2d Gravity With Torsion, Oriented Matroids And 2+2 Dimensions

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We find a link between oriented matroid theory and 2d gravity with torsion. Our considerations may be useful in the context of noncommutative phase space in a target spacetime of signature (2+2) and in a possible theory of gravity ramification.

Keywords: 2d-gravity, 2t physics, 2+2 dimensions.

As it is known, the theory of matroids is a fascinating topic in mathematics [1]. Why should not be also interesting in some scenarios of physics? We are convinced that matroid theory should be an essential part not only of physics in general, but also of M-theory. In fact, it seems that the duality concept that brought matroid theory from a matrix formalism in 1935, with the work of Whitney (see Ref. [2] and references therein), is closely related to the duality concept that brought M-theory from string theory in 1994 (see Refs. [3-11] for connections between matroids and various subjects of high energy physics and gravity). These observations are some of the main motivations for the proposal [12] of considering oriented matroid theory as the mathematical framework for M-theory. In this paper, we would like to report new progress in our quest of connecting matroid theory with different scenarios of high energy physics and gravity. Specifically, we find a connection between matroids and $2d$ gravity with torsion and $2+2$ dimensions. In the route we find many new directions in which one can pursue further research, such as tame and wild ramification [13], nonsymmetric gravitational theory (see Ref. [14] and references therein) and Clifford algebras (see Ref. [15] and references therein). We believe that our results may be of particular interest not only for physicists but also for mathematicians.

In order to achieve our goal we first show that a 2×2-matrix function in two dimensions can be interpreted in terms of a metric associated with 2d gravity with torsion. Let us start by writing a complex number z in the traditional form [16]

$$z = x + iy, \tag{1}$$

where x and y are real numbers and $i^2 = -1$. However, there exist another, less used, way to write a complex number, namely [17]

$$\begin{pmatrix} x & y \\ -y & x \end{pmatrix}. \tag{2}$$

In this case the product of two complex numbers corresponds to the usual matrix product. These two representations of complex numbers can be linked by writing (2) as

$$\begin{pmatrix} x & y \\ -y & x \end{pmatrix} = x\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + y\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{3}$$

Since $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ one finds from (1) and (3) that the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ can be identified with the imaginary unit i.

It turns out that the matrices $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ can be considered as two of the matrix bases of general real 2×2 matrices which we denote by $M(2,R)$. In fact, any 2×2 matrix Ω over the real can be written as

$$\Omega = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = x\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + y\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + r\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + s\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{4}$$

where

$$x = \tfrac{1}{2}(a+d), \quad y = \tfrac{1}{2}(b-c), \quad r = \tfrac{1}{2}(a-d), \quad s = \tfrac{1}{2}(b+c). \tag{5}$$

Let us rewrite (4) in the form

$$\Omega_{ij} = x\delta_{ij} + y\varepsilon_{ij} + r\eta_{ij} + s\lambda_{ij}, \tag{6}$$

where

$$\delta_{ij} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \varepsilon_{ij} \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \eta_{ij} \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \lambda_{ij} \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{7}$$

Considering this notation, we find that (1) becomes

$$z_{ij} = x\delta_{ij} + y\varepsilon_{ij}. \tag{8}$$

Comparing (6) and (8), we see that (8) can be obtained from (6) by setting $r = 0$ and $s = 0$. If $ad - bc \neq 0$, that is if $\det\Omega \neq$

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0, then the matrices in $M(2,R)$ can be associated with the group $GL(2,R)$. If we further require $ad-bc=1$, then one gets the elements of the subgroup $SL(2,R)$. It turns out that this subgroup is of special interest in 2t physics [18-20].

Now, consider the following four functions $F(x,y,r,s), G(x,y,r,s), H(x,y,r,s)$ and $Q(x,y,r,s)$, and construct the matrix

$$\gamma = \begin{pmatrix} F & G \\ H & Q \end{pmatrix}. \tag{9}$$

By setting

$$u = \tfrac{1}{2}(F+Q), \quad v = \tfrac{1}{2}(G-H), \qquad w = \tfrac{1}{2}(F-Q), \quad \xi = \tfrac{1}{2}(G+H), \tag{10}$$

we get that γ can be written as

$$\gamma = u\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + v\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + w\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \xi\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{11}$$

or

$$\gamma_{ij} = u\delta_{ij} + v\varepsilon_{ij} + w\eta_{ij} + \xi\lambda_{ij}. \tag{12}$$

We can always decompose the matrix γ_{ij} in terms of its symmetric $g_{ij} = g_{ji}$ and antisymmetric $A_{ij} = -A_{ji}$ parts. In fact, we have

$$\gamma_{ij}(x,y,r,s) = g_{ij}(x,y,r,s) + A_{ij}(x,y,r,s). \tag{13}$$

From (11) or (12) we find that we can write $g_{ij}(x,y,r,s)$ in the form

$$g_{ij}(x,y,r,s) = u(x,y,r,s)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + w(x,y,r,s)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \xi(x,y,r,s)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{14}$$

while

$$A_{ij}(x,y,r,s) = v(x,y,r,s)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{15}$$

