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GENERALIZATING PATH AND FAN GRAPHS: SUBCOLORING AND TOUGHNESS

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ABSTRACT. Two graph classes are presented; the first one (k-ribbon) generalizes the path graph and the second one (k-fan) generalizes the fan graph. We prove that they are subclasses of chordal graphs and so they share the same structural properties of this class. The solution of two problems are presented: the determination of the subchromatic number and the determination of the toughness. It is shown that the elements of the new classes establish bounds for the toughness of k-path graphs.

Keywords: k-path graphs, k-ribbon graphs, k-fan graphs, subcoloring, toughness.

1 INTRODUCTION

When researching new algorithmic problems in graphs it is usual to start examining well-known classes for which it will be easy to explore and rehearse a variety of approaches. Among these classes, the complete graphs, the path graphs, the wheels, the bipartite graphs are frequently assessed. A further approach is to define new subclasses for which the problems can be solved in polynomial time.

Two classes are defined here with this purpose: the *k-ribbon graphs* and the *k-fan graphs* generalize the path graphs and the fan graphs, respectively. We prove that they are subclasses of chordal graphs and so they share the same structural properties of this class. Efficient solutions of two problems are presented: the determination of the subchromatic number and the determination of the toughness. The study of the new subclasses lead to the establishment of bounds for the toughness of k-path graphs.

2 k-RIBBONS AND k-FANS

In this section we present the definition of two classes of graphs. The first one, the k-ribbon graph, is based on the power of a graph. The second one, the k-fan graph, is the result of the join of two basic graphs.

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Given a graph $G = (V, E)$ and a positive integer d, the *d-th power* of G is the graph $G^d = (V, E')$ in which two vertices are adjacent when they have distance at most d in G (Diestel [5]). Clearly $G = G^1 \subseteq G^2 \subseteq \ldots$.

Definition 2.1. *A graph G is a k-ribbon graph when it is the k-th power of the path graph P_n, $1 \leq k < n$.*

The *join of two graphs* $G = (V, E)$ and $G' = (V', E')$ is the graph $G + G' = (V \cup V', E \cup E' \cup \{uv \mid u \in V \text{ and } v \in V'\})$ (Gross & Yellen [9]).

Definition 2.2. *Consider the complete graph K_ℓ, $1 \leq \ell \leq k - 1$ and $k \geq 2$, and $R_{k-\ell}$ the $(k - \ell)$-ribbon graph. The graph $G = K_\ell + R_{k-\ell}$, $n \geq k + 1$, is a k-fan graph.*

Theorems 3 and 5 show that a class that generalizes the k-ribbons and the k-fans is the k-path graphs, a subclass of k-trees which in turn constitutes a subclass of chordal graphs. Several of their properties will be useful in the proofs of the paper.

Definition 2.3 (Pereira *et al.* **[14]).** *A k-path graph, $k > 0$, can be inductively defined as follows:*

- *Every complete graph with $k + 1$ vertices is a k-path graph.*
- *If $G = (V, E)$ is a k-path graph, $v \notin V$ and $Q \subseteq V$ is a k-clique of G containing at least one simplicial vertex, then $G' = (V \cup \{v\}, E \cup \{vw \mid w \in Q\})$ is also a k-path graph.*
- *Nothing else is a k-path graph.*

Theorem 1 (Markenzon *et al.* **[11]).** *Let $G = (V, E)$ be a k-tree with $n > k + 1$ vertices. G is a k-path graph if and only if G has exactly two simplicial vertices.*

A *clique-tree* of G is a tree T whose vertices are the maximal cliques of G such that for every two maximal cliques Q and Q', each clique in the path from Q to Q' in T contains $Q \cap Q'$. Lemma 2 shows an important property concerning the clique-tree of a k-path graph which enables us to conclude that every k-path graph is an interval graph.

Lemma 2 (Pereira *et al.* **[14]).** *Every k-path graph admits a unique clique-tree and this clique-tree is a path.*

Let $G = (V, E)$ be a k-path graph and $\mathbf{Q}$ be the set of maximal cliques of G. Some properties of k-path graphs are:

1. $|Q| = k + 1$, $Q \in \mathbf{Q}$;
2. $|\mathbf{Q}| = n - k$;
3. If $Q, Q' \in \mathbf{Q}$ are adjacent vertices in a clique-tree of G then $|Q \cap Q'| = k$.

