

Monitoring a wandering mean with an np_x chart

Linda Lee Ho^{a,*}, Antonio Fernando Branco Costa^b

^{a,*}lindalee@usp.br, EPUSP, Brasil

^bfbranco@feg.unesp.br, UNESP, Brasil

Abstract

This article considers the np_x chart proposed by Wu et al. (2009) to control the process mean, as an alternative to the use of the \bar{X} chart. The distinctive feature of the np_x chart is that sample units are classified as first-class or second-class units according to discriminating limits. The standard np chart is a particular case of the np_x chart, where the discriminating limits coincide with the specification limits and the first (second) class unit is the conforming (nonconforming) one. Following the work of Reynolds Junior, Arnold and Baik (1996), we assume that the process mean wanders even in the absence of any specific assignable cause. A Markov chain approach is adopted to investigate the effect of the wandering behavior of the process mean on the performance of the np_x chart. In general, the np_x chart requires samples twice larger (the standard np chart requires samples five or six times larger) to outperform the \bar{X} chart.

Keywords

Discriminating limits, np_x chart. First order autoregressive model AR (1). Quality control. Attribute and variable control chart. Attribute inspection.

1. Introduction

Control charts are designed to detect assignable causes that may occur in production processes. When the traditional \bar{X} chart is used, there is an implicit assumption that the process mean is a variable that can assume only two values: its in-control value and a second value given by the in-control value plus a shift resulting of the assignable cause occurrence. However, in some situations, it may be more realistic to assume that the process mean wanders, even in the absence of any specific assignable cause. Following the work of Reynolds Junior, Arnold and Baik (1996) and Lu and Reynolds (1999a, 1999b, 2001), we consider that the wandering behavior of the process mean fits to a first-order autoregressive AR (1) model. They used a complex approach involving Markov chains and integral equation methods to study the properties of the \bar{X} chart. This method is commonly used to evaluate the properties of the EWMA and CUSUM charts. Lin and Chou (2008), Zou, Wang and Tsung (2008), and Lin (2009), respectively, applied this method to study the \bar{X} charts with variable sampling rates, with variable sampling schemes at fixed times, and with variable parameters. We propose a simpler approach, named the pure Markov approach (COSTA; MACHADO, 2011),

to compare the \bar{X} chart and the np_x chart proposed by Wu et al. (2009), in terms of the speed with which they signal.

As an alternative to the use of the \bar{X} chart, we investigate the performance of the np_x chart in signaling changes in the position of a wandering process mean. This chart signals when there are more than m units, classified as second-class units, among the n units that make up the sample. A unit is dichotomously classified as a first or a second-class unit if its X value falls close to or far away from μ_0 , the target value of the process mean, according to discriminating limits. The standard np chart is a particular case of the np_x chart, where the discriminating limits coincide with the specification limits and the first (second) class unit is the conforming (nonconforming) one. According to Wu et al. (2009), a second-class unit is not necessarily defective; consequently, the np_x chart often provides an indication of impending trouble and allows operators to take corrective action before any defective is actually produced. The idea of monitoring the process mean with a chart for attributes was also explored by Wu and Jiao (2008). Their chart signals when the interval between two suspect samples is

smaller than a pre-specified value. A suspect sample is the one with more than m out of n units in the sample classified as second-class units.

This paper is organized as follows: the assumptions and the description of the np_x chart are in section 2; the speed with which the np_x chart and the \bar{X} chart signal is discussed in section 3. Finally, the example and final remarks are drawn respectively in sections 4 and 5.

2. The np_x chart

When the np_x chart is in use for monitoring a quality characteristic X , samples of size n_0 are chosen every h_0 hours, and their units are dichotomously classified as first-class or second-class units. A first (second) class unit has its X value close to (far away from) μ_0 , the target value of the process mean, according to discriminating limits. Figure 1 presents a "GO/NO GO" ring gage that dichotomously classifies the shafts. A first (second) class shaft has its X radius value falling inside (outside) the interval (LDL, UDL); being LDL the Lower Discriminating Limit and UDL the Upper Discriminating Limit.

