On the Delta Function Normalization of the Wave Functions of the Aharonov-Bohm Scattering of a Dirac Particle

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In a previous paper, we found the most general boundary conditions for the Aharonov -Bohm scattering of a Dirac particle. We found the resulting wave functions but we did not worry about delta normalizing them. As is well know, in practice, it is not easy to evaluate the diverging integrals occurring in the process. The purpose of this paper is to evaluate those integrals and present the resulting delta normalized eigenfunctions.

I Introduction

In a previous article [1] we considered the Hamiltonian operator H of a Dirac particle of mass $m > 0$, moving in two dimensions in the presence of an infinitely thin magnetic flux tube at the origin, formally defined as

$$H = \left[\overrightarrow{p} + \frac{e}{c}\overrightarrow{A}\right] \cdot \overrightarrow{\alpha} + \beta m \tag{1}$$

where $\overrightarrow{p} = (p_x, p_y)$, $\overrightarrow{\alpha} = (\alpha_1, \alpha_2)$,

$$\alpha_i = \begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix}, \qquad i = 1, 2,$$

and

$$\beta = \begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix}, \tag{2}$$

where σ_1, σ_2 and σ_3 are the Pauli matrices. The vector potential, in the Coulomb, Gauge is

$$\frac{e}{c}\overrightarrow{A} = \frac{\phi}{r^2}(-y, x). \tag{3}$$

We considered also the helicity operator given by

$$\Lambda = \left[\overrightarrow{p} + \frac{e}{c}\overrightarrow{A}\right] \cdot \overrightarrow{\Sigma}, \tag{4}$$

where $\overrightarrow{\Sigma} = (\Sigma_1, \Sigma_2)$, and

$$\Sigma_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \qquad i = 1, 2. \tag{5}$$

In this previous article [1], we found the most general domains where the Hamiltonian H is a self-adjoint operator. The Hamiltonian operator and the Helicity operator Λ admit a four parameter family of self-adjoint extensions in one-to-one correspondence with the boundary conditions (BC's) to be satisfied by the eigenfunctions at the origin. The actions of the Helicity operator Λ and the Hamiltonian operator H commute before specification of the BC's. Although this occurs, to ensure commutativity and consequently to obtain common eigenfunctions, it is not sufficient to take the same BC's for both operators as claimed in reference [2]. This fact occurs because both operator H and Λ, when acting in a common domain, do not let it invariant.

The Helicity conservation can be obtained by the imposing a formal condition that leads to certain relations between the parameters of the extensions. In other words the formal condition we impose defines new domains with the parameters obeying certain relations. These new domains we found are the most general domains where both operators H and Λ are self-adjoint and effectively commute with consequently common eigenfunctions. In reference [1], we wrote down these most general common eigenfunctions, but we did not delta normalize them. In this paper we present this calculation, that completes the results of the article [1]. This paper is organized as follows. In Section II, we present the computation of the normalization constant of the most general common eigenfunctions of H and Λ that satisfies the more general BC's [1]. In Section III we show that by imposing the orthonormality condition for the eigenfunctions of H we can obtain a special boundary condition that makes H self-adjoint. This boundary condition depends on one parameter and is the boundary condition obtained in reference [2] . This is not a coincidence, but is related to the fact that a self-adjoint operator possesses complete set of orthonormal eigenfunctions. Unfortunately this procedure can not be used to obtain the most general boundary conditions, but only the above mentioned special case.

