STABILITY ANALYSIS OF TAKAGI-SUGENO FUZZY SYSTEMS VIA LMI: METHODOLOGIES BASED ON A NEW FUZZY LYAPUNOV FUNCTION

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RESUMO

Análise da estabilidade de sistemas fuzzy de Takagi-Sugeno via LMI: Metodologia baseda numa nova função de Lyapunov fuzzy
A análise de estabilidade de sistemas fuzzy TS pode ser aprimorada com o uso de funções de Lyapunov fuzzy, uma vez que as mesmas são parametrizadas por funções de pertinência e podem definir melhor a ca-racterística variante no tempo de tais sistemas através do uso da informação relacionada à primeira derivada temporal das funções de pertinência. Neste trabalho uma função de Lyapunov fuzzy aperfeiçoada é usada com o intuito de se desenvolver condições de estabilidade que avaliam também a segunda derivada temporal das funções de pertinência, aprimorando a caracterização do aspecto variante no tempo de sistemas TS. Novos testes no formato de LMIs são desenvolvidos usando diferentes estratégias para incorporar tais derivadas e empregando algumas ferramentas numéricas que desacoplam as matrizes do sistema daquelas da função de Lyapunov fuzzy. Exemplos numéricos ilustram a eficácia dessas metodologias.

PALAVRAS-CHAVE: sistemas fuzzy Takagi-Sugeno, Desigualdades Matriciais Lineares, Análise de Estabilidade

ABSTRACT

Stability analysis of TS fuzzy systems can be much improved by resorting to fuzzy Lyapunov functions, since they are parameterized by membership functions and can better characterize the time-varying feature of these systems by means of the information regarding the first time-derivative of the membership functions. In this paper an enhanced fuzzy Lyapunov function is used to develop stability conditions that evaluate also the second time-derivative of membership functions, improving the time-varying characterization of TS systems. By using different strategies to consider the information regarding such derivatives and employing some numerical tools that decouple system from Lyapunov function matrices new LMI tests are developed. Numerical examples illustrate the effectiveness of those methodologies.

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KEYWORDS: Takagi-Sugeno fuzzy systems, Linear Matrix Inequalities, Stability analysis

1 INTRODUCTION

Stability analysis and control design for Takagi-Sugeno fuzzy systems (Takagi and Sugeno, 1985) have been routinely formulated as feasibility and optimization problems in LMI (Linear Matrix Inequalities) form (Tanaka and Wang, 2001). Recently powerful SOS formulations have been proposed by Tanaka et al. (2009) enhancing numerical performance. Nonetheless, a source of conservativeness that remains is the choice of an appropriate candidate Lyapunov function.

For some time it has been noticed that parameterized Lyapunov functions are a good choice to deal with time-varying systems, deserving attention some seminal works on this subject (Gahinet et al., 1996; Feron et al., 1996; Fierro et al., 1996). The idea is to use an affine combination of quadratic functions ($x^T Px$), which individually are not necessarily Lyapunov, parameterized by uncertain time-varying parameters. This combination leads to a Lyapunov function. The fuzzy Lyapunov functions proposed in Jadbabaie (1999) and Rhee and Won (2006) fall into this category but the parameterization is given in terms of the membership functions.

The reason because this parametrization reduces conservativeness is twofold. To guarantee stability in the Lyapunov sense the information regarding the first time-derivative of the parameters must be taken into account. This is an advantage in comparison with quadratic stability where regardless the time-varying feature of a system stability is imposed to it for arbitrary rates of change. In other words, quadratic stability considers only polytopic characteristic. Another reason is that using multiple Lyapunov functions more degrees of freedom are granted for the LMI problem whereas for quadratic stability a single Lyapunov function is available.

The fuzzy Lyapunov function in Jadbabaie (1999) and in Tanaka et al. (2003) posses both these features whereas the function used in Rhee and Won (2006) and in Mozelli, Palhares, Avellar and dos Santos (2010) does not. Rhee and Won (2006) proposed a line integral fuzzy Lyapunov function, similar to those obtained through the variable gradient method for nonlinear systems (Haddad and Chellaboina, 2008), that does not rely on the first time-derivative of the membership functions. Although it is a multiple Lyapunov function, which is an improvement and includes quadratic stability as a particular case, it can be noticed that for some systems the missing information regarding parameter variation can produce more conservative results (Mozelli, Palhares, Souza and Mendes, 2009; Mozelli, Palhares and Avellar, 2009).

Motivated by the relevance inherent to information regarding the time-derivative of the membership functions a new Lyapunov function is employed in this paper parameterized not only by the membership functions but also by the first time-derivative of them. By doing that the information concerning the second time-derivative of the membership functions becomes available to describe better the time-varying characteristic of TS systems. The new function was previously presented in a conference paper (Mozelli and Palhares, 2010) and good results were achieved. In this paper extra numerical tools are employed to reduce conservativeness even more as the computational effort is kept low. Numerical examples are performed to illustrate the advantages in using this new kind of function together with the strategies discussed in Mozelli, Palhares and Mendes (2010).

1.1 Notation

The notation used throughout is standard. The superscript ^{T} indicate transposition of vectors and matrices; for matrices $M > 0$ (≥ 0) indicates that M is positive definite (nonnegative definite); $M_{(i,j)}$ denotes the i-th line and j-th column element; the subsets $\{1, 2, \ldots, r\} \subset \mathbb{N}^*$, $\{1, 2, \ldots, p\} \subset \mathbb{N}^*$ and $\{1, 2, \ldots, m\} \subset \mathbb{N}^*$ are represented by $\mathcal{R}$, $\mathcal{P}$ and $\mathcal{M}$, respectively.

2 FUNDAMENTS ON TS MODELS

Takagi-Sugeno (TS) dynamic fuzzy systems are described by means of a set of fuzzy rules (Tanscheit et al., 2007; Teixeira and Assunção, 2007)

$$\begin{aligned} R_i: \quad & \text{IF} \;\; z_1(t) \text{ is } \mathcal{M}_1^i \text{ and } \cdots \text{ and } z_p(t) \text{ is } \mathcal{M}_p^i \\ & \text{THEN} \;\; \dot{x}(t) = A_i x(t) \end{aligned} \tag{1}$$

where the state vector is $x(t) \in \mathbb{R}^n$; the number of local models and fuzzy rules R_i is given by r; A_i are real matrices of appropriate dimension; for this model, the premise variables vector is $z(t) = [z_1(t) \cdots z_p(t)]$ and the input sets are indicated by $\mathcal{M}_j^i, i \in \mathcal{R}, j \in \mathcal{P}$.