The np_x chart with UCL - Upper Control Limit = m , signals when there are more than m units, classified as second-class units, among the n units that make up the sample. The standard np chart is a particular case of the np_x chart, where the discriminating limits coincide with the specification limits and the first (second) class unit is the conforming (nonconforming) one.

The model proposed by Reynolds Junior, Arnold and Baik (1996) is used to investigate the effect of the wandering behavior of the process mean μ on the performance of the np_x chart. According to this model, μ_i can be expressed in terms of μ_{i-1} :

$$\mu_i = (1 - \phi)\xi + \phi\mu_{i-1} + \alpha_i \quad i = 1, 2, \dots \quad (1)$$

where $\alpha_i \sim N(0; \sigma_\alpha^2)$ and ϕ is the correlation between μ_{i-1} and μ_i , which are, respectively, the values of the process mean when the $(i - 1)$ th and i th samples are

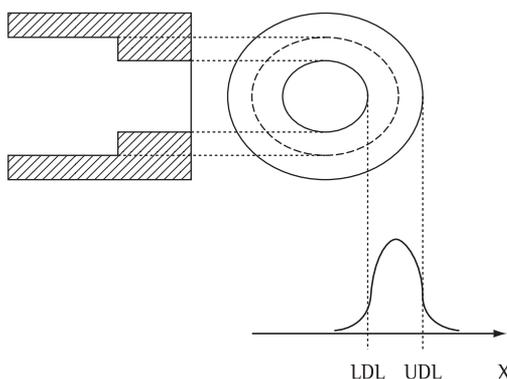


Figure 1. Go/No Go Ring Gage.

taken. According to Reynolds Junior, Arnold and Baik

(1996), if $\mu_0 \sim N(\xi; \sigma_\mu^2)$ then $\sigma_\mu^2 = \frac{\sigma_\alpha^2}{(1 - \phi^2)}$. If the time

to obtain a sample is negligible relative to the time between samples, then the j th observation of the i th sample can be written as

$$X_{ij} = \mu_i + e_{ij} \quad i = 1, 2, 3, \dots, \text{ and } j = 1, 2, \dots, n_i \quad (2)$$

where $e_{ij} \sim N(0; \sigma_e^2)$ is the j th random error at sampling time t_j . The j th unit of the i th sample has probability p_i ($1 - p_i$) of being a second (first) class unit:

$$p_i = 1 - \Pr[LDL < X_{ij} < UDL | X_{ij} \sim N(\mu_i; \sigma_e^2)] \quad (3)$$

When the np_x chart is in use, the monitoring statistic is Y - the number of second-class units encountered in each sample. As the chart signals whenever Y exceeds the Upper Control Limit (UCL),

$$p = \Pr[Y > UCL] = 1 - \sum_{y=0}^{UCL} \binom{n}{y} p_i^y (1 - p_i)^{n-y} \quad i = 1, 2, 3, \dots \quad (4)$$

is the signaling probability.

The effect of the correlation, among observations of the process mean, on the performance of the np_x chart is evaluated in terms of ϕ and σ_α^2 . Without loss of generality, we assume $\sigma_e^2 = 1.0$.

3. Performance of the np_x chart

When the interval between samples is fixed, the average run length (ARL) is the parameter used to assess the chart's performance (MONTGOMERY, 2005). Before the assignable cause occurrence, $\xi = \mu_0$, the ARL is named ARL_0 . The ARL_0 measures the average number of samples between false alarms (MONTGOMERY, 2005). After the assignable cause occurrence, $\xi = \mu_0 + \delta\sigma_x$, and the ARL, named ARL_1 , measures the average number of samples the control chart requires to signal a $\delta\sigma_x$ shift in the position of the process mean.

