IMPROVED $\mathcal{H}_2$ AND $\mathcal{H}_\infty$ CONDITIONS FOR ROBUST ANALYSIS AND CONTROL SYNTHESIS OF LINEAR SYSTEMS

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ABSTRACT

This paper proposes improved H-2 and H-infinity conditions for continuous-time linear systems with polytopic uncertainties based on a recent result for the discrete-time case. Basically, the performance conditions are built on an augmented-space with additional multipliers resulting in a decoupling between the Lyapunov and system matrices. This nice property is used to develop new conditions for the robust stability, performance analysis, and control synthesis of linear systems using parameter dependent Lyapunov functions in a numerical tractable way.

KEYWORDS: Robustness, $\mathcal{H}_2/\mathcal{H}_\infty$ norms, convex optimization.

RESUMO

Este artigo propõe novas condições para as normas H-2 e H-infinito para sistemas lineares incertos utilizando as idéias originalmente propostas para sistemas discretos. Basicamente, as condições de desempenho são obtidas em espaço de estados aumentado onde novas variáveis livres são adicionadas ao problema resultando em uma separação entre as matrizes do sistema e de Lyapunov. Este propriedade é utilizada no desenvolvimento de novos critérios para análise de estabilidade robusta e desempenho, e síntese de controle para sistemas lineares utilizando funções de Lyapunov dependente de parâmetros que podem ser resolvidas através de problemas de otimização convexa.

PALAVRAS-CHAVE: Robustez, normas $\mathcal{H}_2$ e $\mathcal{H}_\infty$, otimização convexa.

1 INTRODUCTION

Recently, de Oliveira et al. (1999) have proposed an enhanced stability test for discrete-time systems for which the controller parametrization does not explicitly depends on the Lyapunov function. This allows for instance the use of parameter dependent Lyapunov function to design a fixed robust $\mathcal{H}_2$ or $\mathcal{H}_\infty$ controller or even to solve mixed control problems in a less conservative way. Moreover, it was shown that their method recover the usual LMI test for quadratic stability as a particular case. Thus the new condition will never give a more conservative result if compared with the usual quadratic stability test. These nice properties are basically due to the LMI characterization in an augmented-space which introduces a new slack variable bypassing the product term that appears in the Lyapunov stability condition for

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discrete-time systems. In other words, the following usual condition

$$A'PA - P < 0 \tag{1}$$

is replaced by the following improved condition

$$\left[\begin{array}{cc} P + G + G' & A'G' \\ GA & -P \end{array} \right] < 0. \tag{2}$$

Notice that the term $A'PA$ in (1) is the main source of conservativeness when dealing with uncertain system and robust design with mixed performance specifications. The reason is that robust performance tests are less conservative when P is parameterized in terms of the uncertainties and the performance indexes, but the robust controller parametrization requires a single P to be used. The improved condition in (2) eliminates the constraint of a single Lyapunov function because the controller parametrization does not depend on P.

Since the publication of the aforementioned paper, several researchers have tried to get a similar result for linear continuous time systems. However, the proposed results are not convex, as for instance the techniques in (Shaked, 2001), (Trofino and de Souza, 2002), and even the convex approach proposed by Apkarian et al. (2001) has shown to be more conservative than the usual quadratic stability test in many cases. At the same time, Ebihara and Hagiwara (2002) have proposed a dilated LMI version of the standard LMI tests for robust control synthesis under mixed performance specifications by means of parameter-dependent Lyapunov functions. As in the previous references, the method is not convex when solving robust control problems because a scalar must be tuned using a line search procedure. Some nice properties of this line search problem are presented in the paper.

