Topological Mass Generation and Duality Transformations in Diverse Dimensions

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In this seminar we shall discuss the issue of duality transformation in the context of f topological mass generation in diverse dimensions. Particular emphasis will be given to the mass generation mechanism as the result of interference between self and anti self-dual components, as disclosed by the soldering formalism. Since this is a gauge embedding procedure derived from an old algorithm of second-class constraint conversion used by the author to approach anomalous gauge theories, a quick review of the subject will be presented. The problem of classification of the electromagnetic duality groups that is closely related will also be briefly discussed. Particular emphasis will be given to a new approach to duality based on the soldering embedding to tackle to problem of mass generation by topological mechanisms in D=3 and D=4 dimensions including the couplings to dynamical matter and nonlinear cases.

1 Preliminaries

The notion of soldering or combining different species of objects to yield a new composite structure is a very recurrent concept in Physics and Mathematics. The fusion process can be established as long as the constituents display either opposite or complementary aspects of some definite property. This simple observation contains the genesis of the technique, which we call the Soldering Formalism. Over the last few years we have systematically developed the soldering formalism and extensively applied to a wide variety of problems, including higher dimensional models [1-18]. In this report we shall basically review our work.

The soldering mechanism is a new technique developed to work with distinct manifestations of self dual aspects of some symmetry, for instance, chirality, helicity and electromagnetic self-duality, which can be soldered. It provides a clean algorithm, inherited from our constraint conversion program[19], for fusing the opposite nature of these symmetries by taking into account interference effects. Therefore, two models carrying the representation of such symmetries being otherwise completely independent, can be fused by this technique, irrespective of dimensional considerations. Indeed while this technique is applied for the quantum mechanical harmonic oscillator[20] it may also be used to investigate the cases of electromagnetic dualities in distinct dimension, including the new and interesting instance of non commutative manifolds. Particular instances of the scalar field theory in two dimensions, the Maxwell theory in four dimensions or the Chern-Simons theory in three dimensions and massive p-form self-duality in arbitrary dimensions are some of the exemples to be discussed here.

The list of applications is indeed quite extensive but I would like to emphasize the following four contributions that brought original solutions to old problems. In [2] a proposal for mass generation as a consequence of interference of massless modes was given that corroborates an old conjecture by Jackiw[21]. It proposes an alternative mechanism for the dynamical mass generation known as Schwinger mechanism as a consequence of left–right interaction. In the same spirit, a proposal for the physical origin of the topological mass in the 3D Maxwell-Chern-Simon model was given in [9] as a consequence of the interference between self-dual and anti self-dual modes of a 3D analogue of chiral bosons. A dispute, lasting more than ten years, regarding the proper way to formulate the dynamics of 2D chiral bosons has been settled in [5] using the dual projection version of the soldering formalism. Finally, I would like to mention the resolution of the equivalence between the non-abelian self-dual model and the Yang-Mills-Chern-Simons model, for any value of the coupling constant g, in [12]. Previous attempts, based on the master action approach, were restricted to $g \to \infty$ limit.

After discussing the general aspects of the method we shall focus on a few applications. Although our initial study in soldering was to investigate the possibility of fusing different chiralities of two independent systems displaying truncated diffeomorphism into a 2D gravity model, we shall concentrate on two different lines of developments. One is to explore the intimate connection between soldering and duality to study the phenomenon of topological mass generation. The other is to study the possibility of simultaneously considering the soldering and bosonization, in different dimensions. In the context of the last, the technique is initially developed to solder the dual aspects of some symmetry following from the bosonization of two distinct fermionic models, thereby leading to new results. Exploiting this technique, the two dimensional chiral determinants with opposite chirality is soldered to reproduce either the usual gauge invariant Schwinger model or the Thirring model. The extension of the analysis for $D \geq 2$ bosonization has also been considered and two independent three dimensional massive Thirring models with same coupling but opposite mass signatures, in the long wavelength limit, combined by the process of bosonization and soldering to yield an effective massive Maxwell theory. The current bosonization formulas are given, both in the original independent formulation as well as the effective theory, and shown to yield consistent results for the correlation functions. Similar features also hold for quantum electrodynamics in three dimensions.

An interpretation of the soldering process has also been offered that discloses the whole process as a canonical transformation, however in the Lagrangian side[22]. We coined it dual projection. This new interpretation has provided us with some practical applications and used to study some controversial issues to date, such as the way to properly couple chiral matter to gauge and gravitational backgrounds[23] or to establish to equivalence between different formulations of chiral modes besides providing new interpretation to their field constituintes. It also offered the opportunity to treat the electromagnetic duality groups under this technique and to disclose their group structure dimensional dependence, both for massless and massive fields and to propose a new physical interpretation for the phenomenon[6]. It also opens up the possibility to deal with electromagnetic duality for arbitrary p-form theory in different dimensions. In a different line of investigation, the formalism was also used to study the duality equivalence of related models, including nonlinear effects, in distinct dimensions and couplings to different matters [9,11-17].

