t_{ij}^s \ge 0 \tag{68}$$

**Proposition 3.1.** *The optimal solution of the dual problem* (55)-(67) *can be found by inspection.*

**Proof.** The solution of (55)-(68) is started by setting $\rho_{ij}^s$ as the shortest path for a given $i - j$ pair considering the installed structure $y^h$ at the state of the world $s$ (see Proposition 2 of [11]).

$$\rho_{ij}^s = \min \left\{ c_{ij}, \min_{k \neq m} \left\{ c_{ijkm} + \beta_k^s + \beta_m^s \right\}, \min_k \left\{ c_{ijkk} + \beta_k^s \right\} \right\}$$

Starting values for $u_{ijk}^s$ are already available, when $k \neq i$ and $k \neq j$, from the algorithm proposed in [11] by adding the respective values of $\beta^s$ to the linear transportation costs. In order to find the optimal values of the other dual variables, please refer to the four steps procedure described in Proposition 1 of [10], repeating it to each state of the world $s$. □

## 4    COMPUTATIONAL EXPERIMENTS

This section is composed by three distinct subsections, each one aiming to explore a different feature of the proposed model and deployed algorithms.  In the first subsection, the superior quality of the stochastic design is illustrated by comparing the outcomes of deterministic and stochastic modeling for a toy instance. The second subsection investigates the scalability of the deployed algorithms, testing them for benchmark data sets. In the third section, the case study is described in detail, and the obtained results are properly discussed.

### 4.1    Deterministic versus stochastic hub-and-spoke network design

The goal here is to show the superior quality of stochastic hub location if compared to deterministic hub network design.  The adopted parameters for the hub congestion functions were $e = 0.0001$ and $b = 2$. These experiences have been made using XPRESS-MP QMIP solver. As working problem instance, an adapted version of *CAB10* has been produced. The generated demand matrix is showed below, having only three distinct sceneries:

Scenery 1:
$$
\begin{bmatrix}
0 & 1617.25 & 1907.25 & 5009 & 1172.5 & 1548.5 & 2922 & 560.75 & 2214.25 & 1812 \\
1617.25 & 0 & 3249.75 & 3423 & 830.5 & 1394 & 969.5 & 800.5 & 1674.75 & 1049.5 \\
1907.25 & 3249.75 & 0 & 8783.75 & 1489 & 3530.25 & 1487.75 & 1442 & 4144.5 & 1060.5 \\
5009 & 3423 & 8783.75 & 0 & 4773.5 & 8779.75 & 5355.75 & 6835.5 & 12835.25 & 3956.5 \\
1172.5 & 830.5 & 1489 & 4773.5 & 0 & 1821 & 775.5 & 390.5 & 1795 & 479.25 \\
1548.5 & 1394 & 3530.25 & 8779.75 & 1821 & 0 & 1255.75 & 878 & 2604.75 & 885.75 \\
2922 & 969.5 & 1487.75 & 5355.75 & 775.5 & 1255.75 & 0 & 2889.25 & 1619.75 & 8565.25 \\
560.75 & 800.5 & 1442 & 6835.5 & 390.5 & 878 & 2889.25 & 0 & 1403.75 & 1773.75 \\
2214.25 & 1674.75 & 4144.5 & 12835.25 & 1795 & 2604.75 & 1619.75 & 1403.75 & 0 & 1112 \\
1812 & 1049.5 & 1060.5 & 3956.5 & 479.25 & 885.75 & 8565.25 & 1773.75 & 1112 & 0
\end{bmatrix}
$$

Scenery 2:
$$
\begin{bmatrix}
0 & 6469 & 7629 & 20036 & 4690 & 6194 & 11688 & 2243 & 8857 & 7248 \\
6469 & 0 & 12999 & 13692 & 3322 & 5576 & 3878 & 3202 & 6699 & 4198 \\
7629 & 12999 & 0 & 35135 & 5956 & 14121 & 5951 & 5768 & 16578 & 4242 \\
20036 & 13692 & 35135 & 0 & 19094 & 35119 & 21423 & 27342 & 51341 & 15826 \\
4690 & 3322 & 5956 & 19094 & 0 & 7284 & 3102 & 1562 & 7180 & 1917 \\
6194 & 5576 & 14121 & 35119 & 7284 & 0 & 5023 & 3512 & 10419 & 3543 \\
11688 & 3878 & 5951 & 21423 & 3102 & 5023 & 0 & 11557 & 6479 & 34261 \\
2243 & 3202 & 5768 & 27342 & 1562 & 3512 & 11557 & 0 & 5615 & 7095 \\
8857 & 6699 & 16578 & 51341 & 7180 & 10419 & 6479 & 5615 & 0 & 4448 \\
7248 & 4198 & 4242 & 15826 & 1917 & 3543 & 34261 & 7095 & 4448 & 0
\end{bmatrix}
$$