From the above scenario, this paper aims to present new LMI conditions for $\mathcal{H}_2$ and $\mathcal{H}_\infty$ analysis of linear uncertain continuous time systems. Similarly to Apkarian et al. (2001) and (Ebihara and Hagiwara, 2002), the proposed approach is convex but in some cases may be more conservative than the usual LMI conditions based on quadratic stability, this behavior relies on the fact that the dilated conditions only recover the standard LMI quadratic stability tests for the nominal case. A strong argument to use the improved conditions instead of the usual LMI tests is its application for designing robust controllers with parameter-dependent Lyapunov functions considering $\mathcal{H}_2$ or $\mathcal{H}_\infty$ specifications and even mixed performance criteria. Aiming to stress that the proposed results can lead to less conservative upper bounds of the performance criteria for uncertain systems, this paper focus on the performance analysis and the results illustrated through exhaustive numerical tests. Basically, the numerical experiments reported in Section 6 reveals that over 16000 randomly generated systems the proposed methodology has achieved a better result in about 83% of cases in the robust $\mathcal{H}_2$ norm against the usual quadratic stability tests and the approach of Apkarian et al. (2001) that surprisingly has got a better result for only one system. The results for the $\mathcal{H}_\infty$ norm of uncertain systems are yet conservative and require further improvements. Section 7 presents the control results for both $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance specifications.

Notation. The notation used throughout this paper is standard. $\mathbb{R}^n$ denotes the set of n-dimensional real vectors, $\mathbb{R}^{n\times m}$ is the set of $n \times m$ real matrices, I_n is the $n \times n$ identity matrix, 0_n is the $n \times n$ matrix of zeros. For a real matrix S, S' denotes its transpose, and $S > 0$ means that S is symmetric and positive-definite. The symbol $\star$ for a block matrix represents its symmetrical block outside the main diagonal. Matrix and vector dimensions are omitted whenever they can be inferred from the context.

2 PROBLEM STATEMENT

Consider the following time-invariant system

$$\mathcal{S} : \left\{ \begin{array}{rcl} \dot{x} & = & Ax + Bw, \\ z & = & Cx + Dw, \end{array} \right. \tag{3}$$

where $x \in \mathbb{R}^n$ denotes the state, $w \in \mathbb{R}^m$ the disturbance input vector, $z \in \mathbb{R}^q$ the performance vector, A, B, C, D are real matrices of compatible dimensions. To represent some system dynamics and parameters that are not precisely known or are difficult to be exactly modelled, suppose the matrices of the system $\mathcal{S}$ can take any value in a given polytope Π as indicated below (Boyd et al., 1994):

$$\Pi = \mathbf{Co}\left\{ \left[\begin{array}{cc} A_i & B_i \\ C_i & D_i \end{array} \right],\ i = 1, \ldots, p \right\} \tag{4}$$

where $\mathbf{Co}\{\cdot\}$ refers to the convex hull of $\{\cdot\}$. For convenience, we may alternatively represent the uncertain system $\mathcal{S}$ by the notation $\mathcal{S} \in \mathbb{S}$ where the set $\mathbb{S}$ is as follows:

$$\mathbb{S} := \left\{ \mathcal{S} \text{ in (3)} : \left[\begin{array}{cc} A & B \\ C & D \end{array} \right] \in \Pi \right\} \tag{5}$$

One can characterize the system performance in terms of input-to-output experiments by means of the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms. When the disturbance signal is impulsive, the $\mathcal{H}_2$-norm is the most indicated (assuming that $D = 0$), on the contrary for square integrable signals the $\mathcal{L}_2$-gain of the input-to-output operator (or simply $\mathcal{H}_\infty$-norm) can be used. For completeness, we provide the following definitions of system norms.

Definition 1 The $\mathcal{H}_2$-norm of system $\mathcal{S}$ is given by

$$\|\mathcal{S}\|_2 \triangleq \sup_{\mathcal{S}\in\mathbb{S}} \sum_{i=1}^{m} \|\bar{z}_i(t)\|_2 \tag{6}$$

where $\bar{z}_i(t)$ is the system response to a unitary impulse in the $i-th$ input channel with $x(0)=0$ and $D=0$.

If the disturbance signal $w(t)$ is a white noise with zero mean value and unitary power density spectra, the $\mathcal{H}_2$-norm may be interpreted as follows

$$\|\mathcal{S}\|_2^2 \triangleq \sup_{\mathcal{S}\in\mathbb{S}} \mathcal{E}(z(t)'z(t))$$

where $\mathcal{E}(z(t)'z(t))$ denotes the mathematical expectation of the random variable $z(t)'z(t)$.