Theorem 3 (Markenzon *et al.* **[11]).** *Every k-ribbon,* $1 \leq k < n$*, is a k-path graph.*

Note that 1-ribbons are actually path graphs and there is a unique k-ribbon of order n. For a given n, it is possible to present $n - 1$ k-ribbons with $1 \leq k \leq n - 1$.

A k-ribbon graph can be inductively defined by simply replacing the second item of Definition 2.3 by the following item.

- *If* $G = (V, E)$ *is a k-ribbon,* $v \notin V$ *and* $Q \subseteq V$ *is a k-clique of G composed by the k most recently included vertices, then* $G' = (V \cup \{v\}, E \cup \{vw \mid w \in Q\})$ *is also a k-ribbon.*

There is an underlying construction in this definition which establishes a labeling for a k-ribbon. The vertices of the first $(k + 1)$-clique are labeled $v_1, \ldots, v_{k+1}$, where v_1 is a simplicial vertex of G. Following the definition, each new vertex is sequentially labeled. It is easy to see that v_n is the other simplicial vertex of G. Figure 1 shows a 5-ribbon and its labeling.

Since k-path graphs are interval graphs, this labeling provides a immediate proof that k-ribbons are also proper interval graphs (interval graphs that does not contain $K_{1,3}$).

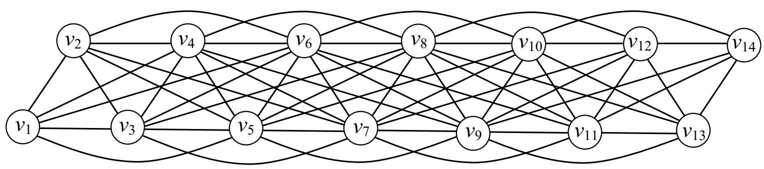

Figure 1 – 5-ribbon of order 14.

Observing this construction, we can determine the degree sequence of a k-ribbon graph, $k \geq 2$.

- if $n = k + 2$, $\langle \underbrace{k + 1, \ldots, k + 1}_{k}, k, k \rangle$.
- if $n = k+2+i$, $1 \leq i \leq k-2$, $\langle \underbrace{k + i + 1, \ldots k + i + 1}_{k-i}, k+i, k+i, \ldots, , k+1, k+1, k, k \rangle$.
- if $n > 2k$, $\langle \underbrace{2k, \ldots, 2k}_{n-2k}, 2k - 1, 2k - 1, \ldots, k + 1, k + 1, k, k \rangle$.

A k-fan graph is also a k-path graph, as proved in Theorem 5.

Lemma 4 (Pereira *et al.* **[13]).** *Let* $G = (V, E)$ *be a k-path graph,* $k > 1$*. Then,*

1. *if v is a universal vertex of G then* $G[V - \{v\}]$ *is a* $(k - 1)$*-path graph;*
2. *if a new vertex* $w \notin V$ *is added to G and a* $(k + 1)$*-path graph* G' *is obtained then w is a universal vertex of* G'*.*

Theorem 5. *Every k-fan,* $k \geq 2$*, is a k-path graph.*

Proof. A k-fan graph $G = K_\ell + R_{k-\ell}$ can be seen as the successive addition of ℓ universal vertices to the $(k - \ell)$-ribbon $R_{k-\ell}$. Pereira *et al.* [13] proved in Lemma 4 that if we add a universal vertex to a k-path graph we obtain a $(k + 1)$-path graph. So, a k-fan is a k-path graph. □

The 2-fan graph $K_1 + P_n$ is the usual fan graph. Let $G = K_\ell + R_{k-\ell}$ be a k-fan graph, $k \geq 3$. For a given $n > 2k + 1$, consider the pair $(\ell, k - \ell)$. There are $k - 1$ non-isomorphic k-fan graphs, one for each pair $(1, k-1)$, $(2, k-2)$, ..., $(k-1, 1)$. Figure 2 shows all possible 5-fans of order 11.