II Normalization of the most general eigenfunctions of H and Λ

The general form of the common eigenfunctions $\psi_{E,\lambda}(kr)$ of H and Λ operators before specification of the domains are [1]

$$\psi_{+|E|,\pm k}(kr) = \frac{1}{N}\begin{pmatrix} J_{-\nu}(kr)+c_+(k)J_\nu(kr) \\ \frac{-ik}{|E|+m}\left(J_{-\nu-1}(kr)-c_+(k)J_{1+\nu}(kr)\right) \\ \frac{\pm k}{|E|+m}\left(J_{-\nu}(kr)+d_+(k)J_\nu(kr)\right) \\ \mp i\left(J_{-\nu-1}(kr)-d_+(k)J_{1+\nu}(kr)\right)\end{pmatrix}, \tag{6}$$

$$\psi_{-|E|,\pm k}(kr) = \frac{1}{N}\begin{pmatrix} J_{-\nu}(kr)+c_-(k)J_\nu(kr) \\ \frac{+ik}{|E|-m}\left(J_{-\nu-1}(kr)-c_-(k)J_{1+\nu}(kr)\right) \\ \frac{\mp k}{|E|-m}\left(J_{-\nu}(kr)+d_-(k)J_\nu(kr)\right) \\ \mp i\left(J_{-\nu-1}(kr)-d_-(k)J_{1+\nu}(kr)\right)\end{pmatrix}, \tag{7}$$

where $\pm|E|$ are the positive and negative eigenvalues of H and $\lambda=\pm k$ ($k=\sqrt{E^2-m^2}$) are the eigenvalues of Λ.

By imposing the most general conditions of self-adjointness and commutativity for H and Λ operators, the coefficients $c_\pm(k)$ and $d_\pm(k)$ must obey [1]

$$c_-(k)=d_-(k)=c_+(k)=d_+(k)=+\frac{\sqrt{2}}{2}\frac{(-1)^{n_1+n_2}\sin\theta}{(1-(-1)^{n_1}\cos\theta\sin\varphi)}\left(\frac{k}{\sqrt{2}m}\right)^{2|\nu|-1}, \tag{8}$$

for $\lambda=+k$, and

$$c_-(k)=d_-(k)=c_+(k)=d_+(k)=-\frac{\sqrt{2}}{2}\frac{(-1)^{n_1+n_2}\sin\theta}{(1-(-1)^{n_1}\cos\theta\sin\varphi)}\left(\frac{k}{\sqrt{2}m}\right)^{2|\nu|-1}, \tag{9}$$

for $\lambda=-k$.

For future use we define

$$a=\frac{\sqrt{2}}{2}\frac{(-1)^{n_1+n_2}\sin\theta}{(1-(-1)^{n_1}\cos\theta\sin\varphi)}\left(\frac{1}{\sqrt{2}m}\right)^{2|\nu|-1}. \tag{10}$$

The parameters θ and φ are the parameters of the extensions that must satisfy some relations (see equations 4.9 and 4.10 of reference[1]).

For orthonormality we must have

$$\int_0^\infty rdr\psi^*_{E,\lambda}(kr)\,\psi_{E',\lambda'}(k'r)=\frac{1}{\sqrt{kk'}}\delta(k-k'). \tag{11}$$

Using the formula developed by Ausdretch, Jasper and Skarzhinsky [3],

$$\int_0^\infty rdrJ_\nu(kr)J_{-\nu}(k'r)=\frac{1}{\sqrt{kk'}}\delta(k-k')\cos\pi\nu+ \frac{2}{\pi}\frac{\sin\pi\nu}{k^2-k'^2}\left(\frac{k}{k'}\right)^\nu, \tag{12}$$

and the well-known formula [4]

$$\int_0^\infty rdrJ_\nu(kr)J_\nu(k'r)=\frac{1}{\sqrt{kk'}}\delta(k-k'), \tag{13}$$

we must show first of all that the non-δ contribution terms that come from equation (12) must vanish in the computation of equation (11).