Alternatively, supposing $x(0)=0$, the greatest energy gain that can be obtained from the disturbance signal $w(t)\in\mathcal{L}_2$ to the output $z(t)$ corresponds to the $\mathcal{H}_\infty$-norm of the uncertain system $\mathcal{S}$ leading to the following definition.

Definition 2 The $\mathcal{H}_\infty$-norm of system $\mathcal{S}$ is given by

$$\|\mathcal{S}\|_\infty \triangleq \sup_{\substack{\mathcal{S}\in\mathbb{S},\\ 0\neq w\in\mathcal{L}_2}} \frac{\|z(t)\|_2}{\|w(t)\|_2} \tag{7}$$

where $\mathcal{L}_2$ denotes the space of square integrable vector functions on $[0,\infty)$.

From the above scenario, the problem to be addressed in this paper is to obtain a new set of less conservative LMI conditions to compute the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ system norms in a numerical tractable way.

3 $\mathcal{H}_2$ ANALYSIS

Assuming that $D=0$, a bound on the $\mathcal{H}_2$-norm of system $\mathcal{S}$ can be computed by the following standard result from the LMI theory and the observability Gramian (Boyd et al., 1994).

Lemma 1 *Consider system $\mathcal{S}$ with $D=0$. Suppose there exist symmetric matrices P and N with appropriate dimensions satisfying the follow optimization problem for all $\mathcal{S}\in\mathbb{S}$.*

$$\min_{P,N} \operatorname{trace}(N) : \begin{cases} P>0,\ N-B'PB>0, \\ A'P+PA+C'C<0. \end{cases} \tag{8}$$

Then, $\mathcal{S}$ is quadratically stable and the following holds:

$$\|\mathcal{S}\|_2^2 < \operatorname{trace}(N),\ \forall\,\mathcal{S}\in\mathbb{S}. \tag{9}$$

From above, notice that the decision variable P multiplies the system matrix, and when A is an affine function of another decision variable, as for instance the control-gain K in design control problems, Lemma 1 can be easily applied for control design if the system matrices are perfectly known. However, for the class of systems considered in this paper a robust controller can be obtained only by considering a single Lyapunov matrix P (Shaked, 2001) which may be very conservative. To overcome this problem, an augmented-LMI version of Lemma 1 is proposed in the following.

Theorem 2 *Consider system $\mathcal{S}$ with $D=0$. Suppose there exist positive definite matrices W,N and a free matrix G, with appropriate dimensions, satisfying the following optimization problem for all $\mathcal{S}\in\mathbb{S}$.*

$$\begin{gathered} \min_{W,N,G} \operatorname{trace}(N) : \\ \begin{bmatrix} N & B'G' \\ GB & W \end{bmatrix} > 0, \\ \begin{bmatrix} W+\tilde{A}'G'+G\tilde{A} & \hat{A}'G' & C' \\ G\hat{A} & -W & 0 \\ C & 0 & -\frac{I_q}{2} \end{bmatrix} < 0. \end{gathered} \tag{10}$$

where $\hat{A}=A+I_n$ and $\tilde{A}=A-I_n$.

Then, $\mathcal{S}$ is quadratically stable and the following holds:

$$\|\mathcal{S}\|_2^2 < \operatorname{trace}(N),\ \forall\,\mathcal{S}\in\mathbb{S}. \tag{11}$$

Proof: Suppose there exist W,G,N such that optimization problem (10) is satisfied.

Applying the Schur complement to the first LMI of (10) leads to

$$N - B'G'W^{-1}GB > 0$$

Now, defining the Lyapunov matrix as

$$P \triangleq G'W^{-1}G, \tag{12}$$

we get $N-B'PB>0$.

Consider the second LMI of (10) and applying the Schur complement yields

$$\hat{A}'G'W^{-1}G\hat{A} + (W+\tilde{A}'G'+G\tilde{A}) + 2C'C < 0. \tag{13}$$

From the fact that[1]

$$W+\tilde{A}'G'+G\tilde{A}+\tilde{A}'G'W^{-1}G\tilde{A} \geq 0,$$

condition (13) implies

$$\hat{A}'P\hat{A} - \tilde{A}'P\tilde{A} + 2C'C = 2(A'P+PA+C'C) < 0$$

taking into account the definition of P in (12).

