Physical Variables of $d=3$ Maxwell-Chern-Simons Theory by Symplectic Projector Method

J. A. Helayel-Neto[a], M. A. Santos[b], and I. V. Vancea[c]
[a] Centro Brasileiro de Pesquisas Físicas, (CBPF) R. Dr. Xavier Sigaud, 150, 22290-180 Rio de Janeiro, RJ, Brazil
[b] Departamento de Física, Universidade Federal do Espirito Santo (UFES),
Av. Fernando Ferrari S/N, Campos de Goiaberas, 29060-900 Vitória, ES, Brazil
[c] Departamento de Física, Universidade Federal Rural do Rio de Janeiro (UFRRJ), BR 465-07-Seropédica, RJ, Brazil

Received on 18 October, 2006

The Symplectic Projector Method is applied to derive the local physical degrees of freedom and the physical Hamiltonian of the Maxwell-Chern-Simons theory in $d=1+2$. The results agree with the ones obtained in the literature through different approaches.

Keywords: Physical variables; Canonical quantization; Maxwell-Chern-Simons

The quantization of the constrained systems is crucial in building realistic theoretical models. A major challenge in the process of quantization is the identification of the physical degrees of freedom of the system. The most general method of quantization, the BRST method, reduces this problem to the task of solving the cohomology of some nilpotent operators associated to the symmetry group [1]. Although elegant and complete, the full construction of BRST is sometimes unnecessary as more intuitive, albeit less general methods, can be used for quantization. In [2], a procedure for separate the physical degrees of freedom for systems with second class constraints, called "symplectic projector method" (SPM), was proposed and it was subsequently developed in [3–6]. The idea behind the SPM is to construct a local projector from the phase space of the constrained system to the surface of constraints and to use it to obtain the local physical coordinates and the unconstrained Hamiltonian. The SPM represents a first step to treat the gauge theories in a strictly canonical way and it has already been applied to particles on holonomic surfaces [7], non-comutative strings [8] and Abelian Chern-Simons systems [9].

One of the most interesting class of models in field theory is described by the so called Maxwell-Chern-Simons theories (MCS) which are important because they are simultaneously massive and gauge invariant. Recently, the MCS models have been used to study various phenomena related to the electric charges in the Standard Model Extension, topological massive electrodynamics and fractional statistics, vortex solutions in topological field theory, Lorentz symmetry breaking, D-brane Universe, large-N field theories, dualities in field theories and quantum Hall effect to mention just some of their applications. Therefore, the quantization of MCS theories represents an interesting problem already addressed in the frameworks of the symplectic quantization [11], geometric representation [13], covariant Coulomb gauge [15], canonical Coulomb gauge [16], Fadeev-Jackiew formalism [17] and BFT formalism [14] (see also [10, 12, 18–21]).

The aim of this letter is to explicitely derive the physical degrees of freedom and the physical Hamiltonian of the $d=3$ MCS theories by using the SPM in the canonical Coulomb gauge without matter. Our result is consistent with the one given in [16] which uses the Dirac quantization procedure. Compared to [16], our approach is simpler and faster. This represents a non-trivial application of the SPM and proves that it is an effective method applicable to interesting field theoretical models.

Let us start by recalling the basic ideas of the SPM. Consider an arbitrary system with second class constraints $\phi^m\left(\xi^M\right)=0$ where $\xi^M=(x^a,p_a)$, $M=1,2,\ldots,2N$ are the coordinates in the phase space which is assumed to be isomorphic to R^{2N} and $m=1,2,\ldots,r=2k$. One can define a symplectic projector from the phase space of the system to the constraint surface [3] by the following relation

$$\Lambda^{MN}=\ \delta^{MN}-J^{ML}\frac{\delta\phi_m}{\delta\xi^L}\Delta^{-1}_{mn}\frac{\delta\phi_n}{\delta\xi^N}. \tag{1}$$