Scenery 3:
$$
\begin{bmatrix}
0 & 11320.75 & 13350.75 & 35063 & 8207.5 & 10839.5 & 20454 & 3925.25 & 15499.75 & 12684 \\
11320.75 & 0 & 22748.25 & 23961 & 5813.5 & 9758 & 6786.5 & 5603.5 & 11723.25 & 7346.5 \\
13350.75 & 22748.25 & 0 & 61486.25 & 10423 & 24711.75 & 10414.25 & 10094 & 29011.5 & 7423.5 \\
35063 & 23961 & 61486.25 & 0 & 33414.5 & 61458.25 & 37490.25 & 47848.5 & 89846.75 & 27695.5 \\
8207.5 & 5813.5 & 10423 & 33414.5 & 0 & 12747 & 5428.5 & 2733.5 & 12565 & 3354.75 \\
10839.5 & 9758 & 24711.75 & 61458.25 & 12747 & 0 & 8790.25 & 6146 & 18233.25 & 6200.25 \\
20454 & 6786.5 & 10414.25 & 37490.25 & 5428.5 & 8790.25 & 0 & 20224.75 & 11338.25 & 59956.75 \\
3925.25 & 5603.5 & 10094 & 47848.5 & 2733.5 & 6146 & 20224.75 & 0 & 9826.25 & 12416.25 \\
15499.75 & 11723.25 & 29011.5 & 89846.75 & 12565 & 18233.25 & 11338.25 & 9826.25 & 0 & 7784 \\
12684 & 7346.5 & 7423.5 & 27695.5 & 3354.75 & 6200.25 & 59956.75 & 12416.25 & 7784 & 0
\end{bmatrix}
$$

The adopted probability vector was $p = [0.333\,0.333\,0.333]^T$. In order to establish a fair comparison, a deterministic equivalent demand matrix has been produced by taking $w_{ij} = \sum_{s \in S} p^s w_{ij}^s$.

This deterministic equivalent instance was solved by formulation (12)-(22). The resulting optimal hub structures ($y$ vectors) are depicted ahead:

Deterministic: $y^* = [0\ 1\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 1]^T$

Stochastic:     $y^* = [0\ 0\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 1]^T$

In spite of the detailed report of a single experience – for the sake of simplicity – several other tests have been carried out. They were done by tuning the probability vector and the hub installation costs. In general, the stochastic hub network design is a bit more conservative, saving fixed costs, when compared to the deterministic solution. It is also true that the same hub structure has been observed many times, meaning that a deterministic solution is almost always optimal or near-optimal. However, considering the costs involved in such networks and the long term impact on its performance, the risk of designing a non-optimal network by disregarding the uncertainty is definitely not affordable.

## 4.2    The GBD Algorithm Scalability

In order to test the performance of the deployed generalized Benders decomposition algorithm, the CAB data set of the US Civil Aeronautics Board and the AP data set introduced by Ernst and Krishnamoorthy [25, 27] have been used. The former includes problems with 10, 15, 20 and 25 nodes, while from the later problems with 10 to 200 nodes can be obtained.

The hub installation costs have been generated for all instances using a gaussian distribution with average equals to $f_o$ and variance set to 40% to assess how the different fixed costs vary in real problems. Where $f_o$ represents the scaled difference in objective value between a scenario in which there is a virtual hub located in the center of mass and a scenario in which all nodes are hubs, as introduced by Ebery *et al.* [23]. For alternative works considering these test beds see [60, 24, 23, 7, 11, 10]. Furthermore, as done by Ebery *et al.* [23], the nodes with higher flows have been selected to have the higher fixed costs, hardening, in general, the selection of potential hubs.

The generalized Benders decomposition algorithm was implemented with the aid of the general purpose mixed integer programming package XPRESS-MP from Dash optimization. The package routines have been used on the solution of the mixed integer master program. The experiments have been carried out on a regular desktop micro-computer equipped with a Intel-Pentium IV 3.0 GHz processor and 2Gb of RAM memory. The value of $\varepsilon$ has been set to 0.0001%.

The attained results are displayed in Table 1. In this table, the columns *Variables* and *Iterations* account for the total number of decision variables of the solved instance (integer, nonlinear and continuous) and the total number of global iterations of GBD algorithm. The column *SP time fraction (%)* accounts the ratio of time spent to solve all the subproblems and the total solution time. The optimization of hub location problems under uncertainty is expensive, specially concerning the subproblem solution where a large scale non-linear program may be solved for each

scenery of each iteration. In spite of this fact, some large problems could be solved in reasonable time, for a long term strategic decision problem.

