From Integral to Derivative Dispersion Relations

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We demonstrate that integral dispersion relations for hadron-hadron scattering amplitudes can be replaced by differential relations, without the usual high-energy approximation. We obtain analytical expressions for the corrections associated with the low energy region and exemplify the applicability of the novel relations in the context of an analytical parametrization for proton-proton and antiproton-proton total cross sections.

Keywords: Elastic hadron scattering; Dispersion relations; High energies

I. INTRODUCTION

Dispersion relations constitute a fundamental mathematical tool in several areas of Physics. In special, in particle-particle and particle-antiparticle interactions, the correlation between the real and imaginary parts of the scattering amplitude play an important role in the investigation of forward elastic hadron-hadron interactions.

Analyticity, Unitarity and Crossing lead to *Integral Dispersion Relations* (IDR) for the scattering amplitudes in terms of a crossing symmetrical variable. For an elastic process, $m+m \to m+m$, in the forward direction, this variable corresponds to the energy of the incident particle in the laboratory system, E. In this context the one subtracted IDR for crossing even $(+)$ and $(-)$ odd amplitudes, above the physical threshold ($E=m$), read [1, 2]

$$\mathrm{Re}\, F_+(E) = K + \frac{2E^2}{\pi} P\int_m^{+\infty} \mathrm{d}E' \frac{1}{E'(E'^2-E^2)}\, \mathrm{Im}\, F_+(E'), \quad (1)$$

$$\mathrm{Re}\, F_-(E) = \frac{2E}{\pi} P\int_m^{+\infty} \mathrm{d}E' \frac{1}{(E'^2-E^2)}\, \mathrm{Im}\, F_-(E'), \quad (2)$$

where K is the subtraction constant.

The connections with the hadronic amplitudes for crossed channels, such as proton-proton (pp) and antiproton-proton ($\bar{p}p$) elastic scattering, are given by the usual definitions:

$$F_{pp} = F_+ + F_- \qquad F_{\bar{p}p} = F_+ - F_-. \quad (3)$$

The main practical use of the IDR concerns simultaneous investigations on the total cross section (Optical Theorem) and the ratio ρ of the real to imaginary parts of the forward amplitude. In terms of the crossing symmetrical variable E these physical quantities are given, respectively, by [2]

$$\sigma_{\mathrm{tot}} = \frac{4\pi}{\sqrt{E^2-m^2}} \mathrm{Im}\, F(E, \theta_{\mathrm{lab}}=0), \quad (4)$$

$$\rho(E) = \frac{\mathrm{Re}\, F(E, \theta_{\mathrm{lab}}=0)}{\mathrm{Im}\, F(E, \theta_{\mathrm{lab}}=0)}, \quad (5)$$

where θ_{lab} is the scattering angle in the laboratory system.

Although originally introduced in the above integral forms, subsequent analyses have made use of differential forms, which are obtained from the integral ones in the limit of high energies, specifically by considering $m \to 0$ in Eqs. (1) and (2) [3]. In terms of the variable E, these *Derivative Dispersion Relations* (DDR) are given by

$$\mathrm{Re}\, F_+(E) = K + E \tan\left[\frac{\pi}{2}\frac{\mathrm{d}}{\mathrm{d}\ln E}\right] \frac{\mathrm{Im}\, F_+(E)}{E}, \quad (6)$$

$$\mathrm{Re}\, F_-(E) = \tan\left[\frac{\pi}{2}\frac{\mathrm{d}}{\mathrm{d}\ln E}\right] \mathrm{Im}\, F_-(E). \quad (7)$$

For a recent critical review on the replacement of IDR by DDR at high energies see Ref. [4].

In this communication we demonstrate that this replacement can be analytically performed without the high-energy approximation, which is an important result since, in principle, it allows the investigation of the total cross section and the ρ parameter at any energy above the physical threshold. We also exemplify the applicability of these novel relations by means of analytic parametrizations for the scattering amplitude in the context of a Pomeron-Reggeon model for pp and $\bar{p}p$ elastic processes.

The paper is organized as follows. In Sec. 2 we discuss the analytical replacement of IDR by DDR, without high-energy approximations. In Sec. 3 we present the results of the fits to forward data on pp and $\bar{p}p$ scattering. The conclusions and some final remarks are the contents of Sec. 4.

II. EXTENDED DERIVATIVE DISPERSION RELATIONS

The developments that follows were inspired in an previous work by Cudell, Martynov and Selyugin [5]. The differences between the derivative representations by these authors and our final results are discussed in [6], as well as other formal aspects involved.