2 General Aspects of Soldering and Canonical Transformations

The basic idea is to raise a global Noether symmetry of the self and anti-self dual constituents into a local one, but for an effective composite system, consisting of the dual components and an interference term that defines the soldered action. Here we adopt an iterative Noether procedure to lift the global symmetries. Assume the symmetries are being described by the local actions

$$S_{\pm}^{(0)}(\varphi_{\pm}^{\eta}) = \int L_{\pm}^{(0)}(\varphi_{\pm}^{\eta})\, dt \tag{1}$$

invariant under a global multi-parametric transformation $\delta\varphi_{\pm}^{\eta} = \alpha^{\eta}$ where the suffix (0) indicates the iterative nature of the analysis. Here η represents the tensorial character of the basic fields and, for simplicity, will be dropped from now on. In general, under local transformations these actions will not remain invariant, and Noether counter terms become necessary in order to reestablish the invariance. These counter terms contain, apart from the original fields, appropriate contributions of auxiliary fields $W^{(N)}$. Thus, after N iterations we obtain,

$$S_{\pm}(\varphi_{\pm})^{(0)} \rightarrow S_{\pm}(\varphi_{\pm})^{(N)} = S_{\pm}(\varphi_{\pm})^{(N-1)} - W^{(N)} J_{\pm}^{(N)} \tag{2}$$

where $J_{\pm}^{(N)}$ are the Noether currents. For the individual self and anti-self dual systems this iterative gauging procedure will not produce gauge invariant systems. However, for the composite system this procedure will, eventually, lead to an effective action of the form,

$$\begin{aligned} S(\varphi_{\pm}, W)^{(N)} &= S_{+}(\varphi_{+})^{(N-1)} + S_{-}(\varphi_{-})^{(N-1)} \\ &+ W^{(N-1)}\Big(J_{+}^{(N-1)} + J_{-}^{(N-1)}\Big) \end{aligned} \tag{3}$$

which turns out to be invariant under the original transformations. The auxiliary fields $W^{(N)}$ are then eliminated in favor of the other variables. The final effective action then becomes a function of the original variables only. In this form the effective action is no longer a function of the individual fields but of some composite as $\phi = \varphi_{+} - \varphi_{-}$ if the fields belong to the algebra of some group or $\phi = \varphi_{+} \cdot \varphi_{-}$ if they are group variables,

$$S(\varphi_{+}, \varphi_{-}, W)^{(N)}|_{W=f(\varphi_{\pm})} \rightarrow S_{eff}(\varphi_{\pm}) = S_{eff}(\phi) \tag{4}$$

so that, $\delta S_{eff} = 0$. This is our cherished effective action which is a result of the soldering or the fusion of the original actions. Also, since the auxiliary fields are used for soldering only, they will be called *soldering fields*.

3 Soldering, dynamical mass generation and interference

Here we discuss the applications of the soldering technique to the problems of quantum field theory where the full power of this approach is manifested. In quantum mechanics see the original papers[20] and applications[24]. We discuss first two-dimensional field theory where the use of the bosonization technique provides us with exact results. The extension to the non Abelian example and a new look at the Polyakov-Weigmann identity[2], as well as the result of different regularization schemes, known to exist in the chiral Schwinger models leading to distinct categorization of second class constraints, has been proposed[7].

Three dimensional models are considered in the sequel. Although bosonization is not exact in these dimensions, yet some expressions are known in particular limiting conditions. Our interest is basically confined to the long wavelength limit where local expressions are available. Under these conditions the massive Thirring model or quantum electrodynamics bosonize to self dual models. Soldering of such models is carried out to get new models. The soldering of models with different coupling parameters is given. Three dimensional gravity has been examined in the literature where the soldering of linearized Hilbert-Einstein actions with Chern-Simons terms is studied[10]. Some attention has also been devoted to a study of W gravity models[15]. The W_2 and W_3 cases, which yields exact soldering, were discussed in details. Finally the case of higher conformal spins $W_n (n > 2)$, which is developed within some perturbative scheme since soldering is not exact in this instance, was considered. We will not give details of all these exemples in this review.

3.1 Fermion Determinants and Bosonization in Two Dimensions.

Bosonization is a powerful technique that maps a fermionic theory into its bosonic counterpart. It was fully explored in the context of two dimensions[25] and later extended to higher dimensions [26-29]. Its importance lies in the fact that it includes quantum effects already at the classical level. Consequently, different aspects and manifestations of quantum phenomena may be investigated directly, that would otherwise be highly nontrivial in the fermionic language.

The basic notions and ideas are first introduced in the context of two dimensions where bosonization is known to yield exact results. Consider an explicit one loop calculation following Schwinger's point splitting method[30] which is known to yield[31],

$$\begin{aligned} W_+[\omega] &= -i\log\det(i\partial\!\!\!/ + eA\!\!\!/_\pm) \\ &= \int d^2x\, \left(\partial_+\omega\partial_-\omega + 2\,e\,A_\pm\partial_\mp\omega + a_\pm\, e^2\, A_+A_-\right) \end{aligned} \tag{5}$$

where $\omega = \varphi, \rho$. Here light cone metric is used and the regularization ambiguity is manifested through $a_\pm$. In the Hamiltonian approach their values define a second-class constrained systems with two and four constraints.