**Table 1** – GBD algorithm scalability test.

| Instance | Variables | Iterations | Time [s] | SP time fraction (%) |
|----------|-----------|------------|----------|----------------------|
| CAB10 | 90020 | 167 | 1960.33 | 42.78 |
| CAB15 | 472530 | 342 | 2435.34 | 52.69 |
| CAB20 | 1520040 | 387 | 3658.35 | 45.13 |
| CAB25 | 3750050 | 170 | 9563.67 | 72.34 |
| AP10 | 90020 | 165 | 1301.45 | 61.84 |
| AP20 | 1520040 | 383 | 2637.35 | 73.11 |
| AP30 | 7830060 | 246 | 9345.78 | 66.92 |
| AP40 | 24960080 | 473 | 27840.45 | 75.83 |
| AP50 | 61250100 | 692 | 73256.34 | 81.24 |

The large number of iterations is explained by the aggressive congestion cost functions that were adopted. For all these tests the parameters $e$ and $b$ were taken as 0.5 and 2 respectively, implying a congestion cost ratio [10] scaling from 20% to 40%. Since GBD is an outer linearization strategy, it is particularly sensitive as the nonlinearities become dominant in a given MINLP.

### 4.3    Designing the Brazilian Air Transportation Network

Brazil is the fifth largest country of the world, and its area is 8.5 millions of squared kilometers, occupying about 47% of the area of South American continent. The population of the country is 184 million of inhabitants. Due to its great territorial extension and to the size its population, the country needs to have a great number of airports to allow the transportation of passengers and cargo. Nowadays, the country has 1759 private airports and 739 public airports, totaling 2498 airports.

Brazil is the second country of the world that more it possesses airports, behind just of the United States of America. The boards of INFRAERO and ANAC, two government agencies, are in charge of the management of 67 of those airports, that concentrate 97% of the commercial air transportation of the country. Those airports move about 2 million landings and takeoffs, and almost 50 million passengers and 1.2 million of tons per year (basis: year 2007).

In the last few years, Brazilian GDP is growing more than the average global GDP increase, around 5% year. This means an expected high demand increase for air transportation, since each 1% of increase on GDP means 1.15% of demand increase in the air transportation matrix. The passenger traffic in Brazil is strongly concentrated in few airports located in the cities of São Paulo, Rio de Janeiro and Brasília. The existent airports in those cities are the current passenger hubs of the Brazilian network.

Since the governmental investment profile has not matched the increasing traffic of the last few years, these hubs are severely overloaded, and the whole network is experiencing huge delays

and several flight cancelations. On the other hand, several other airports appear to have idle capacity, such as the airports in the cities of Belo Horizonte, Salvador and Curitiba.

In order to use the deployed algorithms, several modeling assumptions have been made. Because of the large computational costs involved, only the 30 larger Brazilian airports were selected to compose the experience. Since INFRAERO and ANAC (Brazilian Civil Aeronautics Board) have historical series of the observed demands for each origin-destination pair in the network, as well as the distance matrix, the data mining phase was not a trouble. Another assumption was the use of the most economic available aircrafts, since the airlines always seek for the lowest operational costs.

The hub installation costs were estimated on the minimal infra-structure investment necessary to promote a node to the hub status. In some cases, this investment means only to improve air traffic control capabilities or to enlarge the passengers terminal or the cargo manipulation area. Sometimes this may imply to enlarge the runway length or to duplicate the existent runway or even to provide a better transportation system to reach the airport. These parameters were established according to the information provided by ANAC and INFRAERO.

**Table 2** – Hub nominal capacity array for the Brazilian case study (passengers/year).

| City | Airport | Capacity | | City | Airport | Capacity |
|------|---------|----------|---|------|---------|----------|
| São Paulo | SBSP | 15000000 | | Belo Horizonte | SBBH | 1500000 |
| Brasília | SBBR | 10000000 | | Goiânia | SBGO | 1000000 |
| Guarulhos | SBGR | 17000000 | | Natal | SBNT | 1000000 |
| Rio de Janeiro | SBGL | 15000000 | | Cuiabá | SBCY | 1000000 |
| Salvador | SBSV | 6000000 | | Campinas | SBKP | 1000000 |
| Rio de Janeiro | SBRJ | 1800000 | | Foz do Iguaçu | SBFI | 1000000 |
| Curitiba | SBCT | 3500000 | | Maceió | SBMO | 1000000 |
| Porto Alegre | SBPA | 4000000 | | Porto Seguro | SBPS | 1000000 |
| Recife | SBRF | 5000000 | | Campo Grande | SBCG | 1000000 |
| Confins | SBCF | 10000000 | | São Luís | SBSL | 1000000 |
| Fortaleza | SBFZ | 1000000 | | Uberlândia | SBUL | 1000000 |
| Belém | SBBE | 2700000 | | Londrina | SBLO | 1000000 |
| Florianópolis | SBFL | 1000000 | | Navegantes | SBNF | 1000000 |
| Vitória | SBVT | 1000000 | | Aracajú | SBAR | 1000000 |
| Manaus | SBMN | 1000000 | | Ribeirão Preto | SBRP | 1000000 |