Let us consider the even amplitude, Eq. (1). Integrating by parts we obtain

$$\begin{aligned} \mathrm{Re}\,F_+(E) &= K - \frac{E}{\pi}\ln\left|\frac{m-E}{m+E}\right|\frac{\mathrm{Im}\,F_+(m)}{m} \\ &\quad - \frac{E}{\pi}\int_m^\infty \ln\left|\frac{E'-E}{E'+E}\right|\frac{\mathrm{d}}{\mathrm{d}E'}\frac{\mathrm{Im}\,F_+(E')}{E'}\mathrm{d}E'. \end{aligned} \tag{8}$$

Following Ref. [5], we define $E' = me^{\xi'}$ and $E = me^{\xi}$ so that the integral term in the above formula is expressed by

$$\frac{me^{\xi}}{\pi}\int_0^\infty \ln\coth\left(\frac{1}{2}|\xi'-\xi|\right)\frac{\mathrm{d}}{\mathrm{d}\xi'}g(\xi')\mathrm{d}\xi', \tag{9}$$

where $g(\xi') = \mathrm{Im}\,F(me^{\xi'})/(me^{\xi'})$. Expanding the logarithm in the integrand in powers of $x = \xi' - \xi$,

$$\ln\left(\cot\frac{1}{2}|x|\right) = \ln\left(\frac{1+\mathrm{e}^{-|x|}}{1-e^{-|x|}}\right) = 2\sum_{p=0}^{\infty}\frac{\mathrm{e}^{-(2p+1)|x|}}{2p+1},$$

and assuming that g *is an analytic function of its argument*, we perform the expansion

$$\begin{aligned} \tilde{g}(\xi') &= \frac{\mathrm{d}}{\mathrm{d}\xi'}g(\xi') = \sum_{n=0}^{\infty}\frac{\mathrm{d}^n}{\mathrm{d}\xi'^n}\tilde{g}(\xi')\bigg|_{\xi'=\xi}\frac{(\xi'-\xi)^n}{n!} \\ &= \sum_{n=0}^{\infty}\frac{\tilde{g}^{(n)}(\xi)}{n!}(\xi'-\xi)^n. \end{aligned}$$

Substituting the above formulas in Eq. (9), integrating term by term and from $\xi = \ln(E/m)$, Eq. (8) can be put in the final form

$$\mathrm{Re}\,F_+(E) = K + E\tan\left(\frac{\pi}{2}\frac{\mathrm{d}}{\mathrm{d}\ln E}\right)\frac{\mathrm{Im}\,F_+(E)}{E} + \Delta^+(E,m) \tag{10}$$

where the correction factor Δ^+ is given by

$$\Delta^+(E,m) = -\frac{E}{\pi}\ln\left|\frac{m-E}{m+E}\right|\frac{\mathrm{Im}\,F_+(m)}{m} + \frac{2E}{\pi}\sum_{k=0}^{\infty}\sum_{p=0}^{\infty}\frac{(-1)^{k+1}\Gamma(k+1,(2p+1)\ln(E/m))}{(2p+1)^{k+2}k!}\frac{\mathrm{d}^{k+1}}{\mathrm{d}(\ln E)^{k+1}}\frac{\mathrm{Im}\,F_+(E)}{E}. \tag{11}$$

With analogous procedure for the odd relation we obtain

$$\mathrm{Re}\,F_-(E) = \tan\left(\frac{\pi}{2}\frac{\mathrm{d}}{\mathrm{d}\ln E}\right)\mathrm{Im}\,F_-(E) + \Delta^-(E,m), \tag{12}$$

where

$$\Delta^-(E,m) = -\frac{1}{\pi}\ln\left|\frac{m-E}{m+E}\right|\mathrm{Im}\,F_-(m) + \frac{2}{\pi}\sum_{k=0}^{\infty}\sum_{p=0}^{\infty}\frac{(-1)^{k+1}\Gamma(k+1,(2p+1)\ln(E/m))}{(2p+1)^{k+2}k!}\frac{\mathrm{d}^{k+1}}{\mathrm{d}(\ln E)^{k+1}}\mathrm{Im}\,F_-(E) \tag{13}$$

Equations (10 - 13) are the novel *Extended Derivative Dispersion Relations* (EDDR), which, in principle, are valid for any energy above the physical threshold $E = m$. We note that the correction factors $\Delta^{\pm} \to 0$ as $E \to \infty$, leading, in this case, to the standard high-energy results, Eqs. (6) and (7).

