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ARTICLES

ROBUST SOLUTION IDENTIFICATION FOR UNCERTAINTY MANAGEMENT IN MOLP – AN INTERACTIVE APPROACH

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ABSTRACT. This paper presents a robust deterministic method for the treatment of uncertainty in decision-making systems based on Sonar Method. It introduces uncertainty into mathematical models as interval numbers. This work proposes an interactive approach that identifies the efficient vertices of the Multiobjective Linear Programming (MOLP), where some or all the coefficients of the objective functions, constraints or limits of the constraints are interval of real numbers. The method identifies robust solutions to MOLP under uncertainty, showing how the interactive approach presented in this paper can provide the possibility of exploring in new decision support methodologies.

Keywords: uncertainty modeling, interval programming, multiobjective linear programming, robust solution, interactive approach.

1 INTRODUCTION

The decision-making in management is always a challenge that requires technical and intellectual development. The use of qualitative and quantitative tools must be added to the prospective vision ability. The dynamics of the relevant facts that need to be considered at the time of decision-making changes quickly, making it difficult to analyze the scenario. The computational developments are constantly accelerating, making it appropriate to use them to solve real problems, since the support of new techniques in this environment can represent sustainable decisions (Lucas, 2012).

Most times, representing a real problem for a mathematical model is not a simple task and considering that the model parameters are under uncertainty, more knowledge about the problem is required. Often, this uncertainty is associated with the dynamics of the competitive environment and sometimes it can be related to the entry data of the research as well as to measurement errors.

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In the scientific literature, some authors present different concepts such as “risk”, “fuzziness”, “uncertainty”, among others. These terms are associated with different areas. For example, in the risk studies, statistics techniques are usually applied. However, in this paper, a deterministic method is presented.

The objective of this paper is to present a new approach, which supports decision-making where uncertainty is introduced into the mathematical model as an interval of real numbers.

Bitran (1980) presents the method called Interval Objective Function that introduces the uncertainty in the coefficients of objective function with interval number in Multiobjective linear programming. This work is very important for further research with interval analysis in multiobjetctive linear programming

Lucas, Antunes & Clímaco (2005) develops the Sonar Method creating a new way of modeling the uncertainty in linear programming problems, where one or more objectives are optimized, opening doors for the development of new interactive decision support methodologies.

Oliveira & Antunes (2007) provide an overview of multiple objective linear programming models with interval coefficients.

Lucas (2009) applies the Sonar Method to the Petrobras’s Plangas System of and obtains the Most Optimist Solution and the Least Optimist Solution for the company’s Production and Sales Planning. The results are the optimal solutions of the linear programming problem where uncertainty is introduced an interval number. The work shows to be interesting for real problems and opens doors for the development of new methodologies to support decision maker.

Hladik (2012) looks for the Optimal Value Range for the Linear Programming Problem with the coefficients of objective function, coefficients and bounds of constraints on interval numbers and intends to determine a tight enclosure to the optimal solution set.

The interactive methodology in this paper aims to identify, whenever possible, robust solutions for the uncertainty in a Multiobjective Linear Programming (MOLP) problem, where some or all of the coefficients of the objective functions, constraints or limits of the constraints can be intervals of real numbers. The decision maker can analyze the solutions with respect to their quality, analyzing the objective function values, and with respect to their feasibility. This approach is based on Sonar Method.

The concept of robustness used in this paper calls a solution robust when it is obtained several times or every time that a set of possible problem coefficient values are implemented (Roy, 1998). The positive side of the interactive approach is having the intervention of the decision maker to ease any practical difficulties that may arise for the process of decision support, avoiding a large number of solutions to be evaluated.

Lucas, Antunes & Clímaco (2006) use the Sonar Method to model the uncertainty in the problem, evaluating the robustness of solution to the uncertainty of the data using the parametric diagram. A robustness index is constructed based on the frequency that basis is generated and the degree of superposition of the indifference regions computed for it with random coefficient generations within the intervals.