An interesting possibility emerges by dimensional reduction of the variables r and s, that is by setting in (13) $r=0$ and $s=0$. We have

$$\gamma_{ij}(x,y) = g_{ij}(x,y) + A_{ij}(x,y), \tag{16}$$

with

$$g_{ij}(x,y) = u(x,y)\delta_{ij} + w(x,y)\eta_{ij} + \xi(x,y)\lambda_{ij} \tag{17}$$

and

$$A_{ij}(x,y) = v(x,y)\varepsilon_{ij}. \tag{18}$$

Of course, according to (8) the expressions (16), (17) and (18) can be associated with a complex structure. This observation can be clarified by using isothermal coordinates in which $w=0$ and $\xi=0$. In this case (16) is reduced to

$$f_{ij}(x,y) = u(x,y)\delta_{ij} + v(x,y)\varepsilon_{ij}, \tag{19}$$

where we wrote $\gamma_{ij}(x,y) \to f_{ij}(x,y)$ in order to emphasize this reduction. In the traditional notation, (19) becomes $f(x,y) = u(x,y) + iv(x,y)$. It turns out that the existence of isothermal coordinates is linked to the Cauchy-Riemann conditions for u and v, namely $\partial u/\partial x = \partial v/\partial y$ and $\partial u/\partial y = -\partial v/\partial x$ [16].

One of the main reason for the above discussion comes from the question: is it possible to identify the symmetric matrix $g_{ij}(x,y)$ with 2d gravity? Assuming that this is the case the next question is then: what kind of gravitational theory describes $\gamma_{ij}(x,y)$? In what follows we shall show that $\gamma_{ij}(x,y)$ can be identified not only with a nonsymmetric gravitational theory in two dimensions but also with 2d gravity with torsion. First, consider the covariant derivative of the metric tensor

$$\nabla_k g_{ij} = \partial_k g_{ij} - \Gamma^l_{ki} g_{lj} - \Gamma^l_{kj} g_{il} = 0. \tag{20}$$

Here, we assume that the symbols Γ^l_{ki} are not necessarily symmetric in the two indices k and i. In fact, if we define the torsion as $T^l_{ki} \equiv \Gamma^l_{ki} - \Gamma^l_{ik}$, one finds that the more general solution of (20) is

$$\Gamma_{kij} = \frac{1}{2}(\partial_k g_{ji} + \partial_i g_{jk} - \partial_j g_{ki}) - \frac{1}{2}(T_{kji} + T_{ijk} - T_{kij}), \tag{21}$$

where $\Gamma_{kij} = \Gamma^l_{ki} g_{lj}$ and $T_{kij} = T^l_{ki} g_{lj}$.

On the other hand, if we consider the expression

$$\frac{1}{2}(\partial_k \gamma_{ji} + \partial_i \gamma_{jk} - \partial_j \gamma_{ik}), \tag{22}$$

by substituting (16) into (22) one gets

$$\tfrac{1}{2}(\partial_k \gamma_{ji} + \partial_i \gamma_{jk} - \partial_j \gamma_{ik}) = \tfrac{1}{2}(\partial_k g_{ji} + \partial_i g_{jk} - \partial_j g_{ki}) + \tfrac{1}{2}(\partial_k A_{ji} + \partial_i A_{jk} - \partial_j A_{ik}). \tag{23}$$

Comparing (23) and (21) one sees that if one sets $T_{kji} = \partial_i A_{kj}$ the expression (23) can be identified with the connection Γ_{kij} which presumably describes 2d gravity with torsion. Since A_{ij} can always be written as (18) we discover that (23) yields

$$\Gamma_{kij} = \frac{1}{2}(\partial_k g_{ji} + \partial_i g_{jk} - \partial_j g_{ki}) + \frac{1}{2}(v_{,k}\varepsilon_{ji} + v_{,i}\varepsilon_{jk} - v_{,j}\varepsilon_{ik}). \tag{24}$$

Here, we used the notation $\partial_k v = v_{,k}$.

The curvature Riemann tensor can be defined as usual

$$\mathcal{R}^{m}_{kij} = \partial_i \Gamma^m_{kj} - \partial_j \Gamma^m_{ki} + \Gamma^m_{ni}\Gamma^n_{kj} - \Gamma^m_{nj}\Gamma^n_{ki}. \quad (25)$$

The proposed gravitational theory, which may be interesting in string theory, can have a density Lagrangian $\mathcal{L}$ of the form $\mathcal{L} \sim \mathcal{R}^2 + \Lambda$ [21], where Λ is a constant. In this context, we have proved that it makes sense to consider the nonsymmetric metric of the form (16)-(18) as a 2d gravity with torsion.