In addition, a slightly different congestion function was used to account the congestion costs. For this set of experiences a function were the congestion effects are perceived only if a flow above 75% of the hub nominal capacity is reached was adopted:

$$\tau_k(g_k) = \max\left\{0, e(g_k - a_k)^b\right\}$$

This assumption is consistent with the most works on the literature about congestion cost functions. The capacity array for the Brazilian airports has been provided by ANAC and INFRAERO and is given in Table 2. All the capacities are expressed in passengers per year.

A standard discount factor $\alpha$ of 0.5 for the inter-hub connections was adopted, since almost all the airlines establish a minimum of 50% of tickets sold by flight leg to be sustainable. The values of the parameters $e$ and $b$ were taken as 1.0 and 2 producing very aggressive congestion effects (see [10]). To mimic the trend of growth of the traffic matrix, a triangular probability distribution was adopted, from 2011 to 2023. The central trend was taken forecasting a mean growth of 5.75% per year, corresponding to 5% of mean growth in GDP. The optimistic and pessimistic forecasts – the extremes of the triangular distribution – were based on the historical data from ANAC. Discrete distributions with ten sceneries were taken as approximations of the adopted triangular distributions. This small number of sceneries is necessary to ensure a reasonable computational cost, dealing with a little loss of accuracy.

The obtained optimal hub structures – y vectors – and the expected total traffic for each hub are shown in Table 3. All the proposed instances have run in computing times under 15 hours of CPU. We must remark that only the flows on the hubs were displayed in Table 3, meaning that if a node is not a hub his traffic is not shown in Table 3. The resulting network designs are plotted graphically in Figures 1, 2, 3 and 4. In these figures, the relative thickness of a given link is a measure of its traffic intensity.

It is important to recall that the capacity array informs just the nominal capacities, since the true capacity of an airport is not known with accuracy. For instance, in 2007 SBSP has registered more than 18 millions of passengers, but its nominal capacity is only 15 millions. So, it is not unexpected to observe flows slightly above that nominal value, if it is preferable to tolerate some congestion to avoid the installation of new hubs. Other effect that is remarkable is the transfer of part of the increasing demand to the direct service links, as a way to save fixed or conges-tion costs. This flow transfer explains why the total hub traffic does not grows in the exact expected rate.

It is possible to see that the infrastructure investment may be directed to the airports of Belo Horizonte (SBCF), Recife (SBRF), Curitiba (SBCT) and Salvador (SBSV). These potential new hubs have idle capacity and may be better exploited in order to reduce the operational costs and flight delays on the Brazilian air transportation network. It is possible to see that it is also necessary to develop a strategy of expansion of all the network hubs, under penalty of a new traffic saturation in the near future. This expansion is particularly easy for SBCF and SBSV, complaining with data of INFRAERO and ANAC.

It was also observed an increasing transfer of traffic to the direct service links in order to avoid the hub congestion costs in 2019 and 2023, what recalls the urgency of a system expansion strategy. Another interesting effect is that the very huge traffic volume of the pair SBSP-SBRJ was not enough to make SBRJ a hub, since this particular airport has no capacity to deal with third party traffic.

Another specially problematic airport is SBSP, having a preferential location and huge demands, but also having an insufficient infra-structure and being hard to expand, since it is inside the city of São Paulo. As the demand is increased from 2007 to 2023, no hub is removed but several hubs