Here, J^{MN} is the symplectic two-form in the original phase space and Δ^{-1}_{mn} is the inverse of the matrix constructed from the Poisson brackets of the constraint functions

$$\Delta_{mn}=\{\phi_m,\phi_n\}. \tag{2}$$

The action of the symplectic projector given by the relation (1) is to project the phase space variables ξ^M onto a set of local variables on the constraint surface ξ^*

$$\xi^{*M}=\Lambda^{MN}\xi^N. \tag{3}$$

¿From these, one can construct the physical Hamiltonian by writing the original Hamiltonian in terms of the physical coordinates (3) which are independent, unconstrained variables that obey the canonical commutation relations. Next, one can derive the equations of motion from the Hamilton-Jacobi equations:

$$\dot{\xi}^*=\{\xi^*,H^*\}, \tag{4}$$

where $\{\ ,\ \}$ are the Poisson brackets. Comparing the Dirac matrix given by the following relation

$$D^{MN}=\{\xi^M\ ,\ \xi^N\}_D=J^{MN}-J^{ML}J^{KN}\frac{\delta\phi_m}{\delta\xi^L}\Delta^{-1}_{mn}\frac{\delta\phi_n}{\delta\xi^K}, \tag{5}$$

with the symplectic projector from (1), one can see [5] that the following relation holds:

$$\Lambda=-DJ. \tag{6}$$

One can quantize the theory starting from the physical Hamiltonian described above. The other observables of the quantum theory are obtained in the same way and they depend on the physical coordinates only.

Now let us apply the above procedure to the MCS theory in Minkowski background. The Lagrangian is given by the following relation

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + m\varepsilon^{\alpha\beta\gamma}A_{\alpha}\partial_{\beta}A_{\gamma}, \tag{7}$$

where $\mu,\nu = 1,2,3$, ε is the antisymmetric tensor in $d=3$ and the metric has the signature $(-1,1,1)$. Here, $F = dA$ and m represents the mass parameter. The canonical Hamiltonian that is obtained from the Lagrangian (7) has the following well known form

$$\mathcal{H} = \int d^2x \left[\frac{1}{2}\pi^i\pi^i + \frac{1}{2}\left(\varepsilon^{ij}\partial^i A^j\right)^2 + \frac{1}{2}m^2 A^k A^k + m\varepsilon^{ij}A^i\pi^j\right]. \tag{8}$$

The system displays second class constraints given by the following relations

$$\Omega^1 = \pi^0 = 0, \tag{9}$$
$$\Omega^2 = \partial^i\pi^i + m\varepsilon^{ij}\partial^j A^i = 0, \tag{10}$$
$$\Omega^3 = A^0 = 0, \tag{11}$$
$$\Omega^4 = \partial^i A^i = 0. \tag{12}$$

The inverse g_{ij} of the matrix

$$g^{ij}(x,y) = \left\{\Omega^i(x), \Omega^j(y)\right\}, \tag{13}$$

constructed from the above constraints (9)-(12) defines a metric in the phase space which has the following form

$$g^{-1} = \begin{pmatrix} 0 & 0 & \delta^2(x-y) & 0 \\ 0 & 0 & 0 & \nabla^{-2} \\ -\delta^2(x-y) & 0 & 0 & 0 \\ 0 & -\nabla^{-2} & 0 & 0 \end{pmatrix}. \tag{14}$$

By using the general formula (1), one can easily show that the local symplectic projector in quantum field theory should be given by the following formula:

$$\Lambda^{\mu}_{\nu}(x,y) = \delta^{\mu}_{\nu}\delta^2(x-y) - \varepsilon^{\mu\alpha}\int d^2r\, d^2\varpi\, g_{ij}(r,\varpi)\,\delta_{\alpha(x)}\Omega^i(r)\,\delta_{\nu(y)}\Omega^j(\varpi), \tag{15}$$

where

$$\delta_{\alpha(x)}\Omega^i(r) \equiv \frac{\delta\Omega^i(r)}{\delta\xi^{\alpha}(x)}. \tag{16}$$

We now have all the ingredients at hand to find out the physical variables of the MCS theory (8). As a first step we compute the symplectic projector of the system by explicitly working out (15) and (14). After some tedious algebra one finds the following result

$$\Lambda = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \delta^2(x-y) - \frac{\partial_1^x\partial_1^y}{\nabla^2} & -\frac{\partial_1^x\partial_2^y}{\nabla^2} & 0 & 0 & 0 \\ 0 & -\frac{\partial_2^x\partial_1^y}{\nabla^2} & \delta^2(x-y) - \frac{\partial_2^x\partial_2^y}{\nabla^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -m\delta^2(x-y) & 0 & \delta^2(x-y) - \frac{\partial_1^x\partial_1^y}{\nabla^2} & -\frac{\partial_1^x\partial_2^y}{\nabla^2} \\ 0 & m\delta^2(x-y) & 0 & 0 & -\frac{\partial_2^x\partial_1^y}{\nabla^2} & \delta^2(x-y) - \frac{\partial_2^x\partial_2^y}{\nabla^2} \end{pmatrix}. \tag{17}$$