The quality and feasibility of the solution identify robust solutions in the sense that “no matter what” (for any setting in which the coefficients are presented within the intervals) the respective values of objective functions are never below certain satisfaction levels suggested by the decision maker. In most cases, the values of the objective functions do not change drastically under possible variations of the coefficients, and in most cases the solution is possible (Lucas, 2009).

This approach looks for solutions to deal with the uncertainty in MOLP problems and it tries not to generate only very conservative solutions (such as the MINIMAX based models).

The main purpose of the presented approach is to help identifying efficient solutions that have a behavior judged as satisfactory by the decision maker, who is confronted with variations of the interval coefficients of the MOLP model, with respect to the quality of the solution, analyzing the objective function values, and with respect to the feasibility of the solution.

2 UNCERTAINTY MANAGEMENT IN SINGLE AND MULTIPLE OBJECTIVE PROGRAMMING

Lodwick & Salles-Neto (2021) define Flexible and Generalized Uncertainty in an excellent work. “Flexibility in the context of an optimization problem, is the relaxation of the meaning of set relationships such as belonging (to the constraint set) or optimizing (in the objective)” and “Generalized uncertainty theory is a mathematical theory of incompleteness or lack of information, lack of specificity, or imprecision”.

The Sonar Method can be used in Flexible and Generalized Uncertainty whenever it is expressed by interval numbers.

Uncertainty is an intrinsic characteristic of real-world problems arising from multiple sources of distinct nature (Antunes, Lucas & Climaco, 2006). Robust optimization (Bertsimas & Sim, 2004; Ben-Tal & Nemirovski, 2008) has been used to treat uncertainty in many classes of problems. In uncertain multiobjective problem (UMOP), several authors have addressed robust solutions. The necessary and sufficient optimality conditions for robust multiobjective efficiency are presented in Kuroiwa & Lee (2012) and Bokrantz & Fredriksson (2017). Optimality conditions and duality theorems for UMOP are investigated in Goberna, Jeyakumar, Li & Vicente-Pérez, (2014), Lee & Lee, (2016); Chen, Köbis & Yao, (2018).

The Minmax (Ben-Tal, Ghaoui & Nemirovski, 2009) robustness concept in UMOP is addressed in Ehrgott, Ide & Schöbel. (2014). They adapted weighted sum scalarization and epsilon-constraints to this new concept. The Tchebycheff function to generate nondominated robust solutions with an interactive weighted procedure is presented in Hassanzadeh, Nemati & Sun (2014). The numerically tractable optimality conditions for highly robust weakly efficient solutions are given in Goberna, Jeyakumar, Li & Vicente-Pérez, (2015). Eichfelder, Krüger & Schöbel (2017) present prove that decision robust efficient solutions can be found by solving a deterministic problem in case of linear objective functions. Different robustness concepts for UMOP are compared in Goberna, Jeyakumar, Li & Vicente-Pérez (2015); Ide & Schöbel (2014).

The conservatism of MinMax method has been observed in several papers (see numerical example, Bokrantz & Fredriksson, 2017). In Minmax, the efficient solutions are found by optimizing in the worst case; hence, it is more conservative and not that good in other scenarios. In a single objective problem, Fischetti & Monaci (2009) developed Light Robust efficiency concept with a "flexible counterpart robust". The model finds the efficient solutions by controlling the loss in the objective function values, with tolerable degradations, in the most typical scenario (Zhou-Kangas & Miettinen, 2019). Kuhn, Raith, Schmidt & Schöbel. (2012, 2016) extended Light robust concept for bi-objective optimization with uncertainty in only one objective. They propose a two-step algorithm that solve a deterministic multiobjective problem and filter among the efficient solutions found robust efficient solutions. Ide & Schöbel (2013, 2016) generalize Light robust for UMOP. The relationships between Minmax robust efficient, lightly robust efficient and nominal efficient solutions are analyzed in Zhou-Kangas & Schöbel (2018). They use two measures 'price to be paid for robustness' and 'gain in robustness' to support the decision maker in considering the trade-offs between robustness and quality.