Finally, the rest of the proof follows from Lemma 1. □

Remark 1 The advantage of considering Theorem 2 instead of Lemma 1 is obvious for polytopic systems, since we can employ parameter-dependent Lyapunov matrices in Theorem 2 by only considering that $W=W_i$ (for $i=1,\ldots,p$) in (10).

[1] See, e.g., (de Oliveira et al., 1999) and (Shaked, 2001).

4 $\mathcal{H}_\infty$ ANALYSIS

From the following LMI version of the Bounded Real Lemma (Boyd et al., 1994), an upper bound on the $\mathcal{H}_\infty$-norm of system $\mathcal{S}$ can be determined.

Lemma 3 *Consider system $\mathcal{S}$. Suppose there exist a symmetric matrix P with appropriate dimension and a scalar γ satisfying the follow optimization problem for all $\mathcal{S} \in \mathbb{S}$.*

$$\begin{gathered}\min_{P} \gamma : \ P > 0, \\ \left[\begin{array}{ccc} A'P + PA & PB & C' \\ B'P & -\gamma I_m & D' \\ C & D & -\gamma I_q \end{array}\right] < 0. \end{gathered} \tag{14}$$

Then, $\mathcal{S}$ is quadratically stable and the following holds:

$$\|\mathcal{S}\|_\infty < \gamma, \ \forall \, \mathcal{S} \in \mathbb{S}. \tag{15}$$

From the same arguments of Section 3, the above Lemma may be conservative for control design of polytopic systems and also for multi-objective performance criteria (Apkarian et al., 2001). To overcome this problem, we give the following improved $\mathcal{H}_\infty$ condition.

Theorem 4 *Consider system $\mathcal{S}$ and suppose there exist a positive definite matrix W, a free matrix G and a positive scalar γ satisfying the following optimization problem for all $\mathcal{S} \in \mathbb{S}$.*

$$\begin{gathered}\min_{W,G,\gamma} \gamma : \ W > 0, \\ \left[\begin{array}{ccccc} W_G & GB & \hat{A}'G' & 0 & C' \\ B'G' & -2\gamma I_m & B'G' & 0 & D' \\ G\hat{A} & GB & -W & 0 & 0 \\ 0 & 0 & 0 & -2\gamma I_m & 0 \\ C & D & 0 & 0 & -\frac{\gamma}{2} I_q \end{array}\right] < 0 \end{gathered} \tag{16}$$

where $W_G = W + G\tilde{A} + \tilde{A}'G'$, $\hat{A} = A + I_n$ and $\tilde{A} = A - I_n$.

Then, $\mathcal{S}$ is quadratically stable and the following holds:

$$\|\mathcal{S}\|_\infty < \gamma, \ \forall \, \mathcal{S} \in \mathbb{S}. \tag{17}$$

Proof: Suppose that Theorem 4 is satisfied. Also, for convenience, define the following notation:

$$\begin{aligned} W_a &= \left[\begin{array}{cc} W & 0 \\ 0 & 2\gamma I_m \end{array}\right], \ G_a = \left[\begin{array}{cc} G & 0 \\ 0 & \gamma I_m \end{array}\right] \\ \hat{A}_a &= \left[\begin{array}{cc} \hat{A} & B \\ 0 & 0_m \end{array}\right], \ \tilde{A}_a = \left[\begin{array}{cc} \tilde{A} & B \\ 0 & -2I_m \end{array}\right], \\ A_a &= \left[\begin{array}{cc} A & B \\ 0 & -I_m \end{array}\right], \ C_a = \left[\begin{array}{cc} C & D \end{array}\right], \\ P_a &= G_a' W_a^{-1} G_a = \left[\begin{array}{cc} P & 0 \\ 0 & \frac{\gamma}{2} I_m \end{array}\right] \end{aligned} \tag{18}$$

From above, notice that we can recast (16) as follows

$$\left[\begin{array}{ccc} W_a + G_a\tilde{A}_a + \tilde{A}_a'G_a' & \hat{A}_a'G_a' & C_a' \\ G_a\hat{A}_a & -W_a & 0 \\ C_a & 0 & -\frac{\gamma}{2} I_q \end{array}\right] < 0$$

From Theorem 2, the above implies the following

$$A_a'P_a + P_aA_a + \frac{1}{\gamma} C_a'C_a < 0$$

Finally, the above LMI is equivalent to (14) and the rest of this proof follows from Lemma 3. □

Observe that the 4-th row and column of (16) can be removed, without loss of generality, reducing the size of the LMI.