By using a standard fuzzy inference method, that is, using a singleton fuzzifier, and center-average defuzzifier the model can be represented in a compact form as (Feng, 2006):

$$\dot{x}(t) = A(h)x(t), \tag{2}$$

with

$$A(h) \triangleq \sum_{i=1}^{r} h_i(z(t))A_i,\ h = [h_1(z(t)) \ \cdots \ h_r(z(t))],$$

where $h_i(z(t))$ are the normalized membership functions:

$$h_i(z(t)) \triangleq \frac{\omega_i(z(t))}{\sum_{i=1}^{r}\omega_i(z(t))},$$

with $\omega_i(z(t))$ given by product fuzzy inference

$$\omega_i(z(t)) \triangleq \prod_{j=1}^{p} \mu_j^i(z_j(t)).$$

The grades of membership of the premise variables in the respective fuzzy sets $\mathcal{M}_j^i$ are given as $\mu_j^i(z_j(t))$. The normalized membership functions satisfy the following properties

$$\begin{aligned} h_i(z(t)) \in [0,1], &\quad \sum_{i=1}^{r} h_i(z(t)) = 1, \\ \sum_{i=1}^{r} \dot{h}_i(z(t)) = 0, &\quad \sum_{i=1}^{r} \ddot{h}_i(z(t)) = 0. \end{aligned} \tag{3}$$

From now on to avoid clutter the time dependency of some variables is dropped out. For instance, $h_i(z(t))$ is replaced by h_i.

3 FUZZY LYAPUNOV FUNCTION PARAMETERIZED BY MEMBERSHIP FUNCTIONS AND THEY TIME-DERIVATIVES

The fuzzy Lyapunov function proposed in Mozelli and Palhares (2010) is given by a double parametrization, using affine combinations of quadratic functions. One set is parameterized by the membership functions and another parameterized by they first time-derivatives:

$$V(x,h,\dot{h}) = x^T \left[P^1(h) + P^2(\dot{h})\right] x, \tag{4}$$

where

$$P^1(h) \triangleq \sum_{i=1}^{r} h_i P_i^1,\ P^2(\dot{h}) \triangleq \sum_{i=1}^{r} \dot{h}_i P_i^2. \tag{5}$$

Notice that the proposed function is indeed a Lyapunov one since

- $V(x,h,\dot{h}) \in \mathcal{C}^1$, case $h_i(z(t)) \in \mathcal{C}^2$
- $V(0,h,\dot{h}) = 0$
- $\|x(t)\| \rightarrow \infty \Rightarrow V(x,h,\dot{h}) \rightarrow \infty$ (radially unbounded)

It remains to ensure that $V(x,h,\dot{h}) > 0, \forall x(t) \neq 0$ being sufficient to meet the following conditions

$$P_i^1 + \sum_{k=1}^{r} \dot{h}_k P_k^2 > 0,\ \forall i \in \mathcal{R} \tag{6}$$

3.1 On the time-derivative of the membership functions

There is more than one way to recast conditions involving the time-derivatives of the membership functions into the LMI framework. The simplest but more conservative form is to consider the worst case scenario taking only the upper bounds, as done in Tanaka et al. (2003).

Due to the properties (3), the time-derivative of the membership functions belong into the convex polytope defined by

$$\begin{aligned} \Omega_\omega &\triangleq \mathrm{co}\{v^1, v^2, \ldots, v^m\} \\ &= \{v^j \in \mathbb{R}^r | -\phi_k^\omega \leq v_k^j \leq \phi_k^\omega, c^T v^j = 0\} \end{aligned} \tag{7}$$

where $c^T = [1,1,\ldots,1] \in \mathbb{R}^r$, k being the k-th coordinate of v^j and $|\dot{h}_k| \leq \phi_k^1$ and $|\ddot{h}_k| \leq \phi_k^2$.

Therefore it is possible to include the information regarding time-derivatives of the membership functions in a less conservative form using a finite number of vectors that indicate the vertices of the polytopes Ω_1 and Ω_2, as done by Geromel and Colaneri (2006) or by Chesi et al. (2009).

In this paper the following matrix is employed to accumulate the vectors that satisfy (7)

Table 1: Number of vertices m according with the dimension r for the politope Ω: exponential growth.

r	2	3	4	5	6	7	8
m	2	6	6	30	20	140	70

$$W^{\omega} \triangleq \left[v^1, \quad v^2, \quad \cdots, \quad \begin{pmatrix} v_1^j \\ v_2^j \\ \vdots \\ v_r^j \end{pmatrix}, \quad \cdots, \quad v^m \right]. \quad (8)$$

Although this approach is less conservative than taking only upper bounds[1] the price paid is an exponential growth in the number of vertices. An closed equation for this growth is hard to be obtained but Table 1 provides the relation between vertices and rules for some values. The results in that Table are obtained using the algorithm proposed in Lara et al. (2009), which is able to provided the vertices of (7), a convex polytope generated by the intersection between the hyperplane $\sum_{k=1}^{r} \dot{h}_k = 0$ and the hyperrectangle imposed by the bounds on the time-derivatives, for instance the hypercube $|\dot{h}_k| \leq \phi^1$.

4 STABILITY CONDITIONS AND LMI TESTS

In this section stability conditions based on the new Lyapunov function are given. Then finite dimension LMI tests are provided based on such conditions using the approaches discussed so far concerning the time-derivative of the membership functions.

The time-derivative of the function (4) with respect to the trajectories of the TS fuzzy system (2) is given by

$$\begin{aligned} \dot{V}(x,h,\dot{h}) = x^T \Big[& \dot{P}^1(h) + \dot{P}^2(\dot{h}) + A^T(h)P^1(h) + \\ & A^T(h)P^2(\dot{h}) + P^1(h)A(h) + P^2(\dot{h})A(h) \Big] x \\ = {} & x^T \Theta(h,\dot{h},\ddot{h}) x \qquad (9) \end{aligned}$$

To ensure stability in the Lyapunov sense the following conditions must hold

$$P^1(h) + P^2(\dot{h}) > 0, \qquad (10)$$

$$\Theta(h,\dot{h},\ddot{h}) < 0 \qquad (11)$$

As discussed in Section 3.1, due to the presence of the time-derivative of the membership functios, there are some ways to recast these conditions into the LMI framework. The first LMI test proposed, established in the following Theorem, is obtained based on the use of upper bounds for these time-derivatives.