Observe that different scalar fields φ and ρ are used to emphasize that the fermionic chiral components are uncorrelated. It is the soldering that abstracts a meaningful combination, this being essentially the simultaneous gauging of a global symmetry of the individual chiral components. Consider, therefore, the gauging of the following global symmetry, $\delta\varphi = \delta\rho = \alpha$, $\delta A_\pm = 0$. The variations in the effective actions (5) are found to be,

$$\delta W_\pm[\eta] = \int d^2x\, \partial_\mp\alpha\; J_\pm(\eta) \tag{6}$$

where the currents are defined as, $J_\pm(\eta) = \frac{1}{2\pi}(\partial_\pm\eta + e\,A_\pm)$; $\eta = \varphi, \rho$. Next we introduce the soldering field $B_\pm$ so that,

$$W_\pm^{(1)}[\eta] = W_\pm[\eta] - \int d^2x\, B_\mp\; J_\pm(\eta) \tag{7}$$

Then it is possible to define a modified action,

$$W[\varphi,\rho] = W_+^{(1)}[\varphi] + W_-^{(1)}[\rho] + \frac{1}{2\pi}\int d^2x\, B_+\, B_- \tag{8}$$

which is invariant under an extended set of transformations that includes the matter fields together with, $\delta B_\pm = \partial_\pm\alpha$. Observe that the soldering field transforms as a vector potential. Since it is an auxiliary field, it can be eliminated from (8). This will naturally solder the otherwise independent chiral components. The relevant solution is found to be, $B_\pm = 2\,J_\pm$. Inserting this solution in (8), we obtain,

$$\begin{aligned} W[\Phi] &= \int d^2x\,\{(\partial_+\Phi\partial_-\Phi + 2eA_+\partial_-\Phi - 2eA_-\partial_+\Phi) \\ &+ (a_+ + a_- - 2)\, e^2\, A_+\, A_-\} \end{aligned} \tag{9}$$

where, $\Phi = \varphi - \rho$. The action is no longer expressed in terms of the different scalars φ and ρ, but only on their specific combination. This combination is gauge invariant.

Let us digress on the significance of the findings. At the classical fermionic version, the chiral Lagrangians are completely independent. Bosonizing them includes quantum effects, but still there is no correlation. The soldering mechanism exploits the symmetries of the independent actions to precisely combine them to yield a single action. Note that the soldering works with the bosonized expressions. Thus the soldered action obtained in this fashion corresponds to a new quantum theory.

We now show that different choices for the parameters a_+ and a_- lead to well known models. To do this consider the variation of (9) under the conventional gauge transformations, $\delta\varphi = \delta\rho = \alpha$ and $\delta A_\pm = \partial_\pm\alpha$. It is easy to see that the expression in parenthesis is gauge invariant. Consequently a gauge invariant structure for W is obtained provided, $a_+ + a_- - 2 = 0$. By functionally integrating out the Φ field from (9), we obtain,

$$W[A_\pm] = \frac{e^2}{4\pi}\int d^2x\{A_+\frac{\partial_-}{\partial_+}A_+ + A_-\frac{\partial_+}{\partial_-}A_- - 2A_+A_-\} \tag{10}$$

which is the familiar structure for the gauge invariant action expressed in terms of the potentials. The opposite chiralities of the two independent fermionic theories have been soldered to yield a gauge invariant action.

Some interesting observations are possible concerning the regularization ambiguity manifested by the parameters a_+ and a_-. Since a single equation cannot fix both the parameters, it might appear that there is a whole one parameter class of solutions for the chiral actions that combine to yield the vector gauge invariant action. Indeed, without any further input, this is the only conclusion. However, Bose symmetry imposes a crucial restriction[32]. This symmetry plays an essential part that complements gauge invariance. In the present case, this symmetry corresponds to the left-right (or + –) symmetry in (5), thereby requiring $a_+ = a_- = 1$. This has important consequences if a Maxwell term was included from the beginning to impart dynamics. Then the soldering takes place among two chiral Schwinger models[33] having opposite chiralities to reproduce the usual Schwinger model[30]. It is known that the chiral models satisfy unitarity provided $a_\pm \geq 1$ and the spectrum consists of a vector boson with mass, $m^2 = \frac{e^2a^2}{a-1}$ and a massless chiral boson. The values of the parameters obtained here just saturate the bound. For the minimal parametrization, the mass of the vector boson becomes infinite so that it goes out of the spectrum. Thus the soldering mechanism shows how the massless modes in the chiral Schwinger models are fused to generate the massive mode of the Schwinger model. It may be observed that the soldering process can be carried through for the non Abelian theory as well, and a relation analogous to (9) is obtained.