Remark 2 From the same arguments of Remark 1, the improvement of Theorem 4 over the standard Bounded Real Lemma (BRL) is clear when we consider parameter-dependent Lyapunov matrices by taking $W = W_i$ (for $i = 1, \ldots, p$) in (16).

5 CONSERVATIVENESS

To show the source of conservativeness of the proposed method, consider the following identity:

$$\Phi := 2(A'P + PA) = \hat{A}'P\hat{A} - \tilde{A}'P\tilde{A}. \tag{19}$$

Without loss of generality let us decompose P as $P = G'W^{-1}G$ for some free matrix G and some $W > 0$ to get

$$\begin{aligned} \Phi &= \hat{A}'G'W^{-1}G\hat{A} - \tilde{A}'G'W^{-1}G\tilde{A} && (20) \\ &\leq \hat{A}'G'W^{-1}G\hat{A} + \tilde{A}'G' + G\tilde{A} + W && (21) \end{aligned}$$

Using the last inequality as an upper bound for $2(A'P + PA)$ we get the following LMI with the Schur complement

$$\left[\begin{array}{cc} \tilde{A}'G' + G\tilde{A} + W & \hat{A}'G' \\ G\hat{A} & -W \end{array}\right] < 0 \tag{22}$$

Clearly (22) implies $\Phi < 0$. However, the converse is not true in general. An exception occurs when the choice $G = -W\tilde{A}^{-1}$ is possible. In this case we get

$$\left[\begin{array}{cc} -W & -\hat{A}'(\tilde{A}')^{-1}W \\ -W\tilde{A}^{-1}\hat{A} & -W \end{array}\right] < 0 \tag{23}$$

which in turn yields

$$\begin{gathered}\hat{A}'P\hat{A} - \tilde{A}'P\tilde{A} = 2(A'P + PA) < 0 \\ \text{where } P = (\tilde{A}')^{-1}W\tilde{A}^{-1} \end{gathered} \tag{24}$$

The choices $G = -W\tilde{A}^{-1}$ and $P = (\tilde{A}')^{-1}W\tilde{A}^{-1}$ are possible whenever the matrix A is not uncertain, because otherwise P, G will depend on the same uncertain parameters leading (22) to be non-convex in these parameters. This shows that (22) is equivalent to the usual LMI quadratic stability test $\Phi < 0$ in the nominal case but it may be conservative in the uncertain case. The same conclusion follows from the results in Section 3, i.e. they may be conservative in the uncertain case but not in the nominal case. However, due to the particular structure of the matrices in (18) the relation $G_a = -W_a\tilde{A}_a^{-1}$ is no longer possible which suggests the results in section 4 may have an additional degree of conservatism. This point needs further investigation.

6 NUMERICAL EXAMPLES

This section reports a conservativeness analysis of the proposed improved conditions by means of exhaustive tests over 16000 linear uncertain systems which are randomly generated using Gaussian distribution with zero mean and unitary variance. Let us start with the results from Theorem 2. For each system $\mathcal{S}$, the robust $\mathcal{H}_2$ norm is compute by:

(i) Theorem 2 with parameter-dependent Lyapunov functions (see Remark 1), refereed to as (N);

(ii) The approach of Apkarian et al. (2001), refereed to as (T);

(iii) The standard $\mathcal{H}_2$-norm computation of Lemma 1 (quadratic stability approach) which is referred to as (Q).