Theorem 1 *Let $|\dot{h}_i| < \phi_i^1, |\ddot{h}_i| < \phi_i^2 \ \forall \ i \in \mathcal{R}$. The TS system (2) is asymptotically stable if there exists symmetric matrices P_i^1, P_i^2, X and Y that satisfy*

$$P_i^2 + Y > 0, \ i \in \mathcal{R}, \qquad (12)$$

$$P_i^1 - \sum_{k=1}^{r} \phi_k^1 \left(P_k^2 + Y \right) > 0, \ i \in \mathcal{R}, \qquad (13)$$

$$P_k^1 + A_i^T P_k^2 + P_k^2 A_i + X > 0, \ i,k \in \mathcal{R}, \qquad (14)$$

$$\begin{aligned} & \sum_{k=1}^{r} \phi_k^2 \left(P_k^2 + Y \right) + \frac{1}{2} \big(\Theta_i + \Theta_j + A_i^T P_j^1 + P_j^1 A_i \\ & + A_j^T P_i^1 + P_i^1 A_j \big) < 0, \ i,j \in \mathcal{R}, \ i \leq j \qquad (15) \end{aligned}$$

with

$$\Theta_i \triangleq \sum_{k=1}^{r} \phi_k^1 \left[P_k^1 + A_i^T P_k^2 + P_k^2 A_i + X \right] \qquad (16)$$

Proof: The demonstration that $V(x,h,\dot{h}) > 0$ is given first. As proposed by (Mozelli, Palhares, Souza and Mendes, 2009), based on (3) it follows that

$$\sum_{k=1}^{r} \dot{h}_k Y = Y \sum_{k=1}^{r} \dot{h}_k = Y(0) = 0, \qquad (17)$$

where Y is any symmetric matrix.

Using this term and relying in the fact that $|\dot{h}_i| \leq \phi_i^1$ with constraints (12) satisfied results in

[1] This can be easily checked for 2 rules. From (3) it follows that $\dot{h}_1 = -\dot{h}_2$. In this case whenever a time-derivative of a membership function reaches it maximum value the other reaches the minimum.

$$P^1(h)+P^2(\dot h)=P^1(h)+\sum_{i=1}^{r}\dot h_i P_i^2 \quad (18)$$

$$=P^1(h)+\sum_{i=1}^{r}\dot h_i\left(P_i^2+Y\right) \quad (19)$$

$$\geq P^1(h)-\sum_{i=1}^{r}\phi_i^1\left(P_i^2+Y\right) \quad (20)$$

Based on the convexity imposed by h_i, to yeld (10) it is sufficient that constraints (13) hold. Back to condition (11), since (12) is satisfied, $|\ddot h_i|\leq\phi_i^2$ and using terms such (17) it follows that

$$\begin{aligned}\Theta(h,\dot h,\ddot h)&=\sum_{k=1}^{r}\ddot h_k P_k^2+A^T(h)P(h)+P(h)A(h)\\&+\sum_{k=1}^{r}\dot h_k\left[P_k+A^T(h)P_k^2+P_k^2A(h)\right]\\&=\sum_{k=1}^{r}\ddot h_k\left(P_k^2+Y\right)+A^T(h)P(h)+P(h)A(h)\\&+\sum_{k=1}^{r}\dot h_k\left[P_k+A^T(h)P_k^2+P_k^2A(h)+X\right]\\&\leq\sum_{k=1}^{r}\phi_k^2\left(P_k^2+Y\right)+A^T(h)P(h)+P(h)A(h)\\&+\sum_{k=1}^{r}\dot h_k\left[P_k+A^T(h)P_k^2+P_k^2A(h)+X\right]\\&=\bar\Theta(h,\dot h,\ddot h) \qquad (21)\end{aligned}$$

Guaranteeing (14) and using the fact $|\dot h_i|<\phi^1$ one has

$$\begin{aligned}\bar\Theta(h,\dot h,\ddot h)&\leq\sum_{k=1}^{r}\phi_k^2\left(P_k^2+Y\right)+A^T(h)P(h)+P(h)A(h)\\&+\sum_{k=1}^{r}\phi_k^1\left[P_k+A^T(h)P_k^2+P_k^2A(h)+X\right]\\&=\sum_{k=1}^{r}\phi_k^2\left(P_k^2+Y\right)+\sum_{i=1}^{r}\sum_{j=1}^{r}h_ih_j\Big\{A_i^TP_j\\&+P_jA_i+\sum_{k=1}^{r}\phi_k^1\left[P_k+A_i^TP_k^2+P_k^2A_i+X\right]\Big\}\\&=\sum_{k=1}^{r}\phi_k^2\left(P_k^2+Y\right)\\&+\sum_{i=1}^{r}\sum_{j=1}^{r}h_ih_j\left\{\Theta_i+A_i^TP_j+P_jA_i\right\} \qquad (22)\end{aligned}$$

with Θ_i defined in (16). Therefore to guarantee (11) it is sufficient to have (15), concluding the proof. □

Corollary 2 *Theorem 6 in Mozelli, Palhares, Souza and Mendes (2009) is a particular case of Theorem 1.*

Proof: By setting $Y=0$ and $P_i^2=0$, $\forall i\in\mathcal{R}$ the constraints in Theorem 1 are the same in Theorem 6 in Mozelli, Palhares, Souza and Mendes (2009). □

Another LMI test is stated in the following Theorem using the vertices of the polytopes (7) that contain the time-derivatives of the membership functions instead of using only their upper bounds.

Theorem 3 *Let $|\dot h_i|<\phi_i^1, |\ddot h_i|<\phi_i^2\ \forall\ i\in\mathcal{R}$. The TS system (2) is asymptotically stable if there exists symmetric matrices P_i^1 and P_i^2 that satisfy*

$$P_i^1+\sum_{k=1}^{r}W_{(k,q)}^1P_k^2>0,\ i\in\mathcal{R},\ q\in\mathcal{M}, \qquad (23)$$

$$\begin{aligned}&\sum_{k=1}^{r}W_{(k,l)}^2\bar P_k+\frac{1}{2}\left[\Theta_i+\Theta_j+A_i^TP_j^1+P_j^1A_i\right.\\&\left.+A_j^TP_i^1+P_i^1A_j\right]<0,\ {\scriptstyle i,j\in\mathcal{R},\ i\leq j,\ l,q\in\mathcal{M}}, \qquad (24)\end{aligned}$$

where

$$\Theta_i\triangleq W_{(k,q)}^1\left[P_k^1+A_i^TP_k^2+P_k^2A_i\right] \qquad (25)$$

with matrices W^1, W^2, having m columns, defined in (8).