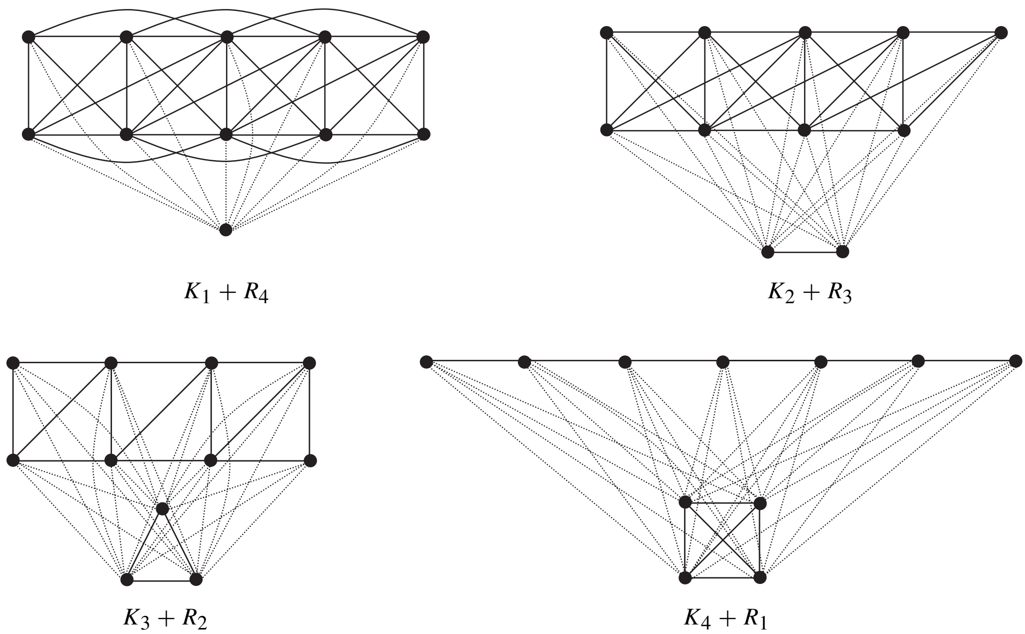

Figure 2 – 5-fans of order 11.

The degree sequence of the k-fan graph $G = K_\ell + R_{k-\ell}$, $k \geq 2$, $1 \leq \ell \leq k - 1$ and $n > k + 3$ can be determined from the sequence degree of $R_{k-\ell}$, incrementing their degrees by ℓ and including ℓ elements with value $n - 1$.

3 SUBCOLORING

A s-coloring of a graph $G = (V, E)$ is a partition of the vertices into s pairwise disjoint sets $V_1, \ldots, V_s$ such that for every $i = 1, \ldots, s$, each color class V_i consists of isolated vertices. This concept can be generalized in several ways and we address one of these, introduced by Albertson *et al.* [1]. A partition $V_1, \ldots, V_s$ is called a *s-subcoloring* of a graph G if each color class induces in G a disjoint union of complete subgraphs. The *subchromatic number* $\chi_s(G)$ is the smallest integer for which G has a s-subcoloring.

It is known that the problem of determining whether $\chi_s(G) \leq k$, $k \geq 2$, is NP-complete (Gimbel & Hartman [8]). So, the recognition of classes of graphs for which it is possible to solve the problem in polynomial time is an interesting research topic. Fiala *et al.* [7] showed that, for a fixed s, recognizing s-subcolorability of graphs with tree decomposition bounded by a constant b can be decided in $O(n2^b s^{b+2})$. Stacho [15] presented a polynomial time algorithm for testing the 2-subcolorability of chordal graphs with time complexity of $O(n^3)$ and Stacho [16] proved that it is NP-complete to decide, for a given chordal graph G, whether or not G admits a s-subcoloring, $s \geq 3$.

A simple result is the subcoloring of paths, $\chi_s(P_n) = \chi(P_n) = 2$, and complete graphs, $\chi_s(K_n) = 1$. This result led us to examine how complete subgraphs are able to affect the subcoloring problem, which other graphs have constant subchromatic number and how the k-ribbons and the k-fans behave.

Albertson *et al.* [1] presented the following results, particularly relevant for the subject addressed here, remembering that indifference graphs are equivalent to proper interval graphs.

Theorem 6 (Albertson *et al.* **[1]).** *Let G be an interval graph which contains no induced $K_{1,c+1}$. Then $\chi_s(G) \leq c$.*

Corollary 7 (Albertson *et al.* **[1]).** *For any indifference graph G, $\chi_s(G) \leq 2$.*

Corollary 7 immediately provides the subchromatic number of a k-ribbon. Theorem 8 offers a constructive proof which also shows the subcoloring of these graphs, which can be obtained in $O(n+m)$. Note that a complete graph with $k+1$ vertices is a k-ribbon graph and its subchromatic number is 1.