When the process mean wanders, the simplest way to study the chart's performance is building a Markov chain that allows us to express the ARL as a function of the expected number the transient states are visited. The value of the process mean at each sampling time is required to define the states of the Markov chain. Thus, to deal with a finite chain, the process mean μ is discretized in w values: $\mu_1, \mu_2, \dots, \mu_w$. We obtain accurate ARLs with $w = 50$. If $Y_i \leq UCL$, being Y_i the value of the monitoring statistic corresponding to the i th sample, then $\mu(i + 1)$, the process mean value when the $(i + 1)$ th sample is taken, defines the transient state of the Markov chain. If $\mu_{i+1} = \mu_k$, then the state (k) is reached, with $k \in \{1, 2, \dots, w\}$. The absorbing state is reached whenever $Y_i > UCL$. The matrix of transient probabilities is given by:

$$Q = [p(r, s)], r, s \in \{1, 2, \dots, w\}$$

where $p(r,s)$ is the transient probability of being in state (r) to reach the state (s) in one step. As the process mean was discretized in w values: $\mu_1, \mu_2, \dots, \mu_w$, with $\mu_j = \mu_0 - [(w - 1) - 2(j - 1)]\Delta$ and $\Delta = 7\sigma_\mu(w - 1)^{-1}$, it follows that

$$p(r,s) = \Pr\{Y_r < UCL \mid X \sim N(\mu_r; \sigma_e^2)\} \times \Pr\left\{\left|\alpha - \mu_s + \phi\mu_r\right| < \Delta \mid \alpha \sim N\left[0; (1 - \phi^2)\sigma_\mu^2\right]\right\} \tag{5}$$

Let $\mathbf{b} = (q_1, q_2, \dots, q_w)$ with $q_i = \Pr\{|\mu - \mu_i| < \Delta \mid \mu \sim N(\mu_0; \sigma_\mu^2)\}$ be the vector of initial probabilities and $\mathbf{t}' = (1, 1, \dots, 1)$, then:

$$ARL = \mathbf{b}'(\mathbf{I} - \mathbf{Q})^{-1} \mathbf{t}$$

To obtain the properties of the \bar{X} chart the expression of $p(r,s)$ should be modified:

$$p(r,s) = \Pr\left\{\left|\bar{X} - \mu_0\right| < L\sigma_{\bar{X}} \mid \bar{X} \sim N\left(\mu_r; \frac{\sigma_e^2}{n}\right)\right\} \times \Pr\left\{\left|\alpha - \mu_s + \phi\mu_r\right| < \Delta \mid \alpha \sim N\left[0; (1 - \phi^2)\sigma_\mu^2\right]\right\} \tag{6}$$

The wandering behavior of the process mean inflates the standard deviation of the sample means

by $f(n, \psi) = \sqrt{1 + \frac{n\psi}{(1-\psi)}}$, where $\psi = \frac{\sigma_\mu^2}{\sigma_X^2} = \frac{\sigma_\mu^2}{(\sigma_\mu^2 + \sigma_e^2)}$.

That is, $\sigma_{\bar{X}} = f(n, \psi) \frac{\sigma_e}{\sqrt{n}}$.

Table 1. The ARL values of the np_x chart ($n = 6$).

		UCL				
		1	2	3	4	5
δ	UDL	2.301	1.720	1.280	0.876	0.432
	0	370.61	370.63	372.29	370.42	371.9
	0.25	115.95	100.14	94.21	93.21	99.90
	0.5	41.2	32.55	29.37	29.25	33.21
	0.75	16.71	12.5	11.22	11.32	13.45
	1	7.76	5.75	5.20	5.32	6.52
	1.25	4.13	3.13	2.88	2.98	3.70
	1.5	2.51	1.99	1.87	1.95	2.41

Table 2. The Effect of the Sample Size on the np_x chart's performance.

		n		
		6	12	24
δ	UCL	3	7	13
	UDL	1.280	0.831	0.750
	0	372.29	371.96	371.67
	0.25	94.21	64.93	44.30
	0.50	29.37	16.20	9.17
	0.75	11.22	5.64	3.11
	1.00	5.20	2.65	1.60
	1.25	2.88	1.61	1.15
	1.50	1.87	1.22	1.03