Proof: To guarantee (10) it is sufficient to explore the convexity of h_i resulting in (6). Since $\dot{h}_i$ belongs to (7) which is also convex, then it is sufficient to meet the constraints (23). The second time-derivatives of the membership functions are also confined to (7) and by using similar arguments it suffices to have (24), concluding the proof. □

There is another way to include the time-derivative of a time-varying parameter, as proposed in Geromel and Colaneri (2006). However Oliveira et al. (2009) discuss how this task can be difficult as successive time-derivatives are considered. Another alternative, present in the conference version of this paper (Mozelli and Palhares, 2010), is to combined both forms employed so far to include the time-derivative of the membership functions. In Mozelli and Palhares (2010) the upper bounds are the choice for the second time-derivative whereas for the first time-derivative convexity of the polytope in which they lie is explored.

Nonetheless numerical tools can still be applied to improve Theorems 1 and 3 in terms of performance without gaging computational time. The idea is to use one of the approaches discussed in Mozelli, Palhares and Mendes (2010) that are able to decouple system matrices from Lyapunov function matrices reducing conservativeness.

The following LMI test is an improvement with respect to Theorem 1.

Theorem 4 *Let $|\dot{h}_i| < \phi_i^1, |\ddot{h}_i| < \phi_i^2\ \forall\ i \in \mathcal{R}$. The TS system (2) is asymptotically stable if there exists symmetric matrices P_i^1, P_i^2, X and Y and any matrices M_1 and M_2 that satisfy (12) to (14) and*

$$\Xi_i < 0,\ i \in \mathcal{R}, \tag{26}$$

with

$$\Xi_i \triangleq \begin{bmatrix} \sum_{k=1}^{r} \phi_k^2 (P_k^2 + Y) + \Theta_i - M_1 A_i - A_i^T M_1^T & \bullet \\ P_i - M_2 A_i + M_1^T & M_2 + M_2^T \end{bmatrix} \tag{27}$$

and Θ_i defined as in (16).

Proof: The following equation can be verified from TS system dynamics (2)

$$2\left[x^T M_1 + \dot{x}^T M_2\right] \times \left[\dot{x} - A(h)x\right] = 0, \tag{28}$$

and it is used to obtain the improved conditions.

The time-derivative (9) can be rewritten as

$$\dot{V}(x,h,\dot{h}) = \xi^T \begin{bmatrix} \dot{P}^1(h) + \dot{P}^2(\dot{h}) + A^T(h) P^2(\dot{h}) + P^2(\dot{h}) A(h) & \bullet \\ P^1(h) & 0 \end{bmatrix} \xi \tag{29}$$

with $\xi^T = [x^T\ \dot{x}^T]$.

Satisfying the constraints (12) and (13) condition (10) is covered. Besides if constraint (14) is satisfied, using terms as in (17) and taking into account $|\dot{h}_i| < \phi_i^1$, $|\ddot{h}_i| < \phi_i^2$ results in

$$\dot{V}(x,h,\dot{h}) \leq \xi^T \sum_{i=1}^{r} h_i \begin{bmatrix} \sum_{k=1}^{r} \phi_k^1 \left(P_k^1 + A_i^T P_k^2 + P_k^2 A_i + X\right) + \sum_{k=1}^{r} \phi_k^2 \left(P_k^2 + Y\right) & \bullet \\ P_i^1 & 0 \end{bmatrix} \xi \tag{30}$$

Finally, with the sum of the null term (28) this inequality remains unaltered

$$\dot{V}(x,h,\dot{h}) \leq \xi^T \sum_{i=1}^{r} h_i \begin{bmatrix} \sum_{k=1}^{r} \phi_k^2 \left(P_k^2 + Y\right) + \Theta_i - M_1 A_i - A_i^T M_1 & \bullet \\ P_i^1 - M_2 A_i + M_1^T & M_2 + M_2^T \end{bmatrix} \xi \tag{31}$$

where Θ_i is defined in (16). Therefore, it is sufficient that (26) hold, concluding the proof. □

Corollary 5 *Theorem 1 in Mozelli, Palhares and Avellar (2009) is a particular case of Theorem 4.*

Proof: By setting $Y = 0$ and $P_i^2 = 0$, $\forall i \in \mathcal{R}$ the constraints in Theorem 4 are the same in Theorem 1 in Mozelli, Palhares and Avellar (2009). □

Theorem 3 can also be improved by resorting to the same numerical tool.

Theorem 6 *Let $|\dot{h}_i| < \phi_i^1, |\ddot{h}_i| < \phi_i^2\ \forall\ i \in \mathcal{R}$. The TS system (2) is asymptotically stable if there exists symmetric matrices P_i^1, P_i^2 and any matrices M_1, M_2 that satisfy (23) and*

$$\Xi_i^{q,l} < 0,\ i \in \mathcal{R},\ q,l \in \mathcal{M} \tag{32}$$

where

$$\Xi_i^{q,l} \triangleq \begin{bmatrix} \sum_{k=1}^{r} \left[W_{(k,q)}^1 P_k^1 + W_{(k,l)}^2 P_k^2\right] -M_1 A_i - A_i^T M_1^T & \bullet \\ \sum_{k=1}^{r} W_{(k,q)}^1 P_k^2 + P_i^1 - M_2 A_i + M_1^T & M_2 + M_2^T \end{bmatrix} \tag{33}$$

Proof: This demonstration follows the same line as for Theorem 3. Condition (10) is guaranteed if (23) is satisfied. The time-derivative (9) can be rewritten as

$$\begin{aligned}\dot{V}(x,h,\dot{h}) &= \xi^T \begin{bmatrix} \dot{P}^1(h) + \dot{P}^2(\dot{h}) & \bullet \\ P^1(h) + P^2(\dot{h}) & 0 \end{bmatrix} \xi \\ &= \xi^T \sum_{i=1}^{r} h_i \begin{bmatrix} \sum_{k=1}^{r} \dot{h}_k P_k^1 + \ddot{h}_k P_k^2 & \bullet \\ P_i^1 + \sum_{k=1}^{r} \dot{h}_k P_i^2 & 0 \end{bmatrix} \xi \end{aligned} \tag{34}$$

Adding the null term and then exploring the fact that h_i and $\dot{h}_i$ are convex, constraints (32) are sufficient, thus concluding the proof. □

Remark 1 The comparison among the proposed Theorems reveals that the number of LMI lines reduces as the numerical tool is employed. As a compromise the number of scalar variables and the size of each LMI raise. The total balance among these effects in terms of computational demand can be seen in Table 2. Notice that there is a small reduction in the numbers of LMI lines and a small increase in the number of variables. Therefore the improvements achieved by the proposed conditions do not reflect as more computational effort and in some cases can even reduce it. Bottom line is that Theorems 1 (3) and 4 (6) have the same complexity order.