Theorem 8. *Let $G = (V, E)$ be a k-ribbon graph, $k \geq 2$ and $n > k + 1$. Then $\chi_s(G) = 2$.*

Proof. Let us consider $V = \{v_1, \ldots, v_n\}$ the set of vertices where v_1 and v_n are the simplicial vertices of the graph. By the properties of k-path graphs and by the definition of k-ribbon, the order of any maximal clique is $k + 1$ and they can be denoted $Q_i = \{v_i, \ldots, v_{i+k}\}$, $i = 1, \ldots, n - k$.

Let q and r be positive integers such that $n = q(k + 1) + r$, $0 \leq r < k + 1$. Consider the following q maximal cliques:

$$\{v_1, \ldots, v_{k+1}\}, \{v_{k+2}, \ldots, v_{2(k+1)}\}, \ldots, \{v_{(q-1)k+q}, \ldots, v_{q(k+1)}\}.$$

By the definition of k-ribbon graph, these cliques are disjoint and they can be disposed alternatively in two classes of colors C_1 and C_2.

The remaining vertices of G form a clique Q. Two cases must be analyzed. If q is even then Q is added to the class C_1 because among the r remaining vertices some are adjacent to vertices of the clique $\{v_{(q-1)k+q}, \ldots, v_{q(k+1)}\} \in C_2$. Otherwise, Q is added to class C_2. So, $\chi_s(G) = 2$. □

Theorem 6 gives a bound to the subchromatic number of a k-fan graph. It is not difficult to see that a k-fan $G = K_\ell + R_{k-\ell}$, no matter ℓ, always contains an induced $K_{1,c}$ formed by a vertex of K_ℓ and a set of cardinality c of independent vertices of $R_{k-\ell}$. However it is a weak bound since as n grows, the value of c also grows.

The subchromatic number and a specific subcoloring of a k-fan graph of order $n \geq k+2$ are presented in the following theorem (the k-fan graph with $k+1$ vertices is a complete graph and its subchromatic number is 1). Note that the subcoloring of the graph is performed in $O(n+m)$ time complexity.

Theorem 9. *Let $G = K_\ell + R_{k-\ell} = (V, E)$ be a k-fan graph, $k \geq 2$ and $n \geq k+2$. Then $\chi_s(G) = 2$ or 3.*

Proof. The subcoloring of a k-fan graph is associated to the number of disjoint maximal cliques of $R_{k-\ell}$ as presented in Theorem 8. These maximal cliques have cardinality $k - \ell + 1$ and $R_{k-\ell}$ has $n - \ell$ vertices. Let q and r be positive integers such that $n - \ell = q(k - \ell + 1) + r$ and $0 \leq r < k - \ell + 1$.

Let $\{v_1, \ldots, v_{n-\ell}\}$ be the vertices of $R_{k-\ell}$ and let v_1 and $v_{n-\ell}$ be the simplicial vertices of the graph.

If $q = 1$, there is only one maximal clique of $R_{k-\ell}$ to be considered. So, $v_1, \ldots, v_{k-\ell+1}$ belongs to color class C_1 and the r remaining vertices $v_{n-\ell-r+1}, \ldots, v_{n-\ell}$ of $R_{k-\ell}$ and the vertices of K_ℓ belong to another color class C_2.

If $q \neq 1$, by Theorem 8, at least two different color classes C_1 and C_2 are required to subcolor the q maximal cliques of $R_{k-\ell}$. So, consider the cases:

- $q = 2$, the set of r remaining vertices $\{v_{n-\ell-r+1}, \ldots, v_{n-\ell}\} \subseteq C_1$ and the vertices of K_ℓ belong to C_2.
- $q = 3$ and $r = 0$, the vertices of K_ℓ belong to C_2.
- Otherwise, the r remaining vertices $v_{n-\ell-r+1}, \ldots, v_{n-\ell}$ and the vertices of K_ℓ form a clique and must belong to the class C_3.

It is possible to summarize all the cases as follows.

$$\chi_s(G) = \begin{cases} 2, & q < 3 \text{ or } q = 3 \text{ and } r = 0 \\ 3, & \text{otherwise.} \end{cases} \qquad \square$$

4 TOUGHNESS

In 1973, Chvátal [4] introduced the concept of toughness. Much research has been carried out on connectivity measures, relating toughness conditions to the existence of hamiltonian cycles (*e.g.* Broersma *et al.* [3]) and comparing measures of vulnerability of the graph (*e.g.* Kratsch *et al.* [10] and Markenzon *et al.* [12]).