III. REGGE PARAMETRIZATION

In order to check the consistences between the IDR and the EDDR in an specific practical example, we consider, as a framework, a Regge Parametrization for the scattering amplitude [7]. For pp and $\bar{p}p$ scattering this analytical model assumes nondegenerate contributions from the even (+) and odd (−) secondary reggeons (a_2, f_2 and ρ, ω, respectively), together with a simple pole Pomeron contribution:

$$\mathrm{Im}\,F(E) = XE^{\alpha_P(0)} + Y_+E^{\alpha_+(0)} + \tau Y_-E^{\alpha_-(0)}, \tag{14}$$

where $\tau = +1$ for pp and $\tau = -1$ for $\bar{p}p$. As usual, the Pomeron and the even/odd reggeon intercepts are expressed by

$$\alpha_P(0) = 1 + \varepsilon, \qquad \alpha_{+/-}(0) = 1 - \eta_{+/-}. \tag{15}$$

The point is to treat simultaneous fits to the total cross section and the ρ parameter from pp and $\bar{p}p$ and compare the results obtained with both IDR and EDDR. Schematically, with parametrization (14-15) we determine $\operatorname{Im} F_{+/-}(E)$ through Eq. (3) and then $\operatorname{Re} F_{+/-}(E)$ either by means of the IDR, Eqs. (1-2) or the EDDR, Eqs. (10-13). Returning to Eq. (3) we obtain $\operatorname{Im} F_{pp}(E)$ and $\operatorname{Im} F_{\bar{p}p}(E)$ and, at last, Eqs. (4) and (5) lead to the analytical connections between $\sigma_{tot}(E)$ and $\rho(E)$ for both reactions.

A. Analytical Results

In the case of IDR we obtain

$$\begin{aligned}\rho(E)\sigma_{\text{tot}}(E) &= \frac{4\pi E}{\sqrt{E^2-m^2}}\left\{\frac{K}{E} \pm Y_- E^{-\eta_-}\cot\left(\eta_-\frac{\pi}{2}\right) + XE^{\varepsilon}\tan\left(\varepsilon\frac{\pi}{2}\right) - Y_+E^{-\eta_+}\tan\left(\eta_+\frac{\pi}{2}\right)\right.\\ &+ \left.\frac{2}{\pi}\sum_{j=0}^{\infty}\left(\pm\frac{Y_-m^{1-\eta_-}}{E(2j+2-\eta_-)} + \frac{Xm^{\varepsilon}}{2j+1+\varepsilon} + \frac{Y_+m^{-\eta_+}}{2j+1-\eta_+}\right)\left(\frac{m}{E}\right)^{2j+1}\right\}\end{aligned} \tag{16}$$

and with the EDDR we have

$$\begin{aligned}\rho(E)\sigma_{\text{tot}}(E) &= \frac{4\pi E}{\sqrt{E^2-m^2}}\left\{\frac{K}{E} \pm Y_- E^{-\eta_-}\cot\left(\eta_-\frac{\pi}{2}\right) + XE^{\varepsilon}\tan\left(\varepsilon\frac{\pi}{2}\right) - Y_+E^{-\eta_+}\tan\left(\eta_+\frac{\pi}{2}\right)\right.\\ &+ \frac{2}{\pi}\sum_{k=0}^{\infty}\left\{\frac{(-1)^{k+1}}{k!}\left[\pm Y_-(1-\eta_-)^{k+1}E^{-\eta_-} + X\varepsilon^{k+1}E^{\varepsilon} + Y_+(-\eta_+)^{k+1}E^{-\eta_+}\right]\sum_{p=0}^{\infty}\frac{\Gamma(k+1,(2p+1)\ln(E/m))}{(2p+1)^{k+2}}\right\}\\ &- \left.\ln\left|\frac{m-E}{m+E}\right|\left[\pm\frac{Y_-m^{1-\eta_-}}{E} + Xm^{\varepsilon} + Y_+m^{-\eta_+}\right]\right\},\end{aligned} \tag{17}$$

where the signs ± apply for pp (+) and $\bar{p}p$ (−) scattering.

B. Fitting and Results

For the experimental data on $\sigma_{tot}(s)$ and $\rho(s)$, we use the Particle Data Group archives [8], to which we added the values of ρ and σ_{tot} from $\bar{p}p$ scattering at 1.8 TeV, obtained by the E811 Collaboration [9]. The statistical and systematic errors were added in quadrature. The fits were performed through the CERN-Minuit code, with the estimated errors in the free parameters corresponding to an increase of the χ^2 by one unit. To fit the data as function of the center-of-mass energy we express the lab energy in the above formulas in terms of s, namely $E = (s - 2m^2)/2m$.

We compiled all the data above the physical threshold, $\sqrt{s} > 2m \approx 1.88$ GeV. However, with the present model, the large number of experimental points just above this threshold allow reasonable statistical results (in terms of the χ^2 per degree of freedom) only for an energy cutoff at $\sqrt{s}_{\text{min}} = 4$ GeV.