**Table 3** – Computational Results for the Brazilian case.

| Year | 2011 | | 2015 | | 2019 | | 2023 | |
|---|---|---|---|---|---|---|---|---|
| Airport | y | Total Traffic | y | Total Traffic | y | Total Traffic | y | Total Traffic |
| SBAR | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| SBBE | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| SBBH | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| SBBR | 1 | 0.80E+07 | 1 | 0.80E+07 | 1 | 0.80E+07 | 1 | 0.80E+07 |
| SBCF | 1 | 1.07E+07 | 1 | 0.36E+07 | 1 | 0.75E+07 | 1 | 1.28E+07 |
| SBCG | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| SBCT | 1 | 0.28E+07 | 1 | 0.20e+07 | 1 | 0.20e+07 | 1 | 0.90e+07 |
| SBCY | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| SBFI | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| SBFL | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| SBFZ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| SBGL | 1 | 1.20E+07 | 1 | 1.09E+07 | 1 | 1.20E+07 | 1 | 1.20E+07 |
| SBGO | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| SBGR | 0 | 0 | 1 | 1.00E+07 | 1 | 1.16E+07 | 1 | 1.36E+07 |
| SBKP | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| SBLO | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| SBMN | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| SBMO | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| SBNF | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| SBNT | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| SBPA | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| SBPS | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| SBRF | 1 | 0.40E+07 | 1 | 0.40E+07 | 1 | 0.40E+07 | 1 | 0.40E+07 |
| SBRJ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| SBRP | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| SBSL | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| SBSP | 1 | 1.20E+07 | 1 | 1.20E+07 | 1 | 1.20E+07 | 1 | 1.20E+07 |
| SBSV | 0 | 0 | 1 | 0.57E+07 | 1 | 0.96E+07 | 1 | 0.96E+07 |
| SBUL | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| SBVT | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