To do this let us consider the two cases: a) when $\lambda=k$ and $\lambda'=k'$(or $\lambda=-k$ and $\lambda'=-k'$) and b) when $\lambda=k$ and $\lambda'=-k'$(or $\lambda=-k$ and $\lambda'=k'$). Considering the forms of $\psi_{E,\lambda}(kr)$ given by equations (6) and (7), the crossing terms of equation (11) are the following for the case a:

$$\frac{1}{N^2}\int_0^\infty rdr\{[c_\pm(k)J_\nu(kr)J_{-\nu}(k'r)+c_\pm(k')J_\nu(k'r)J_{-\nu}(kr)]-$$

$$-\frac{kk'}{(E+m)(E'+m)}[c_\pm(k)J_{\nu+1}(kr)J_{-\nu-1}(k'r)+c_\pm(k')J_{\nu+1}(k'r)J_{-\nu-1}(kr)]+$$

$$+\frac{kk'}{(E+m)(E'+m)}[d_\pm(k)J_\nu(kr)J_{-\nu}(k'r)+d_\pm(k')J_\nu(k'r)J_{-\nu}(kr)]-$$

$$-[d_\pm(k)J_{\nu+1}(kr)J_{-\nu-1}(k'r)+d_\pm(k')J_{\nu+1}(k'r)J_{-\nu-1}(kr)]\}. \tag{14}$$

Using the formula given by equation (12) the non δ contribution terms of the above equation are

$$\frac{1}{N^2}\frac{2}{\pi}\sin\pi\nu(E+E')\left\{\frac{k^\nu k'^{-\nu}}{k^2-k'^2}[c_\pm(k)\frac{1}{E'+m}+d_\pm(k)\frac{1}{E+m}\left(\frac{k}{k'}\right)]-\right.$$

$$\left.\frac{k'^\nu k^{-\nu}}{k^2-k'^2}[c_\pm(k')\frac{1}{E+m}+d_\pm(k')\frac{1}{E'+m}\left(\frac{k'}{k}\right)]\right\}. \tag{15}$$

Taking $c_\pm(k)=d_\pm(k)$ and $c_\pm(k')=d_\pm(k')$ given by equations (8), (9) and (10) for the case a, we see that the resulting non δ-contribution of the above equation vanishes as it should,

$$\frac{1}{N^2}\frac{2}{\pi}\sin\pi\nu(E+E')\frac{k^{-\nu}k'^{-\nu}}{k^2-k'^2}\{\left(\frac{1}{(E'+m)k}+\frac{1}{(E+m)k'}\right)a$$

$$-\ \left(\frac{1}{(E+m)k'}+\frac{1}{(E'+m)k}\right)a\}=0. \tag{16}$$

Considering the forms of $\psi_{E,\lambda}(kr)$ given by equations (6) and (7), the crossing terms of equation (11) are, for the case b, the following :

$$\frac{1}{N^2}\int_0^\infty rdr\{[c_\pm(k)J_\nu(kr)J_{-\nu}(k'r)+c_\pm(k')J_\nu(k'r)J_{-\nu}(kr)]-$$

$$-\frac{kk'}{(E+m)(E'+m)}[c_\pm(k)J_{\nu+1}(kr)J_{-\nu-1}(k'r)+c_\pm(k')J_{\nu+1}(k'r)J_{-\nu-1}(kr)]+$$

$$-\frac{kk'}{(E+m)(E'+m)}[d_\pm(k)J_\nu(kr)J_{-\nu}(k'r)+d-_\pm(k')J_\nu(k'r)J_{-\nu}(kr)]-$$

$$+[d_\pm(k)J_{\nu+1}(kr)J_{-\nu-1}(k'r)+d_\pm(k')J_{\nu+1}(k'r)J_{-\nu-1}(kr)]\}\ . \tag{17}$$

Using the formula given by equation (12) the non δ contribution terms of the above equation are

$$\frac{1}{N^2}\frac{2}{\pi}\sin\pi\nu\,(E+E')\left\{\frac{k^\nu k'^{-\nu}}{k^2-k'^2}\left[c_\pm(k)\frac{1}{E'+m}-d_\pm(k)\frac{1}{E+m}\left(\frac{k}{k'}\right)\right]\right.$$

$$\left.-\frac{k'^\nu k^{-\nu}}{k^2-k'^2}\left[c_\pm(k')\frac{1}{E+m}-d_\pm(k')\frac{1}{E'+m}\left(\frac{k'}{k}\right)\right]\right\}. \tag{18}$$