The next step is to apply the above projector to the field variables. Since the symplectic structure is most conveniently displayed in a symmetric notation, let us rename the field variables as follows

$$\left(A^0, A^1, A^2, \pi^0, \pi^1, \pi^2\right) \Leftrightarrow \left(\xi^1, \xi^2, \xi^3, \xi^4, \xi^5, \xi^6\right). \tag{18}$$

The Hamiltonian (8) in this notation has the following form

$$\mathcal{H}=\int d^2x\left[\frac{1}{2}\left(\xi_5^2+\xi_6^2\right)+\frac{1}{2}\left(\partial_1\xi_3-\partial_2\xi_2\right)^2+\frac{1}{2}m^2\left(\xi_2^2+\xi_3^2\right)+m\left(\xi_2\xi_6-\xi_3\xi_5\right)\right]. \tag{19}$$

We denote the physical variables by $\xi_\mu^*(x)$. The definition (3) now obviously reads

$$\xi^{\mu *}(x)=\int d^2y\Lambda_\nu^\mu(x,y)\,\xi^\nu(y). \tag{20}$$

By using the relation (17) in to the equation (20) one obtains the following physical fields of the MCS theory

$$\begin{aligned}\xi^{1*}(x)&=\xi^{4*}(x)=0, &(21)\\ \xi^{2*}(x)&=A_1^\perp(x), &(22)\\ \xi^{3*}(x)&=A_2^\perp(x), &(23)\\ \xi^{5*}(x)&=\pi_1^\perp(x)-mA_2^\perp(x), &(24)\\ \xi^{6*}(x)&=\pi_2^\perp(x)+mA_1^\perp(x). &(25)\end{aligned}$$

By using the physical coordinates (21)-(25) in the relation (19) we obtain the following projected Hamiltonian

$$\mathcal{H}^*=\int d^2x\left[\frac{1}{2}\left(\xi_5^{*2}+\xi_6^{*2}\right)+\frac{1}{2}\left(\partial_1\xi_3^*-\partial_2\xi_2^*\right)^2+\frac{1}{2}m^2\left(\xi_2^{*2}+\xi_3^{*2}\right)+m\left(\xi_2^*\xi_6^*-\xi_3^*\xi_5^*\right)\right]. \tag{26}$$

From the projected Hamiltonian (26) one derives the equations of motion by using the Hamilton-Jacobi equations:

$$\begin{aligned}\ddot{\xi}_2^* &= -2m^2\xi_2^*+\partial_2\partial_2\xi_2^*-\partial_1\partial_2\xi_3^*-2m\xi_6^*, &(27)\\ \ddot{\xi}_3^* &= -2m^2\xi_3^*+\partial_1\partial_1\xi_3^*-\partial_1\partial_2\xi_2^*-2m\xi_5^*, &(28)\\ \ddot{\xi}_5^* &= -2m^2\xi_5^*+\partial_2\partial_2\xi_5^*-\partial_1\partial_2\xi_6^*+m\left[2m^2-\nabla^2\right]\xi_3^*, &(29)\\ \ddot{\xi}_6^* &= -2m^2\xi_6^*+\partial_1\partial_1\xi_6^*-\partial_1\partial_2\xi_5^*-m\left[2m^2-\nabla^2\right]\xi_2^*. &(30)\end{aligned}$$

In order to compare our results with the ones given in the literature we go back to the standard field notation in which the physical Hamiltonian (26) has the form

$$\mathcal{H}^*=\int d^2x\left[\frac{1}{2}\left(\pi_i^\perp\pi_i^\perp+4m^2A_i^\perp A_i^\perp\right)+\frac{1}{2}\left(\xi^{ij}\partial_iA_j^\perp\right)^2+2m\left(A_1^\perp\pi_2^\perp-A_2^\perp\pi_1^\perp\right)\right]. \tag{31}$$