are added to the network. This result is quite interesting and could not be anticipated *a priori*. Additional research may investigate how these network designs are affected by allowing capacity expansions. However, capacity expansion costs are hard to estimate for several of the airports under study, except for SBCF and SBSV (where these costs are assumed small).
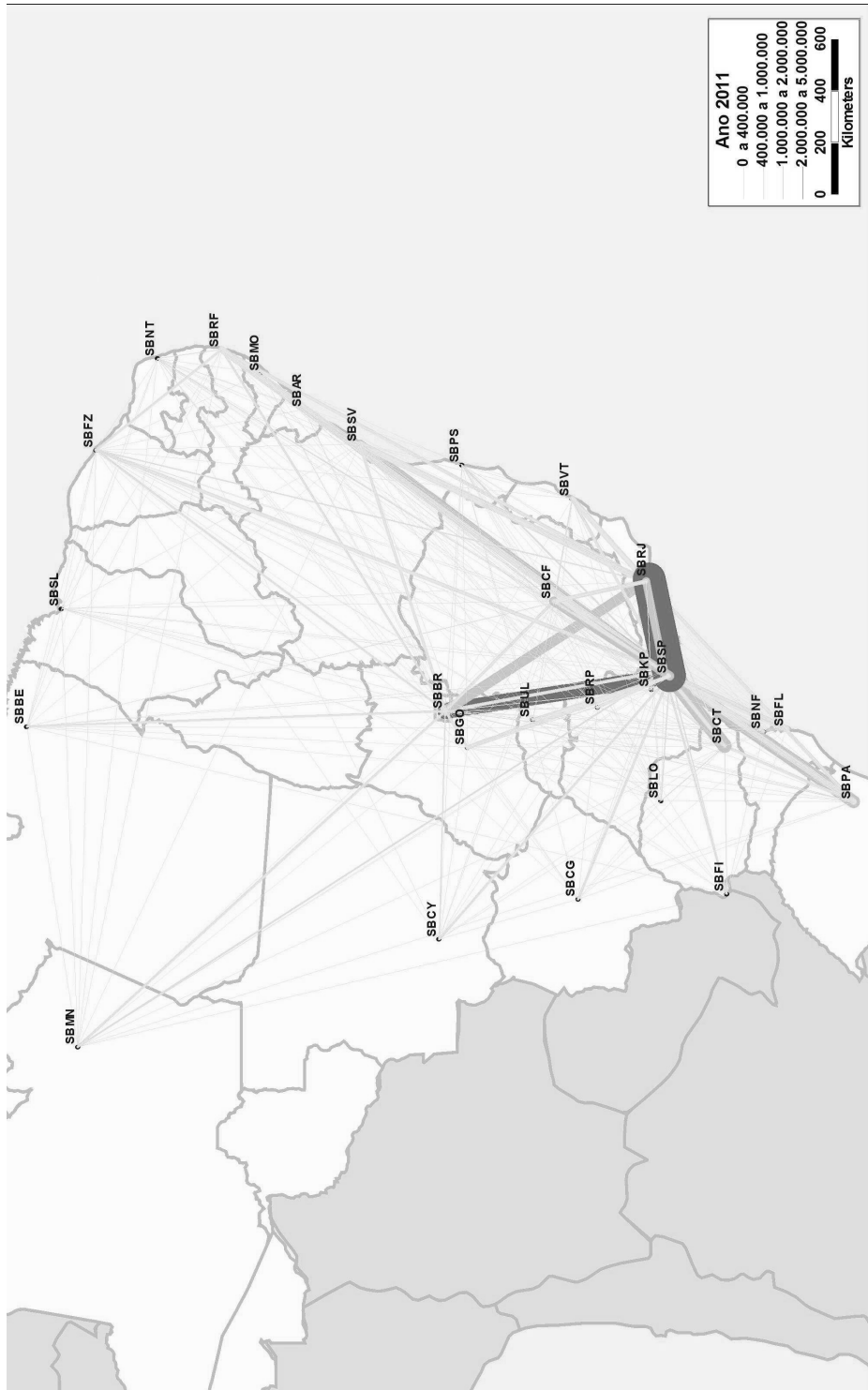
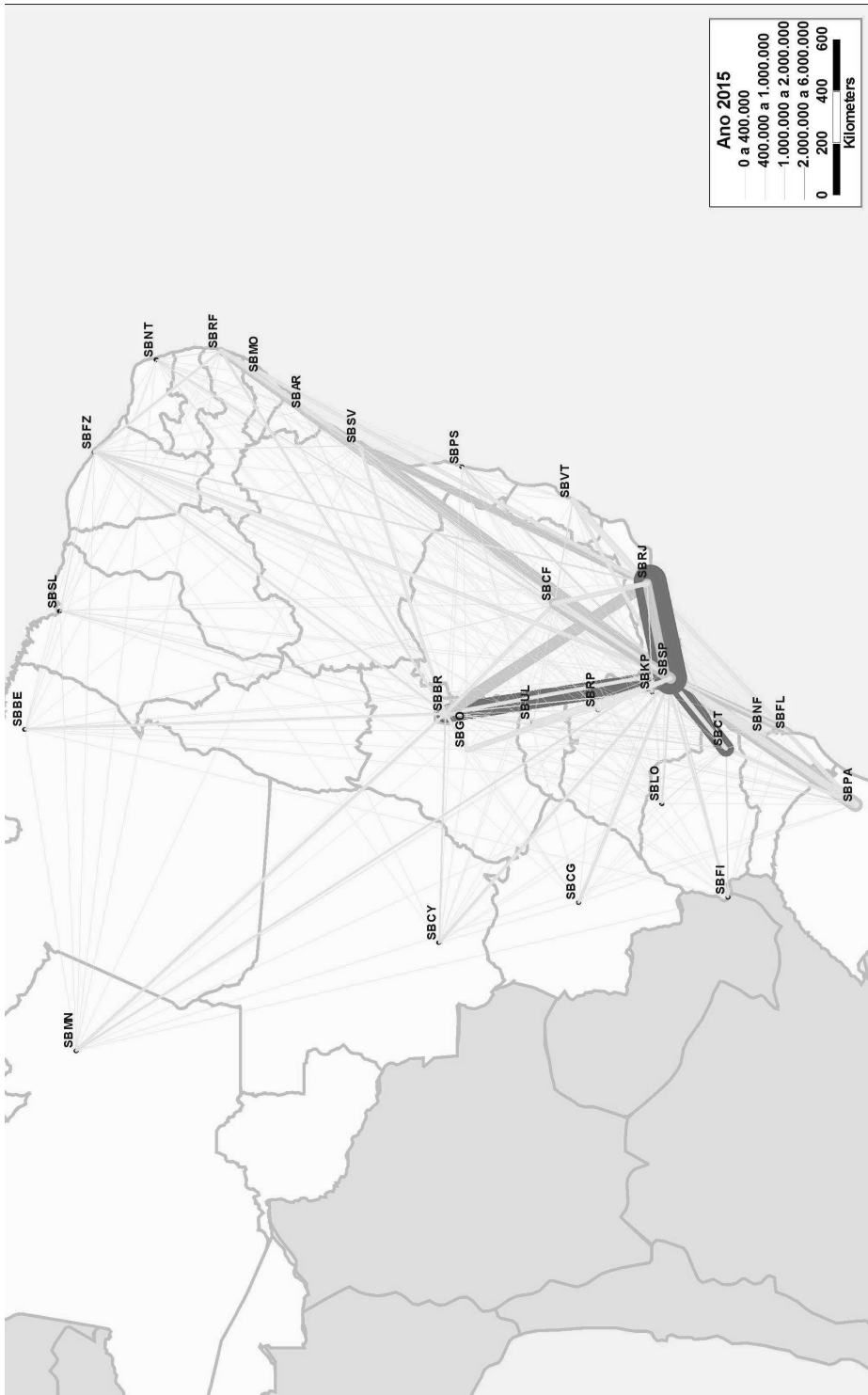**Figure 1** – Optimal Brazilian network for 2011.