Taking $c_\pm(k)=d_\pm(k)$ and $c_\pm(k')=d_\pm(k')$ given by equations (8), (9) and (10) for the case b, we see that the resulting non δ-contribution of the above equation vanishes as it should:

$$\frac{1}{N^2}\frac{2}{\pi}\sin\pi\nu(E+E')\frac{k^{-\nu}k'^{-\nu}}{k^2-k'^2}\left\{\left(\frac{1}{(E'+m)k}-\frac{1}{(E+m)k'}\right)a\right.$$

$$\left.+\ \left(\frac{1}{(E+m)k'}-\frac{1}{(E'+m)k}\right)a\right\}=0. \tag{19}$$

So we see that the most general common eigenfunctions given by equations (6) to (10) of reference [1] are normalizable, since the non-δ function contribution vanishes in the computation of equation (11). Let us find out the normalization constant.

To do this we have to take all δ contributions terms.

The δ function contribution of the crossing terms of equation (11), after some mathematical manipulations using equations (8), (9), (10) and (12) can be written as

$$\frac{1}{N^2}\cos\pi\nu\frac{1}{\sqrt{kk'}}\delta(k-k')\{1+\frac{kk'}{(E+m)(E'+m)}\}\{c_\pm(k)+c_\pm(k')\}$$
$$+\frac{1}{N^2}\cos\pi\nu\frac{1}{\sqrt{kk'}}\delta(k-k')\{1+\frac{kk'}{(E+m)(E'+m)}\}\{d_\pm(k)+d_\pm(k')\}$$
$$=\frac{1}{N^2}\frac{4E}{E+m}[2a(k)^{2|\nu|-1}]\frac{\cos\pi\nu}{k}\delta(k-k')\,. \tag{20}$$

Collecting now the direct terms of equation (11) we have

$$\frac{1}{N^2}\int_0^\infty rdr\{[J_{|\nu|}(kr)J_{|\nu|}(k'r)+c_\pm(k)c_\pm(k')J_{-|\nu|}(k'r)J_{-|\nu|}(kr)]-$$
$$+\frac{kk'}{(E+m)(E'+m)}[J_{|\nu|-1}(kr)J_{|\nu|-1}(k'r)+c_\pm(k)c_\pm(k')J_{1-|\nu|}(k'r)J_{1-|\nu|}(kr)]+$$
$$+\frac{kk'}{(E+m)(E'+m)}[J_{|\nu|}(kr)J_{|\nu|}(k'r)+d_\pm(k)d_\pm(k')J_{-|\nu|}(k'r)J_{-|\nu|}(kr)]-$$
$$+[J_{|\nu|-1}(kr)J_{|\nu|-1}(k'r)+d_\pm(k)d_\pm(k')J_{1-|\nu|}(k'r)J_{1-|\nu|}(kr)]\}. \tag{21}$$

Using the formula (13) the above equation, after some mathematical manipulations using equations (8), (9) and (10) gives

$$\frac{1}{N^2}\frac{4E}{E+m}[1+a^2(k)^{4|\nu|-2}]\frac{\delta(k-k')}{k}. \tag{22}$$

Considering the δ contribution of the crossing terms given by equation (20) and of the direct terms given by equation (21), the normalization condition of equation (11) turns out to be

$$N=\sqrt{\frac{E+m}{4E}}\frac{1}{[1+2a\cos(\pi\nu)(k)^{2|\nu|-1}+a^2(k)^{4|\nu|-2}]^{\frac{1}{2}}}. \tag{23}$$

III The orthonormality condition and the one parameter family of self-adjoint extension for H