The compensatory analysis between robustness and quality, or solution to the worst case or solutions that are also valid in other scenarios, has been a concern in almost all papers. Raith, Schmidt, Schöbel & Thom (2017) notes that the number of sub-problems that have to be solved increases with decreasing values of parameter Price of Robustness (Bertsimas & Sim, 2004).The interactive method of Robust solutions for MOLP problems using interval programming present in this work can be suitable to solve this dilemma.

The method presented is able to search robust solutions on the most important region of decision space in a model of multiobjective linear programming that is the area near of the contour of the feasible set. Other important contribution is to introduce uncertainty through interval coefficients, considering that the interval number is easy to understand and to implement. If an expert is asked about the uncertainty of an important parameter in a specific real problem, the expert can indicate the lower and upper bound to solve the difficult situation and these bounds can be presented in interval number. That option is easier than to require previous knowledge about the probability distributions of the target problem data.

This paper is subdivided into six sections. Initially, in section 2 and 3, the principal's points of Multi-Objective Linear Programming theory and Interval Analysis are introduced. The fourth section shows the new Robust Interactive approach to MOLP based on reference points. Section 5 considers future possibilities of this approach and in the last section, conclusions and future possibilities of this approach are presented.

3 MINIMIZING THE DISTANCE FROM A REFERENCE POINT IN MULTIOBJECTIVE PROGRAMMING

The search for technical decision-making requires knowledge of the target problem, considering the relevance of each of the aspects of possible issues, and techniques that can systematize and enhance the scope of a rational decision (Zeleny, 1974).

The multicriteria problems are usually subdivided into two major groups, which are associated with their methodological approaches: the multi-attribute problems and multiobjective problems.

In multi-attribute analysis, permissible alternatives are explicitly known and finite in number. In this context, one can distinguish among the selection, ordering or characterization problems (Roy, 1990).

In multiobjective programming, the constraints define the set of feasible solutions forms a continuous (as well as the mono-objective programming). However, the set of feasible solutions within the decision variables is mapped in the space of objective functions, so that each alternative is associated with a vector whose components are the values of the objective functions corresponding to this alternative (Steuer, 1986; Cohon, 1978).

According to the process used for aggregating the decision makers' preferences, Multiobjective Programming methods can be classified into three categories: methods where an a priori aggregation of preferences is made; methods of progressive articulation of preferences (interactive methods) and methods in which no articulation of preferences is made (generating methods) (Antunes, Alves & Clímaco, 2016, p3-4).

3.1 MOLP Problem

The MOLP problem presents $\boldsymbol{p}$ -linear objective functions subject to a set of linear constraints.

It is considered that the objective functions are all to maximize:

$$
\begin{array}{c}
\max f_1\ (\underline{x})\ =\ \underline{c}_1\underline{x} \\
\max\ f_2\ (\underline{x})\ =\ \underline{c}_2\,\underline{x} \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\max\ f_p\ (\underline{x})\ =\ \underline{c}_p\,\underline{x} \\
s.t\ \underline{x}\in X\ \{\underline{x}\in\Re^n\,|\,\underline{x}\ \geq\ 0\,,\,A\underline{x}\ =\ \underline{b}\,,\,\underline{b}\in\ \Re^m\,\} \\
\text{or} \\
\text{ital}\ Max\,\underline{f}\ (\underline{x})\ =\ C\,\underline{x} \\
s.\,t\ \ x\in X
\end{array}
$$

where $\boldsymbol{p}$ is the number of objective functions, $\boldsymbol{n}$ is the number of variables, $\boldsymbol{m}$ is the number of constraints, x is the vector of decision variables, $\boldsymbol{C}$ is the matrix of objectives (dimension $\boldsymbol{p} \times \boldsymbol{n}$), whose rows are the vectors $\boldsymbol{c_p}$ (coefficient of objective function $\boldsymbol{f}_{\mathrm{p}}$), $\mathbf{A}$ is the matrix of technological coefficients ($\boldsymbol{m} \times \boldsymbol{n}$) , $\boldsymbol{b}$ is the vector of independent terms, $\boldsymbol{X}$ is the feasible region in the space of variables (Steuer , 1986; Climaco, Antunes & Alves, 2003).