The robust methods to compute the $\mathcal{H}_2$-norm can be applied whenever the underlying system is robustly stable. In this sense, it is required the Hurwitz stability of each matrix $\mathcal{A}$ generated from the convex combination of $A_1, \ldots, A_p$ to numerically verify $\|\mathcal{S}\|_2$ by means of a fine grid procedure. In addition, whenever necessary, we have to modify $\mathcal{A}$ so that the Hurwitz stability is guaranteed for all $\mathcal{A} \in \Pi_A \triangleq \mathbf{Co}\{A_1, \ldots, A_p\}$. To minimize the computational burden, such procedure has been applied only to the extreme values of $\mathcal{A}$. Specifically, whenever the randomly generated matrix A_i is not Hurwitz stable, $(\sigma+1)I_n$ has been subtracted from A_i, where σ is the maximum real part of the eigenvalues of A_i. Although the later procedure cannot ensure the Hurwitz stability of $\mathcal{A}$ over Π_A, it can significantly reduce the number of generated systems that are not asymptotically stable over Π_A.

To carry out the statistical tests, we assume four different values for n and p which are respectively the state vector dimension and the vertex number. Precisely, we have taken 1000 systems for each situation $n = 3, \ldots, 6$ and $p = 2, \ldots, 5$, and our attention is focused on comparing:

n	3			4		
p	N	T	Q	N	T	Q
2	984	963	977	991	965	992
3	956	887	948	977	895	879
4	922	819	915	960	793	950
5	893	721	882	924	705	917
n	5			6		
p	N	T	Q	N	T	Q
2	992	958	995	995	960	993
3	869	978	825	991	854	991
4	977	763	977	979	672	987
5	948	611	957	972	466	978

Table 1: Number of systems for which the methods successfully found a solution for the $\mathcal{H}_2$ case

(a) The number of times that each method provides a feasible solution; and

(b) The number of times that each method achieve the best performance.

Table 1 shows the number of systems that lead to feasible solutions to the methods (N), (T) and (Q). Observe that, the new method (N) and the approach (Q) present similar number of feasible solutions. Moreover, one can note that the approach (T) has worse performance.

Table 2 shows the number of systems in which the approaches (N), (T) and (Q) achieved the lowest upper-bound on $\|\mathcal{S}\|_2$. It turns out that the proposed $\mathcal{H}_2$ condition has achieved the best performance in a large majority of cases, and the improvement is even better with the increasing of the number of vertices, even though the standard quadratic stability test demonstrates a better performance in some situations. In addition, the methodology proposed by Apkarian et al. (2001) has outperformed our approach and the quadratic method for only one system.

A similar study is now carried out to analyze the conservativeness of the proposed $\mathcal{H}_\infty$ condition. To this end we solve the robust $\mathcal{H}_\infty$ problem by using the following approaches:

(i) Theorem 4 with parameter-dependent Lyapunov functions, refereed to as (N_∞);

(ii) The standard $\mathcal{H}_\infty$-norm computation of Lemma 3 (quadratic stability approach) which is referred to as (Q_∞),

n	3			4		
p	N	T	Q	N	T	Q
2	745	1	245	741	0	255
3	768	0	182	818	0	167
4	818	0	120	854	0	115
5	801	0	123	835	0	109
n	5			6		
p	N	T	Q	N	T	Q
2	723	0	273	727	0	268
3	825	0	162	830	0	165
4	844	0	141	868	0	123
5	851	0	117	842	0	143

Table 2: The number of systems corresponding to the approach achieving the lowest bound on $\|\mathcal{S}\|_2$.

To the authors' knowledge, there is no other convex approach to robust $\mathcal{H}_\infty$-norm computation in the continuous-time context.

Assuming $n = 5$, $p = 3$ the number of feasible solution over 1000 systems are 985 with the proposed method (N_∞) and 986 with the usual method (Q_∞). The quadratic approach (Q_∞) has achieved the best performance in 542 cases, and the new approach (N_∞) in 303 cases. The same $\mathcal{H}_\infty$ upper bound was obtained with both methods in 141 cases. The situation is similar for other values of p and n. These numerical experiments confirm the claim in the previous section that the results for $\mathcal{H}_\infty$ are yet conservative and need to be improved.