Table 2: Numerical complexity of the proposed conditions: L number of lines; V number of scalar variables; n system order; r number of rules; m number of columns in (8).

	L	V
Th.1	$(1{,}5r^2 + 2{,}5r)n$	$(n^2 + n)(r + 2)$
Th.4	$(r^2 + 4r)n$	$(n^2 + n)(r + 2) + 2n^2$
Th.3	$(r^2 + r)nm^2/2 + rnm$	$(n^2 + n)r$
Th.6	$(m^2 + m)rn$	$(n^2 + n)r + 2n^2$

5 NUMERICAL TESTS

In this section numerical examples are performed to illustrate the main features and effectiveness of the proposed LMI tests to check stability of TS systems.

Example 1 Consider the following nonlinear system (Tanaka et al., 2003; Tanaka et al., 2009)

$$\begin{aligned}\dot{x}_1(t) &= -\left(\frac{7}{2} + \frac{3}{2}\sin(x_1(t))\right) x_1(t) - 4x_2(t) \\ \dot{x}_2(t) &= \left(\frac{19}{2} - \frac{21}{2}\sin(x_1(t))\right) x_1(t) - 2x_2(t)\end{aligned}$$

By means of the sector nonlinearity approach (Ohtake et al., 2001) an exact TS model can be obtained whose local models or vertices are

$$A_1 = \begin{bmatrix} -5 & -4 \\ -1 & -2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -2 & -4 \\ 20 & -2 \end{bmatrix}.$$

and the membership functions are

$$h_1 = \frac{1 + \sin(x_1)}{2}, h_2 = \frac{1 - \sin(x_1)}{2}$$

Although many trajectories convergent to the origin are observed (see the phase portrait and some trajectories in Figure 1), indicating stability, none quadratic Lyapunov function exists for this system (Tanaka et al., 2003; Tanaka et al., 2009).

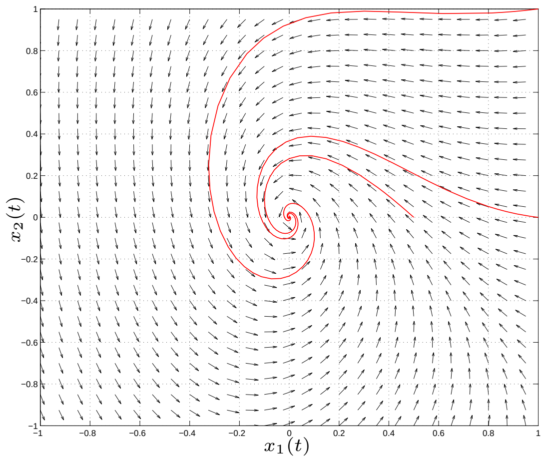

Figure 1: Phase portrait for the nonlinear system of Example 1.

Lyapunov functions that certificate stability for the system described in Example 1 are sought using the proposed LMI tests. To this end the upper bounds must be selected. Using a grid procedure into the space (x_1, x_2) with a step of 0.1, considering the universe of discourse $|x_1| \leq \pi/2$, $|x_2| \leq \pi/2$, the maximum values of the first and second time-derivatives of h_1, given respectively as:

$$\dot{h}_1 = -\frac{\cos(x_1)}{2}\begin{bmatrix}1 & 0\end{bmatrix}\sum_{i=1}^{2} h_i A_i x \tag{35}$$

$$\begin{aligned}\ddot{h}_1 = &-\frac{\cos(x_1)}{2}\begin{bmatrix}1 & 0\end{bmatrix}\left[\sum_{i=1}^{2}\dot{h}_i A_i x - \left(\sum_{i=1}^{2} h_i A_i\right)^2 x\right]\\ &+\frac{\sin(x_1)}{2}\left(\begin{bmatrix}1 & 0\end{bmatrix}\sum_{i=1}^{2} h_i A_i x\right)^2\end{aligned} \tag{36}$$

could be easily computed as 3.56 as and 99.93. Thus with a safe margin the choices for upper bounds are $\phi^1 = 3.60$ and $\phi^2 = 100$. The maximum value for wich the methodology in Tanaka et al. (2003) can guarantee stability is $\phi^1 = 2.57$. The polytopes in which the time-derivatives of the membership functions lie have 2 vertices (they are lines). Thus for Theorems 3 and 6 $m = 2$ and

$$W^1 = \begin{bmatrix}+3.60 & -3.60\\ -3.60 & +3.60\end{bmatrix}, \quad W^2 = \begin{bmatrix}+100 & -100\\ -100 & +100\end{bmatrix}$$

The following matrices are obtained

$$\text{Th. 1: } P_1^1 = \begin{bmatrix}1.2297 & 0.0346\\ 0.0346 & 0.8584\end{bmatrix}, P_2^1 = \begin{bmatrix}2.3058 & 0.0373\\ 0.0373 & 0.5917\end{bmatrix},$$
$$P_1^2 = \begin{bmatrix}2.0402 & -0.2968\\ -0.2968 & 0.0015\end{bmatrix}, P_2^2 = \begin{bmatrix}2.0393 & -0.2978\\ -0.2978 & 0.0006\end{bmatrix}$$

$$\text{Th. 3: } P_1^1 = \begin{bmatrix}0.2292 & 0.0276\\ 0.0276 & 0.1802\end{bmatrix}, P_2^1 = \begin{bmatrix}0.4211 & 0.0186\\ 0.0186 & 0.0970\end{bmatrix},$$
$$P_1^2 = 10^{-3}\begin{bmatrix}0.1108 & 0.7318\\ 0.7318 & 0.5271\end{bmatrix}, P_2^2 = 10^{-12}\begin{bmatrix}0.0153 & -0.1131\\ -0.1131 & 0.0408\end{bmatrix}$$

$$\text{Th. 4: } P_1^1 = \begin{bmatrix}1.1071 & 0.1181\\ 0.1181 & 0.7006\end{bmatrix}, P_2^1 = \begin{bmatrix}1.8104 & 0.0550\\ 0.0550 & 0.4702\end{bmatrix},$$
$$P_1^2 = \begin{bmatrix}1.7686 & -0.2631\\ -0.2631 & 0.0013\end{bmatrix}, P_2^2 = \begin{bmatrix}1.7680 & -0.2639\\ -0.2639 & 0.0006\end{bmatrix}$$

$$\text{Th. 6: } P_1^1 = \begin{bmatrix}0.3143 & 0.0455\\ 0.0455 & 0.1912\end{bmatrix}, P_2^1 = \begin{bmatrix}0.4925 & 0.0206\\ 0.0206 & 0.1170\end{bmatrix},$$
$$P_1^2 = 10^{-3}\begin{bmatrix}0.4645 & 0.2907\\ 0.2907 & 0.0947\end{bmatrix}, P_2^2 = 10^{-14}\begin{bmatrix}0.9039 & 0.2009\\ 0.2009 & -0.1173\end{bmatrix}$$