Methods to solve MOLP are presented by Luc (2016, p 241,264,289): Multiobjective Simplex, Normal Cone and Outcome Space.

3.2 Efficient Solution and Non-Dominated Solution

A solution is called efficient for a multiobjective problem if and only if there is no other feasible solution that enhances the value of an objective function without worsening the value of at least one other objective function (Pareto, 1986; Climaco, Antunes & Alves, 2003).

$$X_E = \left\{\underline{x} \in X \mid \underline{x}' \in X : \underline{f}\left(\underline{x}'\right) \geq \underline{f}\left(\underline{x}\right)\right\}$$

where $\underline{f}\left(\underline{x}'\right) \geq \underline{f}\left(\underline{x}\right)$ if and only if$\underline{f}\left(\underline{x}'\right) \geq \underline{f}\left(\underline{x}\right)$ and $\underline{f}\left(\underline{x}'\right) \neq \underline{f}\left(\underline{x}\right)$.

In MOLP, the feasible set produces another vector space called the Criteria Space, which is the image of the feasible set. The criteria space has ***p*** dimensions and its size varies according to the number of objective functions.

The non-dominated solution is the image of efficient solutions in the Criteria Space.

3.3 Ideal Solutions

The ideal solution is the point **z*** in the space of objective functions that would optimize all functions simultaneously. Each component of the ideal solution is the optimal value of each objective function, individually optimized in the feasible region. When the objective functions are in conflict, the ideal solution is beyond the feasible region, but each $\mathbf{z_k}$***** is individually accessible. The ideal solution can be used as the reference point in surrogate scale functions that are designed to calculate the nearest non-dominated solution, according to a given metric. Note that there can be no **x*** solution in the space of decision variables (even not feasible) having **z*** as image within the objective functions space, so the ideal solution is defined only in this space (Climaco, Antunes & Alves, 2003).

The pay-off table contains the values of the objective functions for each non-dominated solution, which is the individual optimal of each function. This table allows to obtain the information on the ranges of values of each objective function in the non-dominated region.

Table 1 – Pay-Off Table.

	$\underline{x}^1$	$\underline{x}^2$		$\underline{x}^k$		$\underline{x}^p$
f_1	$z^{1,1} = z^{*,1}$	$z^{2,1}$	...	$z^{k,1}$	...	$z^{p,1}$
f_2	$z^{1,2}$	$z^{2,2} = z^{*,2}$	...	$z^{k,2}$	...	$z^{p,2}$
	...	...	...	...	...	...
f_k	$z^{1,k}$	$z^{2,k}$	...	$z^{k,k} = z^{*,k}$	...	$z^{p,k}$
	...	...	...	...	...	...
f_p	$z^{1,p}$	$z^{2,p}$	...	$z^{k,p}$	...	$z^{p,p} = z^{*,p}$

The pay-off table has the following form, where $z^{i,k} = c_k x^i$, with $z^{i,i} = c^i x^i = z^{*,i}$ where the columns are represented by x_i solutions that are ordered in the same sequence of their objective functions, i.e. in column 1 the value of f_1 is calculated with its optimal solution and displayed in the cell z_1,1 and the other lines the optimal solution of f_1 is assigned to the other objective functions . In the second column, the solution which optimizes f_2 is introduced and so on.

The ideal solution can be identified in the diagonal of the individual optimal table (elements $z^{*,k} = z^{k,k}$).

However, for problems with more than two objective functions, the worst values of each objective function in the non-dominated region may not be available in this table. It may be that the worst value of a given objective function in the non-dominated region is not reached in a solution that is the individual optimal of another objective function (and only these solutions are shown in the table).