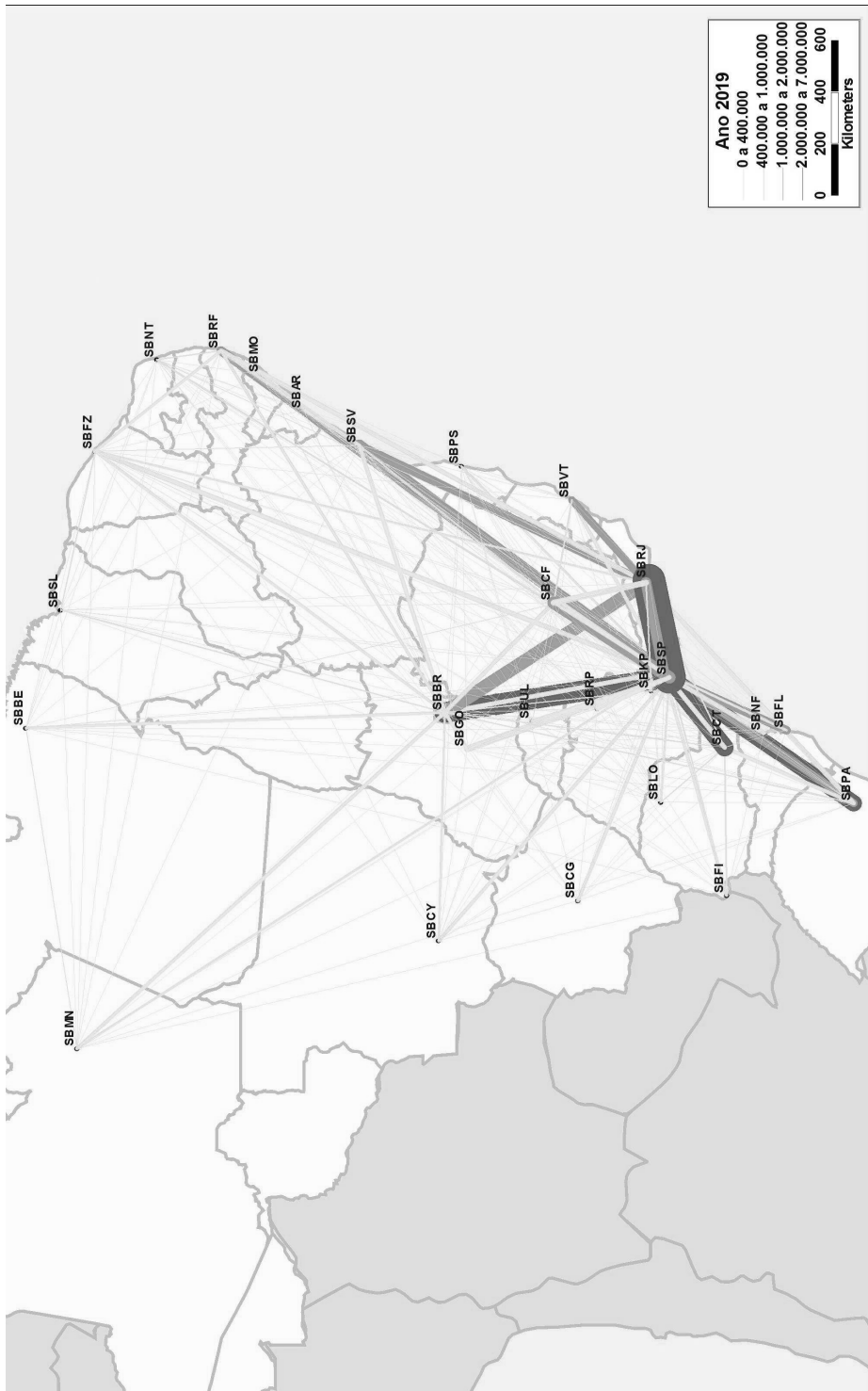**Figure 2** – Optimal Brazilian network for 2015.