We can obtain a one parameter family of self-adjoint extensions of H operator (the BC's of reference [2]) by imposing orthonormality for the eigenfunctions of this operator. It is not necessary to do the complicated calculations of refs.[1] and [2]. This is not a coincidence, but it occurs because a self-adjoint operator always has a complete set of orthonormal eigenfunctions. Let us consider the general form of an eigenfunction of H given by equations (6) and (7). The non-δ contribution crossing terms of the upper component spinor in the computation of equation (11), after using the formula given by equation (12), gives

$$\frac{1}{N^2}\frac{2}{\pi}\sin(\pi\nu)\,c_\pm(k)\frac{k^\nu k'^{-\nu}}{k^2-k'^2}\{1+\frac{E-m}{E'+m}\}-$$
$$\frac{1}{N^2}\frac{2}{\pi}\sin(\pi\nu)\,c_\pm(k')\frac{k'^\nu k^{-\nu}}{k^2-k'^2}\{1+\frac{E'-m}{E+m}\}. \tag{24}$$

Imposing the orthonormality condition this contribution must vanish. Then we have

$$c_\pm(k)=\frac{k^{2|\nu|}}{E+m}\tan\alpha\frac{1}{m^{2|\nu|-1}}, \tag{25}$$

where the constant $\frac{1}{m^{2|\nu|-1}}$ was introduced for dimensional reasons and $\tan\alpha$ is a free parameter of the extension.

The non-δ contribution crossing terms of the lower component spinor in the computation of equation (11), after using the formula given by equation (12) , gives:

$$\frac{1}{N^2}\frac{2}{\pi}\sin(\pi\nu)\,d_\pm(k)\frac{k^\nu k'^{-\nu}}{k^2-k'^2}\{\frac{k}{k'}+\frac{kk'}{(E'+m)(E+m)}\}-$$

$$\frac{1}{N^2}\frac{2}{\pi}\sin(\pi\nu)\,d_\pm(k')\frac{k'^\nu k^{-\nu}}{k^2-k'^2}\{\frac{k'}{k}+\frac{kk'}{(E'+m)(E+m)}\}. \tag{26}$$

Imposing that this contribution vanish we get

$$d_\pm(k)=\frac{k^{2|\nu|}}{E-m}tg\gamma\frac{1}{m^{2|\nu|-1}}, \tag{27}$$

where the constant $\frac{1}{m^{2|\nu|-1}}$ was introduced for dimensional reasons and $\tan\gamma$ is a free parameter.

The results of equations (25) and (27) correspond to the BC's of reference [2] and also of reference [5]. In the case of reference[2] the boundary conditions for the two top components become decoupled from the boundary conditions for the two botton components. In the case of reference[5], only the two top components are considered.

We can also obtain the normalization constant in this case, by computing all the δ function contribution terms by the crossing and direct terms and then imposing the normalizability of equation (11). For the two components of reference[5], we get

$$N=\sqrt{\frac{E+m}{4E}}\frac{1}{[1+2\tan\alpha\cos(\pi\nu)\frac{k^{2|\nu|}}{E+m}\frac{1}{m^{2|\nu|-1}}+\tan^2\alpha\frac{k^{4|\nu|}}{(E+m)^2}\frac{1}{m^{4|\nu|-2}}]^{\frac{1}{2}}}. \tag{28}$$

One can check that this result is the same presented by Sousa Gerbert for the two component spinor in reference [5]

In our more general case, imposing the commutativity condition for H and Λ that is $c_\pm(k)=d_\pm(k)$ for all k, we have

$$N=\sqrt{\frac{E+m}{4E}}. \tag{29}$$

References

[1] V.S. Araujo, F.A.B. Coutinho, and J. Fernando Perez, J. Phys. A: Math. Gen. **34**, 1 (2001).

[2] F.A.B. Coutinho and J. Fernando Perez, Phys. Rev. D **49**, 2092 (1994).

[3] J. Ausdretsch, U. Jasper, and V.D. Skarzhinsky, J. Phys. A: Math. Gen **28**, 2359 (1995).

[4] M. Abramowitz and I. A. Stegun, *Handbook Mathematical Functions*, Dover Publications, New York (1968).

[5] Ph. de Sousa Gerbert, Phys. Rev. D **40,** 1346 (1989).