An interesting feature of fuzzy Lyapunov functions is shown in Figures 2 to 7. The quadratic functions that are parameterized either by the membership functions or by they time-derivatives do not need to be Lyapunov themselves, only the fuzzy combination does. In Figures 2 and 3 the individual quadratic functions $V_i^j(x) \triangleq x^T P_i^j x$ are shown for Theorem 1. The same functions are shown in Figures 5 and 6 for Theorem 3. Notice how the decay is not monotonic for some functions considered individually.

Figures 4 and 7 portrait the fuzzy Lyapunov functions given by Theorems 1 and 3, respectively, obtained through the combination

$$\begin{aligned}V(x,h,\dot{h}) = x^T(t)\Big\{&h_1(x_1(t))P_1^1 + h_2(x_1(t))P_2^1\\ &+\dot{h}_1(x_1(t))P_1^2 + \dot{h}_2(x_1(t))P_2^2\Big\}x(t),\end{aligned} \tag{37}$$

whose decay is monotonic. Therefore they are indeed Lyapunov functions.

Finally, the level curves of these functions are shown together with the phase portrait of the system in Figures 8 and 9. Notice that the level curves are convex and that the system trajectory goes from one contour to another innermost. These level curves are not perfect ellipsoids as one would expect in the case of quadratic stability. The parametrization by the membership functions and they time-derivatives allow the appearance of different convex curves that adapt to the nonlinear dynamic.

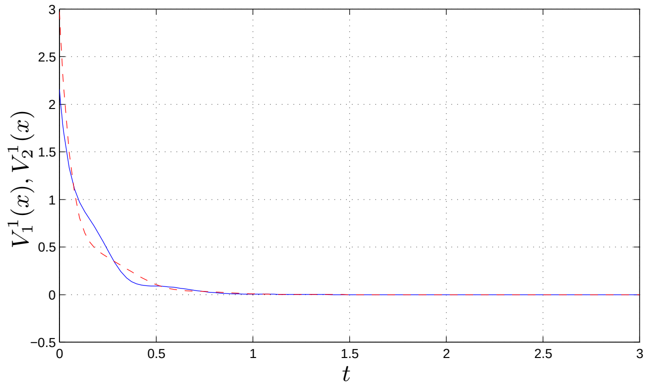

Figure 2: Time evolution of each quadratic function considered apart: solid line $V_1^1(x)$; dashed line: $V_2^1(x)$. Functions obtained with Theorem 1. Initial conditions $x = [1\ 1]^T$.

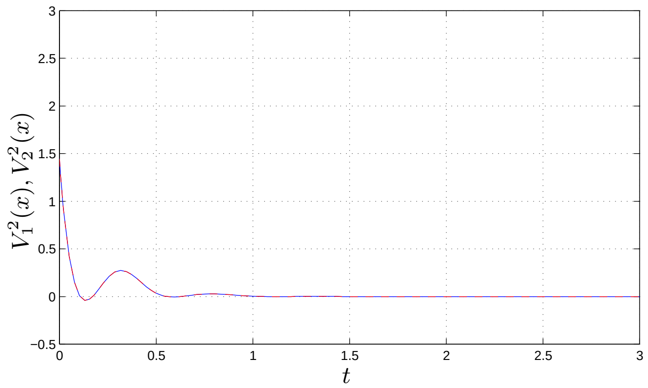

Figure 3: Time evolution of each quadratic function considered apart: solid line $V_1^2(x)$; dashed line: $V_2^2(x)$. Functions obtained with Theorem 1. Initial conditions $x = [1\ 1]^T$.

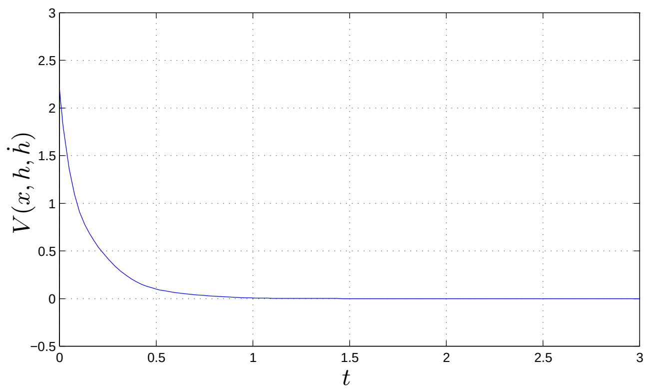

Figure 4: Time evolution of the fuzzy Lyapunov function. Function obtained with Theorem 1. Initial conditions $x = [1\ 1]^T$. Notice the monotonic decay.

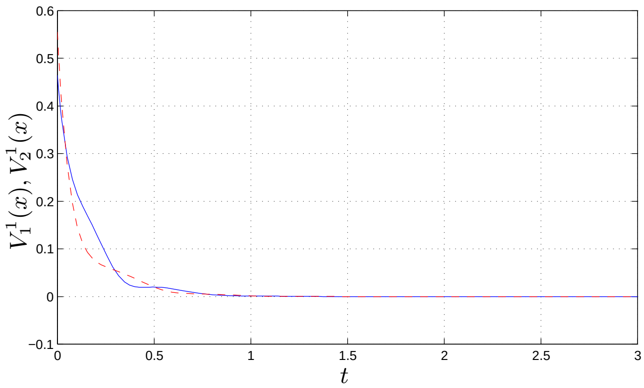

Figure 5: Time evolution of each quadratic function considered apart: solid line $V_1^1(x)$; dashed line: $V_2^1(x)$. Functions obtained with Theorem 3. Initial conditions $x = [1\ 1]^T$.