3.4 Adding Multiple Objectives

The optimization of the substitutive scalar function that, temporarily, adds the multiple objective functions, results in the non-dominated solutions of the original problem and includes information of preference of the decision maker parameters (Wierzbicki, 1986; Vanderpooten, 1989, 1990; Vanderpooten & Vincke , 1989; Antunes, 1991; Climaco, Antunes & Alves, 2003).

In the context of interactive methods, the substitutive scalar function can be used to evaluate the non-dominated solutions guided by the preference of the decision makers but without considering as the true analytical representation of their preference (Clímaco, Antunes & Alves, 2003).

3.4.1 Minimizing the distance from a reference point

This process of minimizing the distance from a reference point determines the efficient solutions, according to the preference of the decision maker. Often the ideal solution is used as this reference point, since each of the ideal solution components is the best possible value for it individually optimized objective function (Clímaco, Antunes & Alves, 2003).

The most commonly used metrics are $L1$, $L2$ (Euclidean) and $L\infty$ (by Chebycheff) metrics. The Figure 1 shows the iso-distance contours (locus of equidistant points) for **z*** for these metrics. In Figure 2, the points $z1$, $z2$ and $z\infty$ minimize the distances from the ideal solution **z*** using the $L1$, $L2$ and $L\infty$ metrics, respectively. Adopting the ideal solution as a reference point, the calculation of the solution which minimizes the Lp metric distance is $\min \; || z^* - f(x) ||_p \quad \text{s. t} \;\; x \in X$.

For $p = 1$ all deviations from the reference point are considered. As p increases, greater deviations will have greater impact on the distance value. For the $L\infty$ metric, the greatest deviation supersedes all others and it is the only one considered (it results in the minimizing of the greatest deviation). In this case, a MINIMAX type of problem occurs that can be transformed into a linear problem by introducing an additional variable.

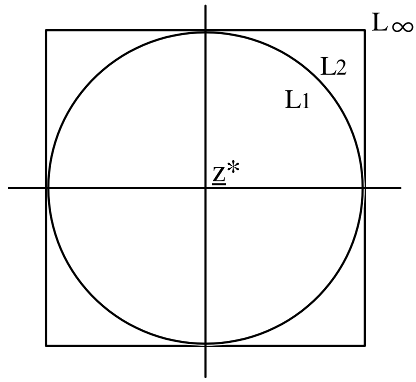

Figure 1 – Iso-distance contours of the L_1, L_2 and L_∞ metrics.

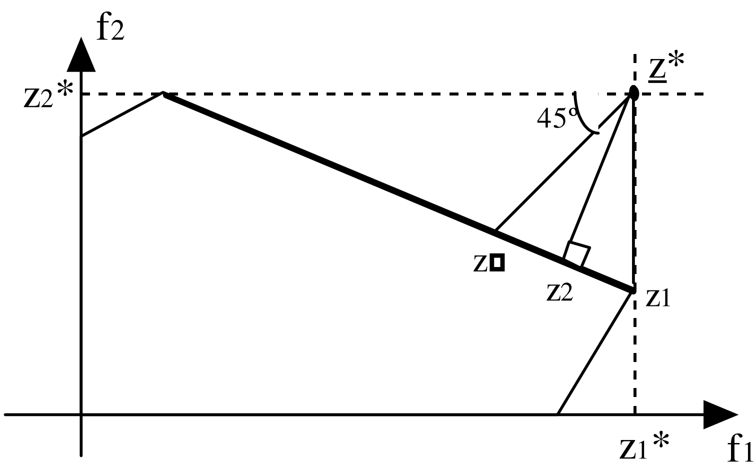

Figure 2 – L_1, L_2 and L_∞ metrics in distance functions.

min v

s. t $v + f_k(x) \geq z_k*$ k=1,...,p

$x \in X$

$v \geq 0.$

If the original multiobjective problem is linear, the substitute scalar problems are only linear with the L1 and L∞ metrics.