Table 1 was built to study the speed, measured by the ARL_x , with which the np_x chart signals changes in the position of a wandering mean. It is considered $n = 6$; $\sigma_\alpha^2 = 0.2$; $\phi = 0.2$; ARL_0 close to 370; $LDL = 0$; $UCL = 1, 2, \dots, 5$ and changes of magnitude $\delta\sigma_x$, with $\delta = 0.25; 0.5; 0.75; 1.0; 1.25$ and 1.5 . After choosing the UCL , a search is undertaken to obtain the Upper Discriminating Limit (UDL) that leads to an ARL_0 close to 370. For example, if $UCL = 2$, the search ends with $UDL = 1.720$, alternatively, if $UCL = 3$, it follows that $UDL = 1.280$. The np_x chart reaches its better overall performance with $UCL = 3$.

Table 2 was built to study the effect of the sample size on the np_x chart's performance. The chart's parameters are the same adopted in Table 1, except the sample sizes that are considered equal to 6, 12 and 24. As expected, the np_x chart signals faster with larger samples.

Table 3. The effect of σ_α^2 and ϕ on the np_x chart's performance.

ϕ		σ_α^2		
		0	0.2	0.4
0	UCL	3	3	3
	UDL	1.136	1.275	1.563
	0	371.41	370.62	370.93
	0.25	74.11	93.39	100.53
	0.50	22.46	28.97	37.40
	0.75	8.54	11.01	15.13
	1.00	4.02	5.09	7.14
	1.25	2.31	2.82	3.90
	1.50	1.58	1.85	2.45
	0.2	UCL	3	3
UDL		1.12	1.280	1.577
0		371.7	372.29	371.05
0.25		68.19	94.21	109.81
0.50		20.97	29.37	38.34
0.75		8.08	11.22	15.72
1.00		3.86	5.20	7.50
1.25		2.25	2.88	4.12
1.50		1.55	1.87	2.57
0.4		UCL	3	3
	UDL	1.023	1.295	1.627
	0	371.28	371.55	371.93
	0.25	41.99	95.43	113.66
	0.50	14.11	30.17	40.91
	0.75	5.92	11.66	17.22
	1.00	3.06	5.43	8.36
	1.25	1.91	3.00	4.60
	1.50	1.40	1.93	2.83
	0.6	UCL	3	3
UDL		0.695	1.335	1.745
0		372.6	371.13	370.77
0.25		10.51	99.82	122.79
0.50		4.71	32.62	47.16
0.75		2.59	12.92	20.89
1.00		1.71	6.06	10.46
1.25		1.31	3.31	5.79
1.5		1.13	2.08	3.50

Table 4. Comparing the np_x and \bar{X} charts throughout their ARL values.

	$\sigma_\alpha^2 = 0.2$		n				
	$\phi = 0.2$	6		12		24	
		np_x	\bar{X} chart	np_x	\bar{X} chart	np_x	\bar{X} chart
UCL	3	2.78	7	2.78	13	2.78	
UDL	1.28		0.831		0.75		
δ	0	372.29	370.45	371.96	370.44	371.67	370.44
	0.25	94.21	76.07	64.93	50.55	44.3	34.43
	0.50	29.37	20.53	16.2	10.78	9.17	6.15
	0.75	11.22	7.19	5.64	3.49	3.11	2.04
	1.00	5.20	3.23	2.65	1.70	1.60	1.20
	1.25	2.88	1.85	1.61	1.19	1.15	1.03
	1.50	1.87	1.32	1.22	1.04	1.03	1.00

Table 3 was built to study the effect of the σ_α^2 and ϕ on the np_x chart's performance. As σ_α^2 increases the ARL₁ values also increase. The same is observed when ϕ increases, except for $\sigma_\alpha^2 = 0$. The wandering behavior of the process mean reduces the ability of the np_x chart in signaling. For instance, when the process mean is fixed (that is, $\phi = 0$ and $\sigma_\alpha^2 = 0$) the np_x chart needs, on average, 8.54 samples to signal a process mean shift of $0.75\sigma_x$; if the process mean wanders (for example, with $\phi = 0.4$ and $\sigma_\alpha^2 = 0.4$) this number increases to 17.22 (more than 100%), see Table 3.