7 STATE-FEEDBACK CONTROL

One of the most advantages of the proposed $\mathcal{H}_2$ and $\mathcal{H}_\infty$ conditions is the possibility of designing a robust state-feedback control law considering parameter-dependent Lyapunov functions. Notice that convex conditions for control design can be easily obtained from the Dual version of Theorems 2 and 4. To extend these theorems for control design, consider $\mathcal{S}$ as defined in (3) with an additional control input $u(t) \in \mathbb{R}^r$, i.e.:

$$\mathcal{S}: \left\{ \begin{array}{rcl} \dot{x} & = & Ax + Bw + Eu, \\ z & = & Cx + Dw + Fu, \\ u & = & Kx, \end{array} \right. \qquad (25)$$

where $K \in \mathbb{R}^{r\times n}$ is the control-gain to be determined, $E \in \mathbb{R}^{n\times r}$, and $F \in \mathbb{R}^{q\times r}$ are uncertain matrices. Similarly to Section 2, the uncertain system $\mathcal{S}$ is represented by the notation $\mathcal{S} \in \mathbb{S}$, where the set $\mathbb{S}$ is redefined accordingly to (25).

7.1 $\mathcal{H}_2$ Design

Setting $D = 0$, a state-feedback control law that stabilizes system $\mathcal{S}$ (while minimizing an upper-bound on its $\mathcal{H}_2$-norm for all $\mathcal{S} \in \mathbb{S}$) can be obtained by means of the dual version of Theorem 2.

The dual version of the improved $\mathcal{H}_2$ condition is devised from the controllability Gramian (Green and Limebeer, 1995) leading to the following result.

Theorem 5 *Consider system $\mathcal{S}$ with $D = 0$. Suppose there exist symmetric matrices W_i, N ($i = 1, \ldots, p$), and non-symmetric ones G, Y with appropriate dimensions satisfying the following optimization problem for all $\mathcal{S} \in \mathbb{S}$.*

$$\min_{W_i,N,G,Y} \operatorname{trace}(N) : \left[\begin{array}{cc} N & CG + FY \\ \star & W_i \end{array} \right] > 0,$$
$$\left[\begin{array}{ccc} \left(\begin{array}{c} W_i + \tilde{A}G + G'\tilde{A}' \\ +EY + Y'E' \end{array} \right) & \star & \star \\ G'\hat{A}' + Y'E' & -W_i & 0 \\ B' & 0 & -\frac{I_m}{2} \end{array} \right] < 0. \qquad (26)$$

where $i = 1, \ldots, p$.

Then, $\mathcal{S}$ is asymptotically stable with

$$u(t) = YG^{-1}x(t),$$

and $\|\mathcal{S}\|_2$ satisfies (11) for all $\mathcal{S} \in \mathbb{S}$.

Proof: The proof of above Theorem is straightforward from the controllability Gramian and the control parametrization $Y = KG$. □

Example 1 Consider a mass/spring/damper system given by the following representation (Zhou, 1998):

$$\dot{x} = A(k_1, k_2)x + Bw + Eu,\ z = Cx + Fu, \qquad (27)$$

where $x = [\ x_1 \quad x_2 \quad x_3 \quad x_4\ ]'$, $k_1 \in [\ 0.8, 1.2\ ]$, $k_2 \in [\ 3.2, 4.8\ ]$, and

$$A = \left[\begin{array}{cccc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k_1 & k_1 & -0.2 & 0.2 \\ \frac{k_1}{2} & -\frac{k_1+k_2}{2} & 0.1 & -0.15 \end{array} \right], B = \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 0.5 \end{array} \right],$$
$$E = \left[\begin{array}{c} 0 \\ 0 \\ 1 \\ 0 \end{array} \right], \quad C = \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right],$$
$$F = \left[\begin{array}{c} 0 \\ 1 \end{array} \right].$$

Applying Theorem 5, we get an upper-bound on the system 2-norm (in closed-loop) of $\lambda = 0.3478$ with a control matrix

$$K = -\left[\ 0.4839 \quad 0.2348 \quad 0.6042 \quad 0.2929\ \right].$$

It turns out that our approach has lead to a better result than the one proposed in (Apkarian et al., 2001) which get an upper-bound of $\lambda = 0.3959$.