Naively it may appear that the soldering of the left and right chiralities to obtain a gauge invariant result is a simple issue since adding the classical Lagrangians $\overline{\psi}D\!\!\!\!/_+\psi$ and $\overline{\psi}D\!\!\!\!/_-\psi$, with identical fermion species, just yields the usual vector Lagrangian $\overline{\psi}D\!\!\!\!/\psi$. The quantum considerations are, however, much involved. The chiral determinants, as they occur, cannot be even defined since the kernels map from one chirality to the other so that there is no well defined eigenvalue problem. This is circumvented by working with $\overline{\psi}(i\not\partial + e\not A_{\pm})\psi$, that satisfy an eigenvalue equation, from which their determinants may be computed. But now a simple addition of the classical Lagrangians does not reproduce the expected gauge invariant form. At this juncture, the soldering process becomes important. It systematically combined the quantized (bosonized) expressions for the opposite chiral components of *different* fermionic species. The importance of this will become more transparent when the three dimensional case is discussed.

At this point it is interesting to examine the impact that different choices of regularizations may have over our result. There are two possible situations left. Here we examine the present regularization and show that different choices for the parameters $a_{\pm}$ in (9) lead to the Thirring model. We have examined an alternative regularization prescription leading to a diverse constraint structure for the Hamiltonian description of the model[7]. Indeed it is precisely when the mass term exists ($i.e.,\ a_+ + a_- - 2 \neq 0$), that (9) represents the Thirring model. Consequently, this parametrization complements that used previously to obtain the vector gauge invariant structure. it is now easy to see that the term in parentheses in (9) corresponds to $\overline{\psi}(i\not\partial + e\not A)\psi$ so that integrating out the auxiliary A_μ field yields,

$$\mathcal{L} = \overline{\psi} i\not\partial\psi - \frac{g}{2}(\overline{\psi}\gamma_\mu\psi)^2 \ \ ; g = \frac{4\pi}{a_+ + a_- - 2} \tag{11}$$

which is just the Lagrangian for the usual Thirring model. It is known[34] that this model is meaningful provided the coupling parameter satisfies the condition $g > -\pi$, so that, $\mid a_+ + a_- \mid > 2$. This condition complements the condition found earlier.

We have therefore explicitly derived expressions for the chiral determinants (5) which simultaneously preserve the factorization property and gauge invariance of the vector determinant. It was also perceived that the naive way of interpreting the chiral determinants as $W[A_+, 0]$ or $W[0, A_-]$ led to the supposed incompatibility of factorization with gauge invariance. Perturbatively we show the lacking of crossing graphs. Classically these graphs do vanish ($P_+P_- = 0$) so that it becomes evident that this incompatibility originates from a lack of properly accounting for the quantum effects. It is possible to interpret this effect, as we will now show, as a typical quantum mechanical interference phenomenon, closely paralleling the analysis in Young's double slit experiment. We also provide a new interpretation for the Polyakov-Wiegman[35] identity. Rewriting (10) in Fourier space as

$$\begin{aligned} W[A_\pm] &= -\frac{N}{2}\int dk \left\{ A_+^* \frac{k_-}{k_+} A_+ + A_-^* \frac{k_+}{k_-} A_- - 2A_+^* A_- \right\} \\ &= -\frac{N}{2}\int dk \mid \sqrt{\frac{k_-}{k_+}} A_+ - \sqrt{\frac{k_+}{k_-}} A_- \mid^2 \end{aligned} \tag{12}$$

immediately displays the typical quantum mechanical interference phenomenon, in close analogy to the optical example. The dynamically generated mass arises from the interference between these movers, thereby preserving gauge invariance. Setting either A_+ or A_- to vanish, destroys the quantum effect, very much like closing one slit in the optical experiment destroys the interference pattern. Although this analysis was done for the Abelian theory, it is straightforward to perceive that the effective action for a non Abelian theory can also be expressed in the form of an absolute square (12), except that there will be a repetition of copies depending on the group index. This happens because only the two-legs graph has an ultraviolet divergence, leading to the interference (mass) term. The higher legs graphs are all finite, and satisfy the naive factorization property.

It is now simple to see that (12) represents an abelianized version of the Polyakov – Wiegman identity by making a familiar change of variables, $A_+ = \frac{i}{e}U^{-1}\partial_+ U$; $A_- = \frac{i}{e}V\partial_- V^{-1}$ where, in the Abelian case, the matrices U and V are given as, $U = \exp\{i\varphi\}$; $V = \exp\{-i\rho\}$; $UV = \exp\{i\Phi\}$ with Φ being the gauge invariant soldered field. It is possible to recast (12), in the coordinate space, as

$$W[UV] = W[U] + W[V] + \int d^2x \left(U^{-1}\partial_+ U\right)\left(V\partial_- V^{-1}\right) \tag{13}$$

which is the Polyakov-Wiegman identity, satisfying gauge invariance. The result can be extended to the non Abelian case since, as already mentioned, the nontrivial interference term originates from the two-legs graph which has been taken into account. It is now relevant to point out that the important crossing piece in either (12) or (13) is conventionally[35, 25] interpreted as a contact (mass) term, or a counter term, necessary to restore gauge invariance. In our analysis, on the contrary, this term was uniquely specified from the interference between the left and right movers in one space dimension, automatically providing gauge invariance. This is an important point of distinction.