Table 4 was built to compare the \bar{X} chart with the np_x chart in terms of the speed they signal changes in the position of a wandering process mean. According to Table 4, the np_x chart requires larger samples to compete with the \bar{X} chart. However, it is simpler and faster to deal with the np_x chart. To compete with the \bar{X} chart, the np_x chart requires samples approximately twice larger. When the proportion of the X variability attributed to the wandering behavior of the process mean increases, that is σ_μ^2 increases, the np_x chart requires larger samples to compete with the \bar{X} chart, see Table 5. The same occurs when σ_α^2 increases.

4. An Example

Wu and Jiao (2008) describe a real field experiment where four operators inspect the diameters X of shafts with nominal value $\mu_0 = 8.00$ mm. Regarding to the attribute inspection, the operators use a simple Mitutoyo ring gage, regarding to the variable inspection, they use a more delicate Mitutoyo digital micrometer to measure the diameters. The average time spent on an individual attribute inspection is 2.125 seconds, against 9.525 seconds, if by variable. The inspection costs are considered to be a linear function of the inspection time. Consequently, in a fair comparison the size of the samples inspected by attribute might be at least four times bigger than the size of the samples inspected by variable. According to

Table 5. np_x and \bar{X} charts with similar performance.

ϕ	Control chart	σ_α^2				
		0.2		0.4		
		np_x	\bar{X} chart	np_x	\bar{X} chart	
0.2	n	10	6	11	6	
	UCL	5	2.78	6	2.78	
	UDL	1.094		1.309		
	δ	0	372.84	370.45	370.47	370.44
		0.25	72.52	76.07	91.99	96.74
		0.50	19.14	20.53	28.55	30.61
		0.75	6.77	7.19	10.96	11.62
		1.00	3.13	3.23	5.11	5.24
		1.25	1.83	1.85	2.84	2.81
		1.50	1.32	1.32	1.84	1.79
0.4	n	11	6	14	6	
	UCL	5	2.78	7	2.78	
	UDL	1.199		1.43		
	δ	0	372.82	370.42	370.52	370.42
		0.25	71.75	76.72	92.27	99.22
		0.50	18.95	20.52	29.11	31.59
		0.75	6.73	6.99	11.39	11.79
		1.00	3.10	3.05	5.35	5.15
		1.25	1.80	1.74	2.94	2.66
		1.50	1.29	1.26	1.87	1.67
0.6	n	13	6	50	6	
	UCL	7	2.78	25	2.78	
	UDL	0.987		1.431		
	δ	0	371.07	370.46	371.85	370.43
		0.25	71.01	78.35	94.15	104.77
		0.50	19.06	20.68	30.73	34.07
		0.75	6.87	6.76	12.35	12.56
		1.00	3.14	2.85	5.78	5.29
		1.25	1.79	1.62	3.05	2.63
		1.50	1.27	1.21	1.85	1.62

Table 5, the np_x chart requires samples approximately twice larger to compete with the \bar{X} chart. We can explore this example in our study once the inspection costs are not affected by the wandering behavior of the process mean.

5. Final Remarks

In this article we compared the speed with which the np_x chart and the \bar{X} chart signal changes in the position of the process mean. The process mean is considered to wander and its successive values are modeled by a first-order autoregressive time series model. In general, the np_x chart requires samples twice larger (the standard np chart requires samples five or six times larger) to outperform the \bar{X} chart. In earlier studies, a complex approach, encompassing Markov chains and numerical quadrature to solve integral equations, were necessary to obtain the properties of the control charts (REYNOLDS JUNIOR; ARNOLD; BAIK, 1996). The aim of this paper was to present ARL as a function of the number of times the transient states of a Markov chain are visited.