4 INTERVAL ANALYSIS

Interval Mathematics is a generalization in which the interval numbers replace the real numbers, the interval arithmetic replaces the real arithmetic and interval analysis replaces the real analysis (Hansen, 1992).

Interval mathematics began with Moore's book "Interval Analysis" in 1966. This work is used as a viable tool for error analysis. Rather than merely treating rounding error, Moore (1966) extended the use of interval analysis to limit the effect of all types of error, including approximation and data error. Since then, numerous studies on interval analysis have been published (Hansen, 1992). An overview of theory and applications of interval analysis is given by Alefeld & Mayer (2000).

4.1 Main definitions

The real interval $X = [a, b]$ considers the interval number X as a number which is a closed interval in the set $\{x\colon\ a \leq x \leq b\}$ of real numbers including the points a and b.

The real number x is equal to an interval $[x, x]$. Such interval is called degenerate interval. When a real number is expressed as an interval, it usually keeps the simple notation. For example, the number 2 is displayed in place of [2, 2] or x in place of $[x, x]$.

The rules of interval arithmetic are simple when one or both terms are degenerate intervals. In this case, it is better to leave a degenerate interval as a real number.

An interval $X = [a, b]$ is said to be positive (or not negative) if $a \geq 0$, strictly positive if the $a >$ 0, negative (or not positive) if $b \leq 0$, and strictly negative if $b < 0$.

Two intervals $[a, b]$ and $[c, d]$ are equal if and only if $a = c$ and b $= d$.

The interval numbers are partially ordered. We have $[a, b] < [c, d]$ if and only if $b < c$. (Hansen, 1992).

4.2 Interval Arithmetic

Addition, subtraction, multiplication and division are indicated by the signs $+$, $-$, $*$, $/$ respectively. If **op** stands for one of these arithmetic operations for the $\boldsymbol{x}$ and $\boldsymbol{y}$ real numbers, so the arithmetic operation corresponding to the interval numbers $\boldsymbol{X}$ and $\boldsymbol{Y}$ (Hansen, 1992) is:

$X \mathbf{\ op\ } Y = \{x \mathbf{\ op\ } y\colon x \in X, y \in Y\}$

Thus, the $\boldsymbol{X}$ op $\boldsymbol{Y}$ interval resulting from this operation includes all numbers that can be formed as x **op** y for each $x \in X$ and each $y \in Y$.

5 ROBUST INTERACTIVE APPROACH BASED ON REFERENCE POINTS FOR MOLP

This approach aims to identify efficient solutions in MOLP models in which all (or only some) coefficients are specified as interval numbers, reflecting the uncertainty associated with the problem.

- For a given "concretization" of the feasible region, the objective function values are never below certain reserve levels imposed by the decision maker.

- For the "majority" of cases, there is a sharp degradation of the objective function values; i.e., a solution that presents a sharp deterioration to any objective function cannot be classified as robust.

- For the "majority" of cases, the solution remains feasible.

This approach requires the establishment of some control parameters by the decision maker who plays a key role in the classification of a solution as robust or not. These parameters introduce an additional degree of subjectivity (practically impossible to avoid in this context unless it goes to very conservative solutions of the MINIMAX type) but with the advantage of involving the decision maker in the process of identifying solutions which are considered robust.

The MOLP problem formulation with interval coefficients is the following, in which the lower (denoted by L) and upper (denoted by U) variation limits are known for each coefficient, assuming that additional information is not available (Lucas, Antunes & Climaco, 2005) :

$$\max \quad z_{\mathrm{k}}(\mathrm{x}) = \sum_{j=1}^{n} \left[c_{\mathrm{kj}}^{L}, c_{\mathrm{kj}}^{U}\right] x_j \qquad \mathrm{k} = 1, \ldots, \mathrm{p}$$

$$\text{s.t} \quad \sum_{j=1}^{n} \left[a_{\mathrm{ij}}^{L}, a_{\mathrm{ij}}^{U}\right] x_j \leq \left[b_i^{L}, b_i^{U}\right] \qquad \mathrm{i} = 1, \ldots, \mathrm{m}$$

$$\mathrm{x_j} \geq 0 \qquad \mathrm{j} = 1, \ldots, \mathrm{n}$$

Starting from this problem, the proposed interactive approach to support the identification of the robust effective solutions is developed as follows. (Lucas, 2009)