3.2 Self-Dual Models and Bosonization in Three Dimensions: The massive Thirring model.

While the bosonization in three dimensions is not exact, nevertheless, for massive fermionic models in the large mass or, equivalently, the long wavelength limit, well defined local expressions are known to exist[27, 28]. Interestingly, these expressions exhibit a self or an anti self dual symmetry that is dictated by the signature of the fermion mass. Clearly, therefore, this symmetry simulates the dual aspects of the left and right chiral symmetry in the two dimensional example, thereby providing a novel testing ground for our ideas. Indeed, two distinct massive Thirring models with opposite mass signatures, are soldered to yield a massive Maxwell theory. This result is vindicated by a direct comparison of the current correlation functions obtained before and after the soldering process.

To effect the soldering consider the bosonization of the massive Thirring model in three dimensions[27, 28] that is therefore reviewed briefly. The relevant partition functional, in the Minkowski metric, is given by,

$$Z=\int D\psi D\overline{\psi}\, e^{i\int d^3x\left[\overline{\psi}(i\partial\!\!\!/+m)\psi-\frac{\lambda^2}{2}j_\mu j^\mu\right]} \tag{14}$$

where $j_\mu = \overline{\psi}\gamma_\mu\psi$ is the fermionic current. As usual, the four fermion interaction can be eliminated by introducing an auxiliary field,

$$Z=\int D\psi D\overline{\psi} D f_\mu\, e^{i\int d^3x\left[\overline{\psi}(i\partial\!\!\!/+m+\lambda f\!\!\!/)\psi+\frac{1}{2}f_\mu f^\mu\right]} \tag{15}$$

Contrary to the two dimensional models, the fermion integration cannot be done exactly. Under large mass limit, however, this integration is possible leading to closed and local expressions[29]. The leading term in this limit was calculated by various means[36] and shown to yield the Chern-Simons three form. Thus the partition functional for the massive Thirring model in the large mass limit is given by,

$$Z=\int D f_\mu\, e^{i\int d^3x\left(\frac{\lambda^2}{8\pi}\frac{m}{|m|}\epsilon_{\mu\nu\lambda}f^\mu\partial^\nu f^\lambda+\frac{1}{2}f_\mu f^\mu\right)} \tag{16}$$

where the signature of the topological terms is dictated by the corresponding signature of the fermionic mass term. The Lagrangian in the above partition function defines a self dual model introduced earlier[42]. The massive Thirring model, in the relevant limit, therefore bosonizes to a self dual model. It is useful to clarify the meaning of this self duality. The equation of motion is given by, $f_\mu = -\frac{\lambda^2}{4\pi}\frac{m}{|m|}\epsilon_{\mu\nu\lambda}\partial^\nu f^\lambda$ from which the following the relations $\partial_\mu f^\mu = 0$ and $\left(\partial_\mu\partial^\mu + M^2\right) f_\nu = 0$; $M = \frac{4\pi}{\lambda^2}$ may be easily verified. A field dual to f_μ is defined as, ${}^*f_\mu = \frac{1}{M}\epsilon_{\mu\nu\lambda}\partial^\nu f^\lambda$ where the mass parameter M is inserted for dimensional reasons. Repeating the dual operation, we find, ${}^*({}^*f_\mu) = \frac{1}{M}\epsilon_{\mu\nu\lambda}\partial^\nu\, {}^*f^\lambda = f_\mu$ thereby validating the definition of the dual field. Combining these results we conclude that, $f_\mu = -\frac{m}{|m|}{}^*f_\mu$. Hence, depending on the sign of the fermion mass term, the bosonic theory corresponds to a self-dual or an anti self-dual model. Likewise, the Thirring current bosonizes to the topological current

$$j_\mu = \frac{\lambda}{4\pi}\frac{m}{\mid m\mid}\epsilon_{\mu\nu\rho}\partial^\nu f^\rho \tag{17}$$

The close connection with the two dimensional analysis is now evident. There the starting point was to consider two distinct fermionic theories with opposite chiralities. In the present instance, the analogous thing is to take two independent Thirring models with identical coupling strengths but opposite mass signatures,

$$\begin{aligned}\mathcal{L}_+ &= \overline{\psi}\left(i\partial\!\!\!/ + m\right)\psi - \frac{\lambda^2}{2}\left(\overline{\psi}\gamma_\mu\psi\right)^2\\ \mathcal{L}_- &= \overline{\xi}\left(i\partial\!\!\!/ - m'\right)\xi - \frac{\lambda^2}{2}\left(\overline{\xi}\gamma_\mu\xi\right)^2\end{aligned} \tag{18}$$