This study might be extended to situations where the assignable cause not only changes the mean position but also increases the process variance. In this case, the np_x chart is compared with the joint \bar{X} and S or R charts.

References

- COSTA, A. F. B.; MACHADO, M. A. G. Variable parameter and double sampling \bar{X} charts in the presence of correlation: the Markov chain approach. *International Journal of Production Economics*, v. 130, n. 2, p. 224-229, 2011.
- LIN, Y. C. The variable parameters control charts for monitoring autocorrelated processes. *Communications in Statistics - Simulation and Computation*, v. 38, p. 729-749, 2009.
- LIN, Y. C.; CHOU, C. Y. The variable sampling rate \bar{X} control charts for monitoring autocorrelated processes. *Quality and Reliability Engineering International*, v. 24, p. 855-870, 2008.
- LU, C. W.; REYNOLDS, M. R. EWMA control charts for monitoring the mean of autocorrelated processes. *Journal of Quality Technology*, v. 31, p. 166-188, 1999a.
- LU, C. W.; REYNOLDS, M. R. Control chart for monitoring the mean and variance of autocorrelated processes. *Journal of Quality Technology*, v. 31, p. 259-274, 1999b.
- LU, C. W.; REYNOLDS, M. R. Cusum charts for monitoring an autocorrelated process. *Journal of Quality Technology*, v. 33, p. 316-334, 2001.
- MONTGOMERY, C. *Introduction to Statistical Quality Control*. New York: Wiley, 2005.
- REYNOLDS JUNIOR, M. R.; ARNOLD, J. C.; BAIK, J. W. Variable sampling interval \bar{X} charts in the presence of correlation. *Journal of Quality Technology*, v. 28, n. 1, p. 12-29, 1996.
- WU, Z.; JIAO, J. A control chart for monitoring process mean based on attribute inspection. *International Journal of Production Research*, v. 46, n. 15, p. 4331-4347, 2008.
- WU, Z. et al. An np control chart for monitoring the mean of a variable based on an attribute inspection. *International Journal of Production Economics*, v. 121, p. 141-147, 2009.
- ZOU, C.; WANG, Z.; TSUNG, F. Monitoring autocorrelated processes using variable sampling schemes at fixed-times. *Quality and Reliability Engineering International*, v. 24, p. 55-69, 2008.

Acknowledgements

The authors would like to acknowledge CNPq for the financial support of this research. We also thank to the anonymous referees of this paper and Professor Philippe Castagliola, the organizer of the First International Symposium on Statistical Process Control (ISSPC 2009) that was held in Nantes, France, on July 16-17, 2009.

Monitoramento da média de processos que oscila através de um gráfico de controle np

Artigo

Este artigo considera um gráfico np_x proposto por Wu et al. (2009) para controle de média de processo como uma alternativa ao uso do gráfico de \bar{X} . O que distingue do gráfico de controle np_x é o fato das unidades amostrais serem classificadas como unidades de primeira ou de segunda classe de acordo com seus limites discriminantes. O gráfico tradicional np é um caso particular do gráfico np_x quando os limites discriminantes coincidem com os limites de especificação e unidade de primeira (segunda) classe é um item conforme (não conforme). Estendendo o trabalho de Reynolds Junior, Arnold e Baik (1996), consideramos que a média de processo oscila mesmo na ausência de alguma causa especial. As propriedades de Cadeia de Markov foram adotadas para avaliar o desempenho do gráfico np_x no monitoramento de média de processos que oscila. De modo geral, o gráfico np_x requer amostras duas vezes maior para superar desempenho do gráfico \bar{X} (enquanto que o gráfico tradicional np necessita tamanho de amostras cinco ou seis vezes maior).

Palavras-chave

Limites discriminantes. Gráfico np_x . Modelo auto-regressivo de primeiro ordem (AR1). Controle de qualidade. Gráfico de controle por atributos e por variáveis. Inspeção por atributos.