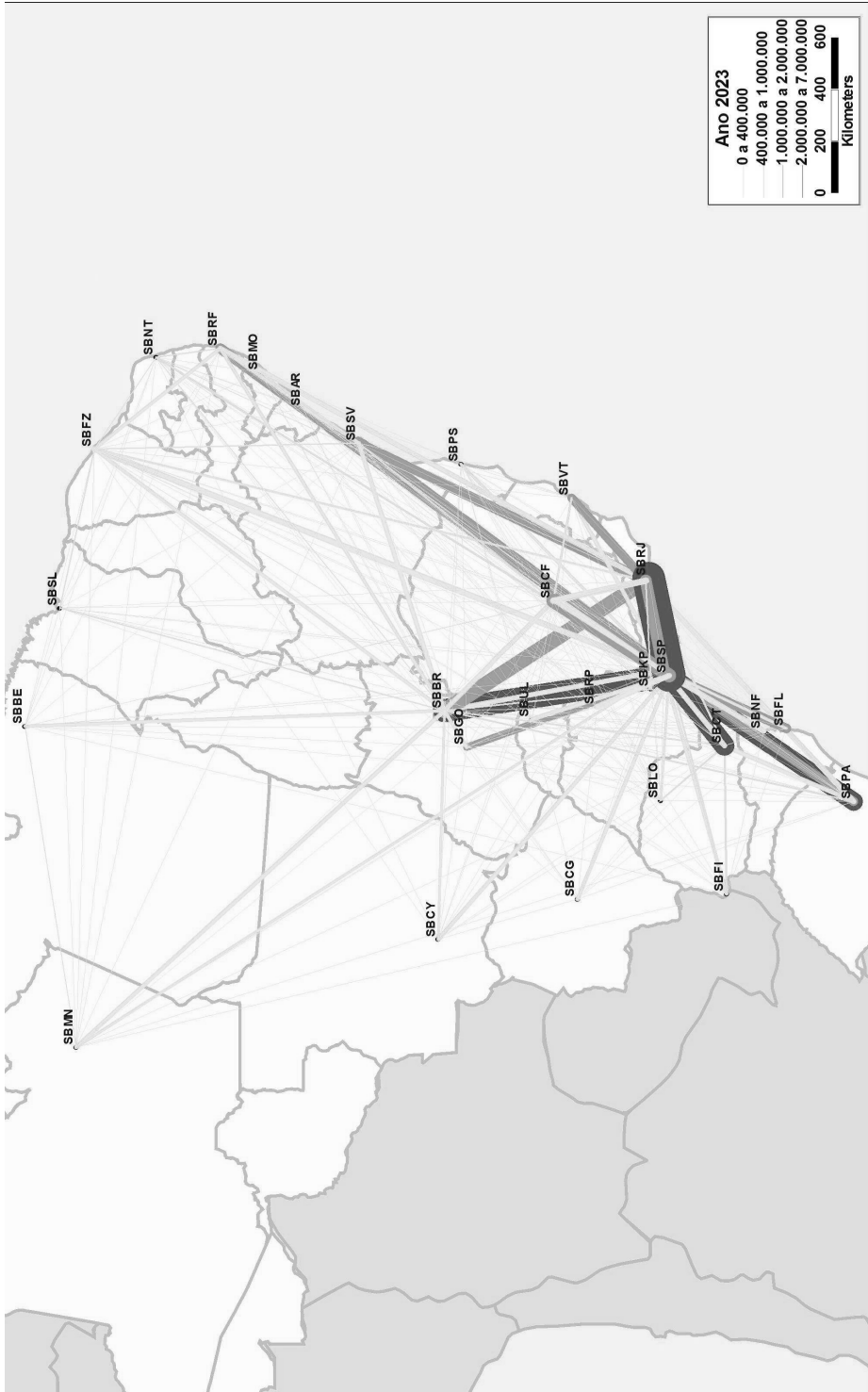**Figure 3** – Optimal Brazilian network for 2019.

**Figure 4** – Optimal Brazilian network for 2023.

## 5   CONCLUDING REMARKS

In this article a model for accounting demand uncertainty and congestion costs for designing multiple allocation hub and spoke networks has been proposed. In addition, the proposed model allows the establishment of direct service links if they are economically interesting.

Generalized Benders decomposition was the selected technique to tackle the proposed problem. This technique has shown enough computing power to overcome large instances in reasonable time, although the computing times observed were expressively superior to those in [10], due to the most stressing congestion cost functions chosen.

The deployed algorithms were also used to redesign the Brazilian air transportation network, and the attained designs were properly discussed. The resulting network designs indicate that is necessary to redistribute flow on several hubs of the network in order to reduce the flight delays and other congestion effects. It has also pointed out that a system expansion is necessary in the near future under penalty of a new saturation of the installed infra-structure.

Future work must be devoted on evaluating how the capacity expansion of the major hubs may impact the proposed optimal hub-and-spoke networks.

## ACKNOWLEDGEMENTS

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