Note that only the relative sign between the mass parameters is important, but their magnitudes are different. From now on it is also assumed that both m and m' are positive. Then the bosonized Lagrangians are, respectively,

$$\begin{aligned}\mathcal{L}_+ &= \frac{1}{2M}\epsilon_{\mu\nu\lambda}f^\mu\partial^\nu f^\lambda + \frac{1}{2}f_\mu f^\mu\\ \mathcal{L}_- &= -\frac{1}{2M}\epsilon_{\mu\nu\lambda}g^\mu\partial^\nu g^\lambda + \frac{1}{2}g_\mu g^\mu\end{aligned} \tag{19}$$

where f_μ and g_μ are the distinct bosonic vector fields. The current bosonization formulae in the two cases are given by

$$\begin{aligned}j_\mu^+ &= \overline{\psi}\gamma_\mu\psi = \frac{\lambda}{4\pi}\epsilon_{\mu\nu\rho}\partial^\nu f^\rho\\ j_\mu^- &= \overline{\xi}\gamma_\mu\xi = -\frac{\lambda}{4\pi}\epsilon_{\mu\nu\rho}\partial^\nu g^\rho\end{aligned} \tag{20}$$

It is now possible to effect the soldering following the general prescription detailed in the last section. The final result, after elimination of auxiliary soldering fields, is

$$\begin{aligned}\mathcal{L}_S &= \mathcal{L}_+ + \mathcal{L}_- - \frac{1}{8}\left(J^+_{\rho\sigma}(f) + J^-_{\rho\sigma}(g)\right)^2\\ &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{M^2}{2}A_\mu A^\mu\end{aligned} \tag{21}$$

where, $A_\mu = \frac{1}{\sqrt{2M}}\left(g_\mu - f_\mu\right)$ is the usual field tensor expressed in terms of the basic entity A_μ. The soldering mechanism has precisely fused the self and anti self dual symmetries to yield a massive Maxwell.

We conclude, therefore, that two massive Thirring models with opposite mass signatures, in the long wavelength limit, combine by the process of bosonization and soldering, to a massive Maxwell theory. The bosonization of the composite current, obtained by adding the separate contributions from the two models, is given in terms of a topological current of the massive vector theory. These results cannot be obtained by a straightforward application of conventional bosonization techniques. The massive modes in the original Thirring models are manifested in the two modes of (21) so that there is a proper matching in the degrees of freedom.

4 Electromagnetic Duality in Different Dimensions

4.1 Duality Groups.

The problem with the space-time dimensionality is a crucial one that has obscured the understanding of duality with technicalities and misconceptions. The distinction among the different even dimensions is manifest by the following double duality relation,

$$^{**}F = \begin{cases} +F, & \text{if } \mathrm{D} = 4\mathrm{k}+2\\ -F, & \text{if } \mathrm{D} = 4\mathrm{k}\end{cases} \tag{22}$$

where $*$ denotes the usual Hodge operation and F is a $\frac{D}{2}$-form. The concept of self duality seems to be well defined only in twice odd dimensions, and not present in the twice even cases and (22) apparently leads to separate consequences regarding the duality groups in these cases. The invariance of the actions in different D-dimensions is preserved by the following groups,

$$\mathcal{G}_d = \begin{cases} Z_2, & \text{if D} = 4\text{k}+2 \\ SO(2), & \text{if D} = 4\text{k} \end{cases} \tag{23}$$

which are called the "duality groups". The duality operation is characterized by a one-parameter SO(2) group of symmetry in D=4k dimensions, while for D=4k+2 dimensions it is manifest by a discrete Z_2 operation. Notice that only the 4 dimensional Maxwell theory and its 4k extensions would possess duality as a symmetry, while for the 2 dimensional scalar theory and its 4k+2 extensions duality is not even definable. We shall discuss the physical origin of this dichotomy.

A solution for the problem came with the recognition of a 2-dimensional internal structure hidden in the space of potentials[37, 38, 39]. Recently this author[6] has developed a systematic method for obtaining and investigating different aspects of duality symmetric actions that embraces all dimensions. A redefinition of the fields in the first-order form of the action naturally discloses the 2-dimensional internal structure hidden into the theory. This procedure produces two distinct classes of dual theories characterized by the opposite signatures of the (2x2) matrices in the internal space. These actions correspond to self dual and anti-selfdual representations of the original theory. Indeed the dichotomy (23) seems to be of much deeper physical origin, since it attributes different group structures to distinct dimensions.

The dual projection operation, that systematically discloses the internal duality space of any theory in D-dimensions is quickly discussed. The distinction of the duality groups is manifest, in the dual projection approach by the following construction. The first-order action for a free field theory is, in general, given as

$$\mathcal{L} = \Pi \cdot \dot{\Phi} - \frac{1}{2}\Pi \cdot \Pi - \frac{1}{2}\partial\Phi \cdot \partial\Pi \tag{24}$$

with Π and Φ being generic free tensor fields in D-1 dimensions and ∂ an appropriate differential operator. For visual simplicity we omit all tensor and space-time indices describing the fields, unless a specific example is considered. The parity of ∂ has a particularly interesting dependence with the dimensionality,

$$P(\partial) = \begin{cases} +1, & \text{if D} = 4\text{k} \\ -1, & \text{if D} = 4\text{k}+2 \end{cases} \tag{25}$$

where parity is defined as,

$$\int \Phi \cdot \partial\Psi = P(\partial) \int \partial\Phi \cdot \Psi \tag{26}$$