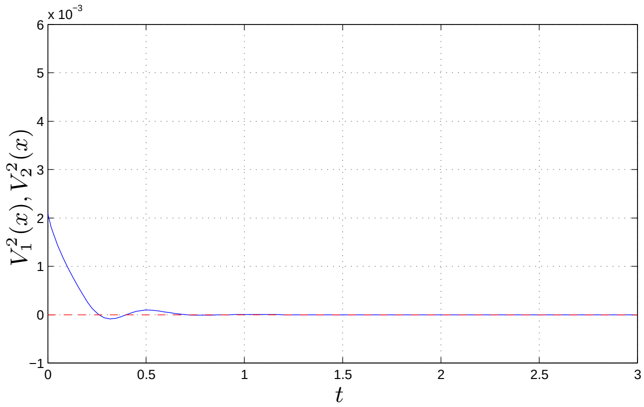

Figure 6: Time evolution of each quadratic function considered apart: solid line $V_1^2(x)$; dashed line: $V_2^2(x)$. Functions obtained with Theorem 3. Initial conditions $x = [1\ 1]^T$.

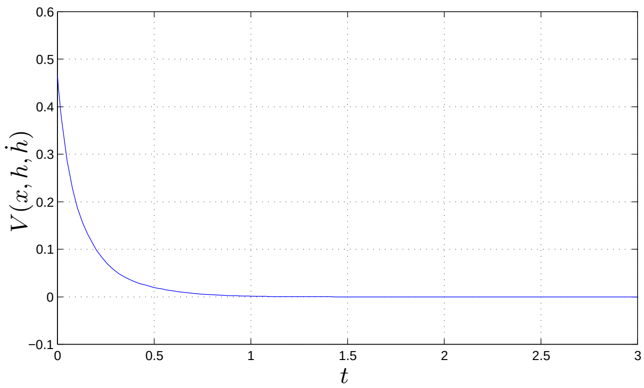

Figure 7: Time evolution of the fuzzy Lyapunov function. Function obtained with Theorem 3. Initial conditions $x = [1\ 1]^T$. Notice the monotonic decay.

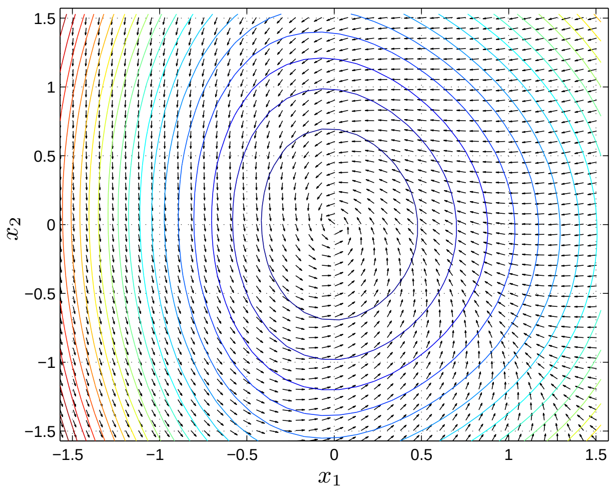

Figure 8: Level curve for the fuzzy Lyapunov functions obtained with Theorem 1.

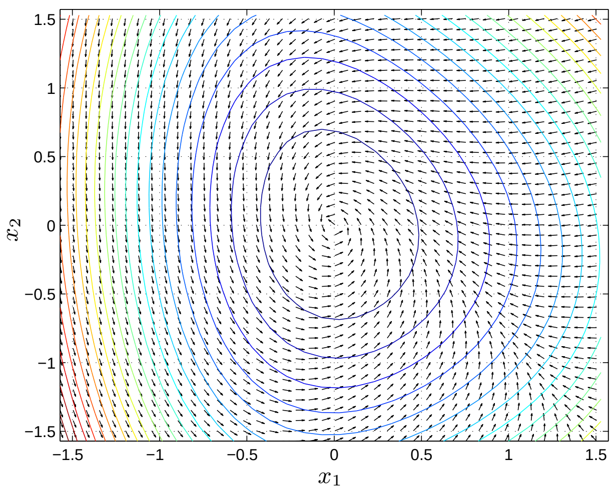

Figure 9: Level curve for the fuzzy Lyapunov functions obtained with Theorem 3.

Example 2 Consider a second order nonlinear system

$$\begin{aligned}\dot{x}_1(t) &= x_2(t)\\ \dot{x}_2(t) &= -2x_1(t) - x_2(t) - f(t)x_1(t),\end{aligned} \qquad (38)$$

where $f(t) \in [0, k]$ is a function that is $\mathcal{C}^1$.

The dynamics in (38) can be exactly modeled by a two rule TS fuzzy system

$$A_1 = \begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix},\ A_2 = \begin{bmatrix} 0 & 1 \\ -2-k & -1 \end{bmatrix},$$

considering as membership functions

$$h_1 = \frac{k - f(t)}{k}, \quad h_2 = \frac{f(t)}{k}.$$

The goal is to find the maximum value of parameter k, namely $k^\star$, for which stability is guaranteed for some bounds on the first and second time-derivative of the membership functions, respectively ϕ^1 and ϕ^2.

Example 2 intends to illustrate the role of the second time-derivative of the membership functions in the proposed LMI tests. Toward this end the new tests are compared with conditions based on the fuzzy Lyapunov function parameterized only by the membership functions (Tanaka et al., 2003), which includes only the information of the first time-derivative. Also there is a comparison with the quadratic Lyapunov function which does not include information regarding time-derivatives of the membership functions whatsoever.

Figure 10 shows in solid line (blue line) the values of $k^\star$ obtained with Theorem 6 in Mozelli, Palhares, Souza and Mendes (2009) for several values of ϕ^1. It also depicts in dash-dotted line (green line) the results obtained by Teixeira et al. (2003) and Montagner et al. (2009), with $g = d = 5$. Since they do not depend on the derivatives of the membership functions, they produce a straight line giving $k^\star = 3.82$ for any ϕ^1. In dashed lines are show the results produced by Theorem 1 for different values of ϕ^2. The improvement achieved by the new test in this example is quite clear, specially for fast changing systems, i.e., those who posses a bigger upper bounds of the first time-derivative indicating that the swicht between rules is more intense. As the value of ϕ^1 is reduced the results of Theorem 6 in Mozelli, Palhares, Souza and Mendes (2009) and Theorem 1 become closer.