7.2 $\mathcal{H}_\infty$ Design

Similarly to the $\mathcal{H}_2$ case, one can obtain a (robust) stabilizing state-feedback controller such that an upper-bound on the $\mathcal{H}_\infty$-norm of $\mathcal{S}$ is minimized by means of a dual version of Theorem 4.

Basically, we redefine the BRL in terms of the Lyapunov matrix inverse and then apply standard results from the matrix theory. These manipulations yields the following result.

Theorem 6 *Consider system $\mathcal{S}$ and suppose there exist symmetric matrices $W_1, \ldots, W_p$, non-symmetric ones, G and Y, and a positive scalar γ satisfying the following optimization problem for all $\mathcal{S} \in \mathbb{S}$.*

$$\min_{W_i, G, \gamma} \gamma : \; W_i > 0,$$
$$\begin{bmatrix} W_{G_i} & \star & \star & 0 & B \\ \Psi_1 & -2\gamma I_q & \star & 0 & D \\ \Psi_2 & \Psi_1' & -W_i & 0 & 0 \\ 0 & 0 & 0 & -2\gamma I_q & 0 \\ B' & D' & 0 & 0 & -\frac{\gamma I_q}{2} \end{bmatrix} < 0 \qquad (28)$$

where $i \in \{1, \ldots, p\}$, and

$$\begin{aligned} W_{G_i} &= W_i + G\tilde{A} + \tilde{A}'G' + YE + E'Y', \\ \Psi_1 &= CG + FY, \\ \Psi_2 &= G'\hat{A}' + Y'E'. \end{aligned}$$

Then, $\mathcal{S}$ is asymptotically stable with

$$u(t) = YG^{-1}x(t),$$

and (17) is satisfied for all $\mathcal{S} \in \mathbb{S}$.

Proof: The proof of above Theorem is straightforward from the dual version of Theorem 4, i.e., by setting $P = P^{-1}, G = G^{-1}, A = A', B = C'$ and $C = B'$, and then applying the control parametrization $Y = KG$. □

Example 2 Consider the following linear time-invariant system borrowed from (Gahinet et al., 1994):

$$\dot{x} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k & k & -f & f \\ k & -k & f & -f \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix},$$

$$z = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0.01 \end{bmatrix} u$$

Approach	BRL	Shaked	Proposed
γ	1.557	1.478	1.498

Table 3: Comparative closed-loop $\mathcal{H}_\infty$-norms.

where $x = [\; x_1 \quad x_2 \quad x_3 \quad x_4 \;]'$, $k \in [\; 0.09, 0.4 \;]$, and $f \in [\; 0.0038, 0.04 \;]$.

The problem to be addressed is to design a state-feedback law $u = Kx$ such that the closed-loop system is robustly stable while minimizing an upper-bound on the worst-case $\mathcal{H}_\infty$-norm of the above system.

Applying Theorem 6, we get an upper-bound $\gamma = 1.498$ with a control-gain given by

$$K = -\begin{bmatrix} 51.523 & 399.593 & 22.331 & 664.714 \end{bmatrix}.$$

Table 3 shows a comparative study among the standard Bounded Real Lemma (BRL) in (Boyd et al., 1994), the improved LMI test of (Shaked, 2001, Lemma 3.1), and our approach. In spite of the fact that our approach seems to be more conservative than the Shaked's one, we stress the fact that our approach is convex while in (Shaked, 2001) a parameter ϵ has to be optimized.

8 CONCLUDING REMARKS

This paper have proposed improved $\mathcal{H}_2$ and $\mathcal{H}_\infty$ conditions for continuous-time linear systems with polytopic uncertainties. Basically, the performance conditions are built on an augmented-space with additional multipliers resulting in a decoupling between the Lyapunov and system matrices. This nice property can be used to design robust state-feedback controllers with parameter dependent Lyapunov functions taking into account both $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms. Statistical numerical tests have proven the advantage of the proposed $\mathcal{H}_2$ approach over previous results from the robust control literature. In addition, we have presented the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ conditions for state-feedback design also considering parameter-dependent Lyapunov functions and a robust control law. Future research will concentrate on improving the $\mathcal{H}_\infty$ results that are yet conservative and extend the results for filtering and control problems.

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