Take for instance the specific cases of 2 and 4 dimensions where ∂ is defined as,

$$\partial = \begin{cases} \partial_x, & \text{if D} = 2 \\ \epsilon_{kmn}\partial_n, & \text{ifD} = 4 \end{cases} \tag{27}$$

which we recognize as odd and even parity respectively. The internal space is disclosed by a suitable field redefinition as $(\Phi\ ,\ \Pi) \rightarrow (A_+\ ,\ A_-)$ that is dimensionally dependent,

$$\begin{aligned} \Phi &\rightarrow A_+ + A_- \\ \Pi &\rightarrow \eta\,(\partial A_+ - \partial A_-) \end{aligned} \tag{28}$$

with $\eta = \pm$ defining the signature of the duality symmetric action. The effect of the dual projection (28) into the first order action is manifest as,

$$\mathcal{L} = \eta\left(\dot{A}_\alpha \sigma_3^{\alpha\beta}\partial A_\beta + \dot{A}_\alpha \epsilon^{\alpha\beta}\partial A_\beta\right) - \partial A_\alpha \cdot \partial A_\alpha \tag{29}$$

We can appreciate the impact of the dimensionality over the structure of the internal space, and the role of the operator's parity in determining the appropriate group for each dimension. For twice odd dimension the dual projection produces the diagonalization of the first-order action into chiral actions, while for twice even dimensions, the result is either a self or an anti-selfdual action, depending on the sign of η,

$$\mathcal{L} = \begin{cases} \eta\dot{A}_\alpha \sigma_3^{\alpha\beta}\partial A_\beta - \partial A_\alpha \cdot \partial A_\alpha, & \text{if D} = 4\text{k}+2 \\ \eta\dot{A}_\alpha \epsilon^{\alpha\beta}\partial A_\beta - \partial A_\alpha \cdot \partial A_\alpha, & \text{ifD} = 4\text{k} \end{cases} \tag{30}$$

By inspection on the above actions one finds that while the first is duality symmetric under the discrete Z_2, the second is invariant under the continuous one-parameter group $SO(2)$. This is in accord with general discussion based on algebraic methods[40]. In fact, using the choice of operators in (27) we easily find that the D=2 case describes a right and a left Floreanini-Jackiw chiral actions[41] if we identify Φ with a scalar field,

$$\mathcal{L} = \eta\dot{A}_\alpha \sigma_3^{\alpha\beta} A'_\beta - A'_\alpha \cdot A'_\alpha \tag{31}$$

The second case, on the other hand, describes either selfdual or anti-selfdual Schwarz-Sen actions, according to the signature of η,

$$\mathcal{L} = \eta\dot{\varphi}_k^\alpha \epsilon^{\alpha\beta} B_k^\beta - B_k^\alpha \cdot B_k^\alpha \tag{32}$$

if we identify Φ with a vector field, $\Phi \rightarrow \varphi_k^\alpha$ and $\partial\Phi \rightarrow \epsilon_{kmn}\partial_m\varphi_n^\alpha = B_k^\alpha$, with B_k^α being the magnetic field.

In summary the above analysis clearly shows that the physical origin for the dimensional dependence of the electromagnetic duality group lies in the parity dependence on dimensionality of a curl-operator naturally defined in the solution of the Gauss law.

4.2 Duality Equivalence: Noether Embedding.

Using the well known equivalence between the self-dual[42] and the topologically massive models[36] proved by Deser and Jackiw[43] through the master action approach, a correspondence has been established between the partition functions for the MTM and the Maxwell-Chern-Simons (MCS) theories. The situation for the case of fermions carrying non-Abelian charges, however, is less understood due to a lack of equivalence between these vectorial models, which has only been established for the weak coupling regime[44]. As critically observed in [45], the use of master actions in this situation is ineffective for establishing dual equivalences.

We propose a new technique to perform duality mappings for vectorial models in any dimensions that is alternative to the master action approach. It is based on the traditional idea of a local lifting of a global symmetry and may be realized by an iterative embedding of Noether counter terms. This technique was originally explored in the context of the soldering formalism[2, 3] and is exploited here since it seems to be the most appropriate technique for non-Abelian generalization of the dual mapping concept.

Using the gauge embedding idea, we clearly show the dual equivalence between the non-Abelian self-dual and the Yang-Mills-Chern-Simons models, extending the proof proposed by Deser and Jackiw in the Abelian domain. These results have consequences for the bosonization identities from the massive Thirring model into the topologically massive model, which are considered here, and also allows for the extension of the fusion of the self-dual massive modes[3] for the non-Abelian case[12]. As mentioned in the introduction this has the advantage of possessing a straightforward extension to the non-Abelian case for all values of the coupling constant.

The non-Abelian version of the vector self-dual model (16), which is our main concern, is given by

$$\mathcal{S}_\chi=\int d^3x\, tr\left[-\frac{1}{2}F_\mu F^\mu+\frac{\chi}{4m}\epsilon^{\mu\nu\lambda}\left(F_{\mu\nu}F_\lambda-\frac{2}{3}F_\mu F_\nu F_\lambda\right)\right] \quad (33)$$

where $F_\mu = F_\mu^a t^a$, is a vector field taking values in the Lie algebra of a symmetry group G and t^a are the matrices representing the underlying non Abelian gauge group with $a = 1,\ldots,\dim G$. The field-strength tensor and the covariant derivative have their usual meaning.