The number of components of a graph $G = (V, E)$ is denoted $\omega(G)$. A graph G is t-tough if $|S| \geq t\, \omega(G-S)$ for every subset $S \subseteq V$ with $\omega(G-S) > 1$. The *toughness* of G, denoted $\tau(G)$, is the maximum value of t for which G is t-tough (taking $\tau(K_n) = \infty$ for all $n \geq 1$). Hence if G is not complete, $\tau(G) = \min\left\{\frac{|S|}{\omega(G-S)}\right\}$, where the minimum is taken over all separators S of vertices in G (see Bauer *et al.* [2]).

Kratsch *et al.* [10] studied the toughness of trapezoid graphs, proving that its determination has time complexity of $O(n^5)$. Trapezoid graphs contain interval graphs and the authors claim that, for this class, the toughness can be determined in $O(n^3)$. However details are not provided and the improvements are not explained. We present results about the toughness of k-ribbons and k-fans showing that for these classes, given n, k and ℓ, the determination takes constant time complexity. The reasoning of the proof provides the procedure for the determination of the vertices of the separator S which takes $O(n + m)$ time complexity. It is noteworthy that these classes establish lower and upper bounds for the toughness of k-path graphs.

Regarding separators, Dirac [6] has proved an important characterization of chordal graphs.

Theorem 10 (Dirac [6]). *G is chordal if and only if every minimal separator of G is a clique.*

Theorems 11, 12 and 13 establish the toughness of k-ribbons and k-fans. Remember that, for a 2-path graph G, it is well known that $\tau(G) = 1$.

Theorem 11. *Let G be a k-ribbon graph, $k \geq 3$ and $n \geq k + 2$. Then, $\tau(G) = \frac{k}{2}$.*

Proof. Every minimal separator S of a k-ribbon has cardinality k and $G - S$ has two connected components. Observing the structure of a k-ribbon, in order to obtain an additional component, at least k vertices must be added to a separator.

Since $\frac{tk}{t+1} - \frac{k}{2} = \frac{k(t-1)}{2(t+1)} > 0 \therefore \frac{k}{2} < \frac{tk}{t+1}$, $t \geq 2$, and the result follows. □

The study of toughness of a k-fan graph G is more interesting, as there are two cases to be considered, taking into account the value of ℓ. A superior bound for the toughness of G is immediate, $\tau(G) = \frac{k}{2}$. In this case, $|S| = k$. However, as the vertices of K_ℓ belong to all separators of the graph, it may be possible to increase S with vertices that belong to separators of $R_{k-\ell}$ in order to determine a smaller value for the toughness. So, we observe that the inequality $\frac{k+k-\ell}{2+1} < \frac{k}{2}$ is true when $k < 2\ell$. Figure 3 presents an example of the determination of $\tau(G) < \frac{k}{2}$.

In the following results, we consider $k \geq 3$, $1 \leq \ell \leq k - 1$ and $n \geq k + 2$.

Theorem 12. *Let $G = K_\ell + R_{k-\ell}$ be a k-fan graph with $k \geq 2\ell$. Then $\tau(G) = \frac{k}{2}$.*

Proof. Since $k \geq 2\ell$, $\frac{k}{2} \leq \frac{k+t(k-\ell)}{t+2}$, $t \geq 1$. Then, $\tau(G) = \frac{k}{2}$. □

Theorem 13. *Let $G = K_\ell + R_{k-\ell}$ be a k-fan graph with $k < 2\ell$.*

If $n < 2k - \ell + 3$ *then* $\tau(G) = \frac{k}{2}$ *else*

$$\tau(G) = \begin{cases} \frac{\ell+(q-1)(k-\ell)}{q} & r = 0 \\ \frac{\ell+q(k-\ell)}{q+1} & otherwise \end{cases}$$

where $n - \ell = q(k - \ell + 1) + r$ *and* $0 \le r < k - \ell + 1$.

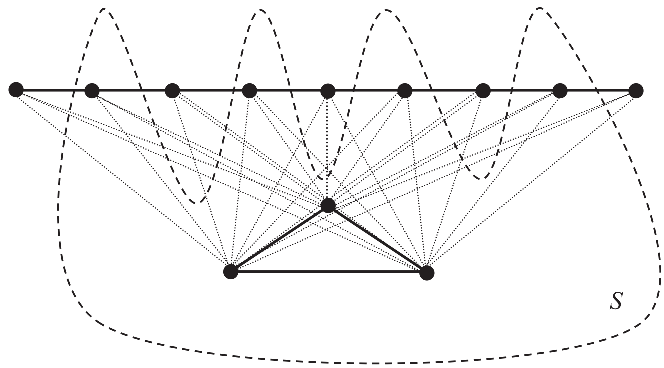

Figure 3 – 4-fan $K_3 + R_1$ of order 12 and $\tau(G) = \frac{7}{5}$.