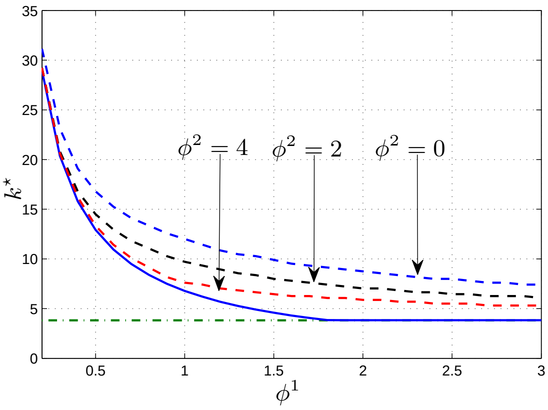

Figure 10: Stability analysis for several Lyapunov functions. Without information regarding time-derivatives of membership functions: dash-dotted (green) line; depending only on the first time-derivative: solid (blue) line; depending on the first and second time-derivatives, Theorem 1: dashed lines.

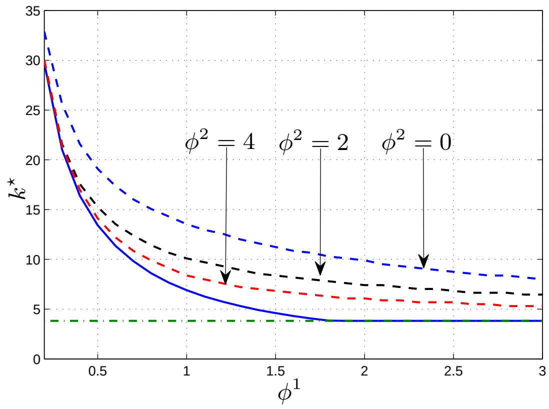

Figure 11: Stability analysis for several Lyapunov functions. Without information regarding time-derivatives of membership functions: dash-dotted (green) line; depending only on the first time-derivative: solid (blue) line; depending on the first and second time-derivatives, Theorem 4: dashed lines.

Figure 11 portraits the comparison between Theorem 1 in Mozelli, Palhares and Avellar (2009) and Theorem 4. The same behavior can be seen again, with a clear advantage for the proposed test over the condition that uses only the first-time derivative of the membership functions. Notice however that with Theorem 4 larger values of $k^\star$ are produced than with Theorem 1. The reason is the use of the numerical tools discussed in Mozelli, Palhares and Mendes (2010), that decouple the system from the Lyapunov function matrices.

Membership functions are responsible for blending a given number of vertices A_i in order to match this combination with a specific nonlinear dynamics, generating an exact or approximate TS model. As discussed in Sala (2009) when stability conditions do not take into account the explicit form of the membership functions and consider only its convex characteristic, as occurs with quadratic or polynomial Lyapunov functions, stability is guaranteed not only to the original nonlinear system and its TS counterpart. The solution is suitable for other nonlinear systems that share the same vertices but are modeled with distinct membership functions satisfying (3). This is a reason why parameterized Lyapunov functions are better than quadratic stability. Quadratic stability concerns only the vertices neglecting membership functions and they time-derivatives. Therefore stability must be guaranteed even for arbitrarily large rates of change. The use of a parameterized Lyapunov function restricts the family of systems for each stability is investigated to those who posses a specific upper bound for the first time-derivative of the membership functions. A great advantage of the proposed function is the fact that group of systems for which stability is investigated is further reduced. They must not only meet a certain upper bound for the first time-derivative but another bound for the second time-derivative of the membership functions. For fuzzy Lyapunov functions parameterized only by the membership functions stability must be guaranteed even for arbitrarily large rates of change of the second time-derivative of the membership functions. This not happens with the proposed function. This is the reason why the performance of Theorem 1 approaches the one of Theorem 6 in Mozelli, Palhares, Souza and Mendes (2009) as the upper bound of the second time-derivative raises. The same with Theorem 4 against Theorem 1 in Mozelli, Palhares and Avellar (2009).

As a final comparison the results of all proposed conditions are shown together in Figure 12 for $\phi^2 = 0$. In the dotted solid line (blue) it is shown the result of Theorem 1 the more conservative among they. This is justified since no numerical tools are employed in Theorem 1 and only the upper bounds of the time-derivatives are considered. In dashed line (green) appears the result obtained with Theorem 6, the least conservative. The explanation for this is the opposite for the performance of Theorem 1, since in Theorem 6 numerical tools are employed and the time-derivative informations are included in a less conservative manner, relying in the convexity of the polytopes (7). Theorem 3 against Theorem 4 there is no clear winner. For low and intermediate values of ϕ^1 Theorem 4, signed with dotted line (black), is better. However as ϕ^1 raises Theorem 3 represented by solid line (red) becomes better.

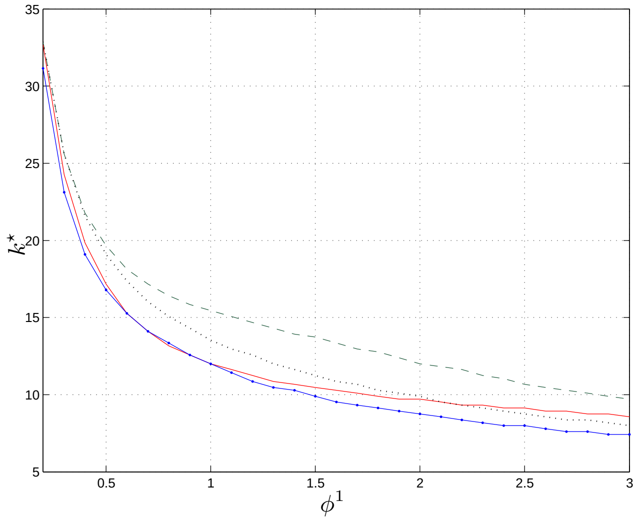

Figure 12: Stability analysis depending on second time-derivatives of the membership functions: solid dotted line (blue) Theorem 1; solid line (red) Theorem 3; dotted line (black) 4; dashed line (green) Theorem 6.

6 CONCLUSION

As this paper has shown, the use of more information regarding the time-varying feature of TS fuzzy system is beneficial for stability analysis. New LMI tests based on the information regarding the second time-derivative of the membership functions were devised which improve stability analysis in comparison with standard parameterized Lyapunov functions. Some conditions proposed in the recent literature can be viewed as particular cases. By resorting to a numerical tool capable to decouple systems and function matrices improved conditions were obtained with respect to the conference version of this paper keeping computational effort in the same order of complexity.

7 ACKNOWLEDGEMENTS

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