Using the master action approach, the action (33) has been shown to be equivalent to the gauge invariant Yang-Mills-Chern-Simons (YMCS) theory $\mathcal{S}$

$$\mathcal{S}=\int d^3x tr\left[\frac{1}{4m^2}F^{\mu\nu}F_{\mu\nu}+\frac{\chi}{4m}\epsilon^{\mu\nu\lambda}\left(F_{\mu\nu}F_\lambda-\frac{2}{3}F_\mu F_\nu F_\lambda\right)\right] \quad (34)$$

only in the weak coupling limit $g\to 0$ so that the Yang-Mills term effectively vanishes. Here we are using the bosonization nomenclature that relates the Thirring model coupling constant g^2 with the inverse mass of the vector model. To study the dual equivalence of (33) and (34) for all coupling regimes, a problem open for more than twenty years, and the consequences over the bosonization program is main contribution of this work.

We are now in position to study the non-Abelian version of the vector self-dual model whose dynamics is given by the action (33). To this end we discuss first in which sense this model possess the self-dual property. Following the same reasoning as in the Abelian case, we define the duality operation as,

$${}^\star F^\lambda \equiv \left[\frac{\chi}{m}\,\epsilon^{\mu\nu\lambda}\,\left(\partial_\mu+F_\mu\right)\right]\,F_\nu\,, \quad (35)$$

where the operator inside the square brackets in the right-hand side acts on the basic field F_ν defining ${}^\star F^\lambda$ as the dual of F_ν. Repeating this operation, and using the equations of motion obtained by varying (33) with respect to F_λ

$$F^\lambda = \frac{\chi}{2m}\,\epsilon^{\mu\nu\lambda}\,F_{\mu\nu}\,, \quad (36)$$

we find, ${}^\star\left({}^\star F_\mu\right) = F_\mu$ thus justifying our terminology and showing the self-dual character of this model.

Likewise the Abelian case, to proceed with the dualization, we begin with the zeroth-iterated action (33) whose variation with respect to F_μ is given by

$$\delta\mathcal{S}_\chi = \int d^3x\, tr\left[J^\mu\delta F_\mu\right]\,, \quad (37)$$

with the Noether currents being defined as,

$$J_\mu = -F_\mu + \frac{\chi}{2m}\,\epsilon_{\mu\nu\lambda}\,F^{\mu\nu}\,. \quad (38)$$

Our goal is to obtain a non-Abelian gauge invariant theory from the above non-invariant self-dual model. To this end we define the first-iterated action by a coupling between the currents J_μ and an auxiliary field B_μ,

$$\mathcal{S}^{(1)} = \mathcal{S}_\chi - \int\, d^3x\, tr\left[\,J^\mu\,B_\mu\,\right]\,, \quad (39)$$

and whose variation is

$$\delta\mathcal{S}^{(1)}=\int d^3x\, tr\left[-\frac{1}{2}\delta(J^\mu J_\mu)-\delta(J^\mu B_\mu)-J^\mu\delta B_\mu\right]\,, \quad (40)$$

where we have used the following transformation rule for the gauging field, $\delta F_\mu = -\delta B_\mu - \delta J_\mu$. This prompt us to define the following second iterated action,

$$\begin{aligned}\mathcal{S}^{(2)}&=\mathcal{S}_\chi + \int\, d^3x\, tr\left[\,\frac{1}{2}\,J^\mu\,J_\mu - F^\mu\,B_\mu\,\right]\\ &= \int d^3x\, tr\left[\frac{\chi}{4m}\,\epsilon^{\mu\nu\lambda}\left(F_{\mu\nu}F_\lambda-\frac{2}{3}F_\mu F_\nu F_\lambda\right)+\frac{1}{2}B_\mu B^\mu\right]\end{aligned}$$

which is gauge invariant after noticing that this transformation rule fixes the B_μ field as $B_\mu = -\frac{\chi}{2m}\,\epsilon_{\mu\nu\lambda}\,F^{\nu\lambda}$. Thanks to the structure of the current (38), this action can finally be put in a more familiar presentation as the gauge invariant theory, $\mathcal{S}^{(2)} \to \mathcal{S}^{YMCS}$ the Yang-Mills-Chern-Simons theory defined in (34). This proves that just as in the Abelian case, the non-Abelian self-dual action defined in (33) is physically equivalent to (34) for all regimes of the coupling constant. In the process we have also shown the duality transformation that correctly defines the inherent self-duality property of action (33).

In summary, a general method has been developed that establishes dual equivalence between self-dual and topologically massive theories based on the idea of gauge embedding over second-class constrained systems. The equivalence has been established using an adaptation of the iterative Noether procedure both for Abelian and non-Abelian self-dual models, including the cases with coupling to dynamical charged fermions.

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