1a- Calculation of solutions that individually optimize each objective function $z_k(x)$ $(k = 1,...,p)$ with the most favorable coefficients of the objective function in the amplified feasible region. For maximizing problems and for restrictions of the type $\leq$, it is

$$\max \quad z_k^{U}(\mathrm{x})$$

$$\text{s.t} \quad \sum_{j=1}^{n} a_{\mathrm{ij}}^{L} x_j \leq b_i^{U}, \ \mathrm{i} = 1, \ldots, \mathrm{m},$$

$$\mathrm{x_j} \geq 0, \ \mathrm{j} = 1, \ldots, \mathrm{n}.$$

These values constitute a reference point $(z_1^{U*}, z_2^{U*}, ..., z_p^{U*})$,under an "optimistic" perspective, since all interval coefficients of the model are considered simultaneously in their most favorable extremes.

1b- Calculation of solutions that individually optimize each objective function $z_k\ (x)\ (k = 1,...,p)$ with the least favorable coefficients of the objective function in the reduced feasible region . For maximizing problems and for restrictions of the type $\leq$, it is

$$max \quad z_k^L(x)$$

$$s.a \quad \sum_{j=1}^{n} a_{ij}^U x_j \leq b_i^L,\ i = 1, \ldots, m,$$

$$x_j \geq 0,\ j = 1, \ldots, n.$$

These values constitute a reference point ($z_1^{L*}, z_2^{L*},..., z_p^{L*}$) under a "pessimistic" perspective, since all interval coefficients of the model are considered simultaneously in their least favorable extremes.

2a- Calculating the solution that minimizes a Chebycheff distance (weighted) for the optimistic reference point using the central coefficients (nominal) of the objective functions , in a feasible region that favors the region between the amplified feasible region and the reduced feasible region (Lucas, Antunes & Clímaco, 2005). In order to mitigate the effects of the magnitude orders of the objective function values, the coefficients of the objective function can be multiplied by a scaling factor, or a normalization can be made; also, we can use calculated weights as in the STEM method (Benayoun at al, 1971). This weighting coefficient is called g_k.

$$\min \quad v$$

$$\text{s.t} \quad \sum_{j=1}^{n} a_{ij}^L x_j \leq b_i^U\ ,\ \text{i} = 1, \ldots, \text{m},$$

$$\sum_{j=1}^{n} a_{ij}^U x_j \geq b_i^L\ ,\ \text{i} = 1, \ldots, \text{m},$$

$$g_k\ (z_k^{U*} - z_k^C(x)) \leq v\ ,\ k = 1, \ldots, p,$$

$$v \geq 0$$

$$x_j \geq 0,\ j = 1, \ldots, n.$$

The solution to this problem is (x^{0U}, z^{0U}).

2b - Calculating the solution that maximizes a Chebycheff distance (weighted) for the pessimistic reference point using the central coefficients (nominal) of the objective functions, in the same feasible region used above.

$$\max \quad v$$

$$\text{s.t} \quad \sum_{j=1}^{n} a_{ij}^L x_j \leq b_i^U\ ,\ \text{i} = 1, \ldots, \text{m},$$

$$\sum_{j=1}^{n} a_{ij}^U x_j \geq b_i^L\ ,\ \text{i} = 1, \ldots, \text{m},$$

$$g_k\ (z_k^C(x) - z_k^{L*}) \geq v\ ,\ k = 1, \ldots, p,$$

$$v \geq 0$$

$x_j \geq 0$, j = 1, ..., n.