The results of the fits with both IDR and EDDR are displayed in Table I and the corresponding curves together with the experimental data on $\sigma_{tot}(s)$ and $\rho(s)$ in Fig. 1. From Table I we see that the numerical results are exactly the same.

TABLE I: Simultaneous fits to σ_{tot} and ρ, $\sqrt{s}_{\text{min}} = 4$ GeV (270 data points), with K as a free parameter and using Integral Dispersion Relations and Extended Derivative Dispersion Relations.

	IDR	EDDR
X (mb)	1.598 ± 0.034	1.598 ± 0.034
Y_+ (mb)	3.957 ± 0.053	3.957 ± 0.053
Y_- (mb)	-2.082 ± 0.080	-2.082 ± 0.080
ε	0.0919 ± 0.0021	0.0919 ± 0.0021
η_+	0.3554 ± 0.0098	0.3554 ± 0.0098
η_-	0.569 ± 0.010	0.569 ± 0.010
K	-2.27 ± 0.28	-2.27 ± 0.28
χ^2	315.3764	315.3788
χ^2/DOF	1.20	1.20

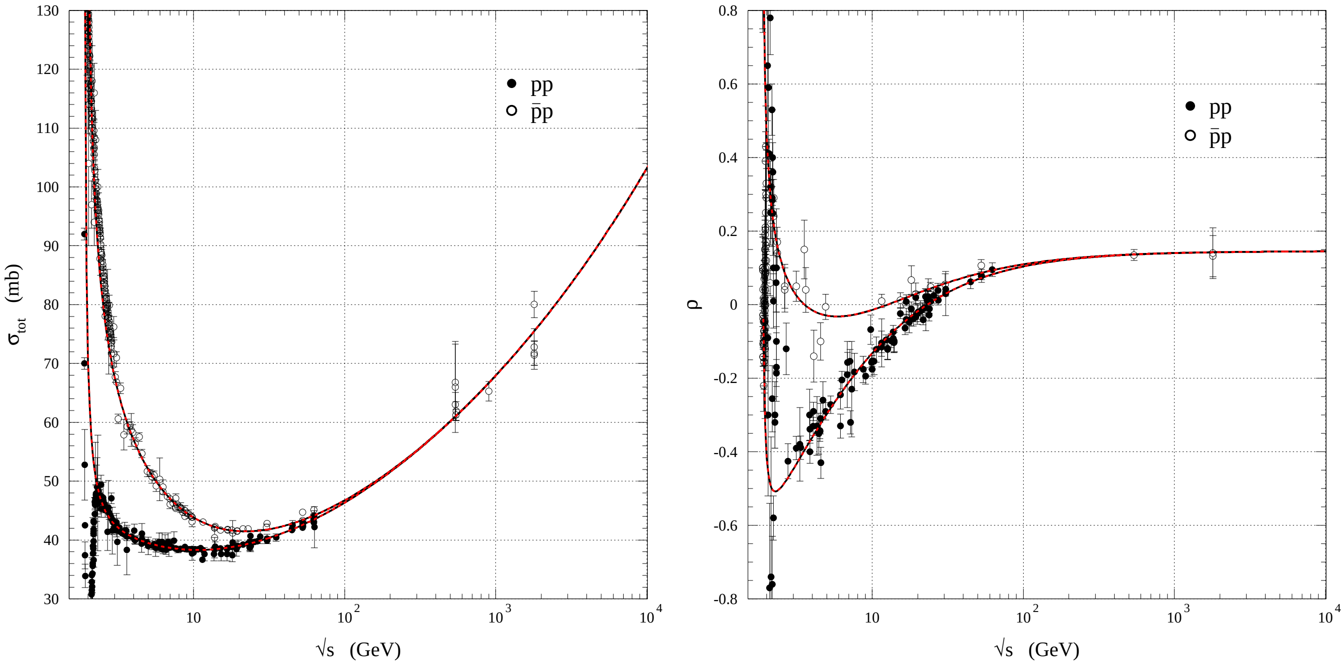

FIG. 1: Results for the total cross sections and $\rho(s)$ obtained through the IDR (solid) and EDDR (dashed) with fit cutoff at $\sqrt{s}_{\rm min} = 4$ GeV. Both curves coincide.

IV. CONCLUSIONS

We have obtained novel expressions for the DDR without any high-energy approximation. These extended DDR are, therefore, intended for any energy above the physical threshold. However, as in the case of IDR, the practical efficiency of the EDDR in the reproduction of experimental data on σ_{tot} and ρ depends, of course, on the model considered. Here we made use of the Extended Regge Parametrization, for which a cutoff at $\sqrt{s} = 4$ GeV was necessary. For example, by considering the full nondegenerated case (four contributions, each one from each meson trajectory, a_2, f_2, ρ, ω), or another model, this cutoff can be reduced [10].

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