Proof. Let us denote the $n - \ell$ vertices of $R_{k-\ell}$ by $v_1, \ldots, v_{n-\ell}$ where v_1 and $v_{n-\ell}$ are the simplicial vertices. The vertices of K_ℓ are $v_{n-\ell+1}, \ldots, v_n$.

A separator of a k-path graph must contain a minimal separator of the graph. So, $|S| \ge k$. Observing the structure of a k-fan graph, a minimal separator generates two connected components.

Since the vertices of K_ℓ are universal in G they must belong to any separator. Let us build the separator S. Consider, without loss of generality, $S = \{v_2, \ldots, v_{k-\ell+1}, v_{n-\ell+1}, \ldots, v_n\}$, *i.e.*, S is composed by $\{v_2, \ldots, v_{k-\ell+1}\}$ (a separator of $R_{k-\ell}$) and the vertices of K_ℓ. At this point the vertex v_1 is a component of $G - S$.

We can repeat the reasoning to the remaining component, which is a $(k-\ell)$-ribbon graph with at least $k-\ell+1$ vertices. In order to obtain a lower value than $\frac{k}{2}$ for the toughness, $n \ge 2k-\ell+3$ since $n \ge k+2$. So, considering this case, the next vertices to be added to S in order to establish a new separated component is the clique $\{v_{k-\ell+3}, \ldots, v_{2(k-\ell)+3}\}$ where the vertex $v_{k-\ell+2}$ is the new component, and so on.

The number of cliques in $R_{k-\ell}$ to be chosen in this process is $\left\lfloor \frac{n-\ell}{(k-\ell)+1} \right\rfloor$. The number of components of the graph is $\left\lceil \frac{n-\ell}{(k-\ell)+1} \right\rceil$ and since each component has exactly one vertex, it is maximum. Consider q and r positive integers such that $n-\ell = q(k-\ell+1)+r$ and $0 \le r < k-\ell+1$. If $r = 0$ then $\left\lfloor \frac{n-\ell}{(k-\ell)+1} \right\rfloor = q$, $\left\lceil \frac{n-\ell}{(k-\ell)+1} \right\rceil = q+1$ and $\tau(G) = \frac{\ell+(q-1)(k-\ell)}{q}$ otherwise $\left\lfloor \frac{n-\ell}{(k-\ell)+1} \right\rfloor = q-1$, $\left\lceil \frac{n-\ell}{(k-\ell)+1} \right\rceil = q$ and $\tau(G) = \frac{\ell+q(k-\ell)}{q+1}$. So, the result follows. □

Table 1 shows the toughness of 7-fan graphs with $n \le 18$.

Table 1 – $\tau(G)$ of 7-fan graphs.

			n									
k	ℓ	$k-\ell$	9	10	11	12	13	14	15	16	17	18
7	1	6										
	2	5					3.5					
	3	4										
	4	3					3.33	3.33	3.33	3.33	3.25	3.25
	5	2				3	3	3	2.75	2.75	2.75	2.6
	6	1			2.67	2.67	2.25	2.25	2	2	1.83	1.83

The smallest toughness of $G = K_\ell + R_{k-\ell}$ for a fixed n is obtained when $\ell = k-1$. The proof of Corollary 14 is immediate.

Corollary 14. *Let G be a k-fan graph such that $k < 2\ell$ and $n \geq 2k - \ell + 3$. The lowest value of toughness is obtained when $\ell = k-1$.*

By the reasoning presented in Theorem 13, vertices that belong to several minimal separators of the graph are key elements of separators which produce a lower toughness. In k-fan graphs these vertices are the universal vertices. For k-path graphs in general, it is easy to see that the same happens. The k-fan graphs are actually the k-path graphs which minimize the number of vertices added to a separator in order to obtain a new component. The next corollary summarizes this observation.

Corollary 15. *Let G be a k-path graph, $k \geq 2$ and the k-fan graph $K_{k-1} + R_1$. Then,*

$$\begin{cases} \dfrac{n+k-2}{n-k+2} \leq \tau(G) \leq \dfrac{k}{2} & \text{if } n-k \text{ is even} \\[2ex] \dfrac{n+k-3}{n-k+1} \leq \tau(G) \leq \dfrac{k}{2} & \text{if } n-k \text{ is odd.} \end{cases}$$

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