The solution to this problem is (x^{0L}, z^{0L}).

In the case of the occurrence of an empty feasible region, the restrictions $\sum_{j=1}^{n} a_{ij}^{U} x_j \geq b_i^L$, i = 1, ..., m, are discarded.

3 - Solutions are calculated by solving the following substitutive scalar problem with coefficients randomly generated within their intervals

min v

s.t $\sum_{j=1}^{n} a_{ij}^{r} x_j \leq b_i^r$, i = 1, ..., m,

$\sum_{j=1}^{n} a_{ij}^{U} x_j \geq b_i^L$, i = 1, ..., m,

$g_k\ (z_k^{U*} - z_k^{r}(x)) \leq v$, k = 1, ..., p,

$z_k^{r}(x) \geq z_k^{L*}$, k = 1, ..., p,

$z_k^{L}(x) \geq z_k^{L*}$, k = 1, ..., p,

$v \geq 0$

$x_j \geq 0$, j = 1, ..., n.

Where ***r*** denotes the randomly generated coefficients with a uniform distribution within their intervals.

Among a set of solution generated through problem solving, we consider as robust solutions those whose values of objective functions, with the central coefficients (nominal):

- Are never worse in a value greater than Δ_k (k = 1, ... , p) , which can be given in absolute terms or as a percentage in relation to the values of objective functions with the most favorable coefficients, and
- Are always better at least in a value ϕ_k (k = 1, ... , p) (possibly $\Delta_k = \phi_k$, for any or all k = 1, ... , p) with respect to their values of objective functions with the least favorable coefficients , and
- Never vary over ε % in relation to the objective function values obtained with random coefficients within the intervals in all simulations (steps 2 and 3).

6 FUTURE POSSIBILITIES

A possible application of this approach is to develop an automatic monitoring system for uncertainty management as follows.

The problem solving in step 3 could be done by repeating ***h*** times the random generation of the coefficients of the objective functions within their intervals.

If the values of the objective functions are located more than $\boldsymbol{q}$ times in a region $\Delta_{\boldsymbol{k}}$ (k=1,...,p) near $z^{0U}=z(x^{0U})$, or near $z^{0L}=z(x^{0L})$, then x^{0U}, or x^{0L} is classified as robust solution given the quality of the values of the objective functions . The $\boldsymbol{q}$ parameter serves as a solution quality threshold. It would also be possible to establish a veto parameter to "veto" the quality of the solution whenever the respective values of the objective functions are situated more than a distance from a desired value, or within a certain distance from an unwanted value.

Additionally, the $\boldsymbol{a_{ij}}$ and $\boldsymbol{b}_{\mathrm{i}}$ coefficients are randomly generated within the intervals. If the solution is not feasible (with respect to the interval restrictions) a number of times lower than s, it is then classified as robust solution according to the feasibility.

If a solution is found to be robust according to the quality (of the values of the objective functions) and according to the feasibility, it is then classified as robust.

The system can be fed automatically and to be considered under control if the results are obtained by robust solution. When there is no robust solution, the system gives a warning sign to update the model under uncertainty.

Another future perspective is the development of a computer application to use the Sonar Method to treat uncertainties.

7 CONCLUSION

The world is changing rapidly. The advancement of Information Technology has changed the life and dynamics of society, people and enterprises through system automation, expert systems, and the use of artificial intelligence.

The modeling of real problems under uncertainty still presents a major scientific challenge. The management of uncertainty needs to be explored in many areas, requiring the development of new techniques appropriate for specific problems.

Thus, the method presented in this paper proposes the use of the interval numbers to exploit the problem uncertainty and to find the robust solution, considering both **quality** and **feasibility** of the solution.

In order to help the decision maker there is a large field of the development of the software to be able uncertainty management to be applied to various scientific areas, such as Health, Sustainability and Public Management, where small changes can have great impact on the solutions of the target problems.

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COMPLIANCE WITH ETHICAL STANDARDS

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