This represents the MCS Hamiltonian in the canonical Coulomb gauge. The result (31) agrees with the transverse expression of the Hamiltonian obtained in [16] along a different line of arguments. Using the standard field notations, the equations of motion given by the relations (27)-(30) take the familiar look

$$\begin{aligned}\left(\Box+4m^2\right)A_1^\perp&=-2m\pi_2^\perp, &(32)\\ \left(\Box+4m^2\right)A_2^\perp&=2m\pi_1^\perp, &(33)\\ \Box\pi_1^\perp&=0, &(34)\\ \Box\pi_2^\perp&=0, &(35)\end{aligned}$$

which amounts to ensuring that

$$\Box\left(\Box+4m^2\right)A_i^\perp=0,\quad(i=1,2). \tag{36}$$

This equation guarantees that the physical excitation is a massive $\left(p^2=4m^2\right)$ transverse vector. The massless quantum ($p^2=0$) is a spurious one: it has no dynamical rôle and does not correspond to any physical mode . Indeed, by coupling the A_μ field propagator to a conserved external current, the current-current amplitude is such that the imaginary part of its residue taken at the pole $p^2=0$ vanishes, wich confirms that the latter does not correspond to any physical excitation. On the other hand, the non-trivial pole $p^2=4m^2$ yields a positive defined residue which enforces its physical character as the only degree of freedom carried by the A_μ field.

The quantization of the system should be performed in the usual fashion, starting from the physical variables (21)-(25). In order to avoid any confusion, we should stress out that the true physical Hamiltonian is the one given in the relation (26) in which the physical variables ξ^*'s from (21)-(25) obey the canonical Poisson brackets [2]. The transverse field variables used above just help us to compare the results obtained from the SPM with the ones in the literature.

In conclusion, we have obtained the physical field variables and the physical Hamiltonian of the MCS theory by projecting the constrained system onto the constraint surface. Our results agree with the one obtained in the literature in the context of quantization of MCS within the canonical formalism.

Acknowledgements

We would like to thank to M. A. De Andrade for useful discussions. I. V. V. acknowledges discussions with M. Botta-Cantcheff.

[1] M. Henneaux, C. Teitelboim, *Quantization of Gauge Systems*, (Princeton University Press, 1992)
[2] C. M. Amaral, 'Nuovo Cim. B **25**, 817 (1975).
[3] P. Pitanga and C. M. Amaral, Nuovo Cim. A **103** (1990) 1529.
[4] M. A. Santos, J. C. de Mello and P. Pitanga, Z. Phys. C **55**, 271 (1992).
[5] M. A. De Andrade, M. A. Santos, and I. V. Vancea, Mod. Phys. Lett. A **16**, 1907 (2001) [arXiv:hep-th/0108197].
[6] M. A. Dos Santos, hep-th/0202078.
[7] M. A. De Andrade, M. A. Santos, and I. V. Vancea, arXiv:hep-th/0308169.
[8] M. A. De Andrade, M. A. Santos, and I. V. Vancea, JHEP **0106**, 026 (2001) [arXiv:hep-th/0104154].
[9] L. R. Manssur, A. L. Nogueira, and M. A. Santos, Int. J. Mod. Phys. A **17**, 1919 (2002) hep-th/0005214. In arbirary dimension:
[10] X. Bekaert and A. Gomberoff, JHEP **0301**, 054 (2003) [arXiv:hep-th/0212099].
[11] S. T. Hong, Y. W. Kim, Y. J. Park, and K. D. Rothe, J. Phys. A **35**, 7461 (2002) [arXiv:hep-th/0204188].
[12] S. Ghosh, Annals Phys. **291**, 1 (2001) [arXiv:hep-th/0009165].
[13] L. Leal and O. Zapata, Phys. Rev. D **63**, 065010 (2001) [arXiv:hep-th/0008049].
[14] M. Fleck, arXiv:hep-th/0007098.
[15] K. Haller and E. Lim-Lombridas, Annals Phys. **246**, 1 (1996) [Erratum-ibid. **257**, 205 (1997)] [arXiv:hep-th/9409133].
[16] F. P. Devecchi, M. Fleck, H. O. Girotti, M. Gomes, and A. J. da Silva, Annals Phys. **242**, 275 (1995) [arXiv:hep-th/9411224].
[17] H. l. Lee, Y. W. Kim, and Y. J. Park, J. Korean Phys. Soc. **30**, 23 (1997) [arXiv:hep-th/9711186].
[18] H. O. Girotti, Int. J. Mod. Phys. A **14**, 2495 (1999) [arXiv:hep-th/9803153].
[19] S. Li and Y. s. Duan, Astrophys. Space Sci. **268**, 455 (1999) [arXiv:gr-qc/9808078].
[20] S. Ghosh, arXiv:hep-th/9901051.
[21] T. Itoh and P. Oh, Phys. Rev. D **63**, 025019 (2001) [arXiv:hep-th/0006163].