From the point of view of complex structure there are a number of interesting issues that arises from the above formalism. One may be interested, for instance, in considering the true degrees of freedom for the metric g_{ij}. In this case one may start with the Teichmuller space associated with the metric g_{ij} and then to determine the Moduli space of such a metric [22]. Another possibility is to consider similarities. In this case one may be interested to associate with the metric g_{ji} either the Riemann-Roch theorem [23] or the tame and wild ramification complex structure [13]. In the later case one may assume that the principal part of the metric g_{ij} looks like

$$g_{ij}(x,y) = (\frac{T_n}{z^n} + \frac{T_{n-1}}{z^{n-1}} + ... + \frac{T_1}{z})\delta_{ij}. \quad (26)$$

In this case the similarities can be identified with solitons associated with black holes. In this scenario our constructed route to 2d gravity with torsion provides a bridge which may bring many ideas from complex structure to 2d gravity with torsion and *vice versa*.

Let us now study some aspects of the above formalism from the point of view of matroid theory. Consider the matrix

$$V_i^A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \end{pmatrix}, \quad (27)$$

with the index A taking values in the set

$$E = \{1,2,3,4\}. \quad (28)$$

It turns out that the subsets $\{\mathbf{V}^1,\mathbf{V}^2\}$, $\{\mathbf{V}^1,\mathbf{V}^3\}$, $\{\mathbf{V}^2,\mathbf{V}^4\}$ and $\{\mathbf{V}^3,\mathbf{V}^4\}$ are bases over the real of the matrix (27). One can associate with these subsets the collection

$$\mathcal{B} = \{\{1,2\},\{1,3\},\{2,4\},\{3,4\}\}, \quad (29)$$

which can be understood as a family of subsets of E. It is not difficult to show that the pair $\mathcal{M} = (E,\mathcal{B})$ is a 2-rank self-dual matroid. The fact that we can express $\mathcal{M}$ in the matrix form (27) means that this matroid is representable (or realizable) [1]. Moreover, one can show that this matroid is graphic and orientable. In the later case the corresponding chirotope [1] is given by

$$\chi^{AB} = \varepsilon^{ij} V_i^A V_j^B. \quad (30)$$

Thus, we get, as nonvanishing elements of the chirotope χ^{AB}, the combinations

$$12+, \; 13-, \; 24-, \; 34+. \quad (31)$$

The relation of this matroid structure with of our previous discussion comes from the identification $\{\mathbf{V}^1,\mathbf{V}^2\} \to \delta_{ij}$, $\{\mathbf{V}^1,\mathbf{V}^3\} \to \eta_{ij}$, $\{\mathbf{V}^2,\mathbf{V}^4\} \to \lambda_{ij}$ and $\{\mathbf{V}^3,\mathbf{V}^4\} \to \varepsilon_{ij}$. The signs in (31) correspond to the determinants of the matrices δ_{ij}, η_{ij}, λ_{ij} and ε_{ij}, which can be calculated using (30). Therefore, what we have shown is that the bases of $M(2,R)$ as given in (4) (or (7)) admit an oriented matroid interpretation. It may be of some interest to consider the weak mapping $\mathcal{B} \to \mathcal{B}_c$ with

$$\mathcal{B}_c = \{\{1,2\},\{3,4\}\}, \quad (32)$$

leading to the reduced pair $\mathcal{M}_c = (E,\mathcal{B}_c)$. When the local structure is considered as in (14)-(18), one needs to rely in the matroid fiber bundle notion (see Refs. [24] and [25] and references therein). Therefore, we have found a link which connect matroid fiber bundle with 2d gravity with torsion.

It is worth mentioning the following observations. It is known that the fundamental matrices $\delta_{ij},\eta_{ij},\lambda_{ij}$ and ε_{ij} given in (7) not only form a basis for $M(2,R)$ but also determine a basis for the Clifford algebras $C(2,0)$ and $C(1,1)$. In fact one has the isomorphisms $M(2,R) \sim C(2,0) \sim C(1,1)$. Moreover, one can show that $C(0,2)$ can be constructed using the fundamental matrices (7) and Kronecker products. It turns out that $C(0,2)$ is isomorphic to the quaternion algebra H. Since all the others $C(a,b)$'s can be constructed from the building blocks $C(2,0)$, $C(1,1)$ and $C(0,2)$, this means that our connection between oriented matroid theory and $M(2,R)$ also establishes an interesting link with the Clifford algebra structure (see Ref. [15].and references therein).

Let us make some final remarks. The above links also apply to the subgroup $SL(2,R)$ which is the main object in 2t physics. In this case it is known that noncommutative field theory of 2t physics [18-20] (see also Ref. [26]) contains a fundamental gauge symmetry principle based on the noncommutative group $U_\star(1,1)$. This approach originates from the observation that a world line theory admits a Lie algebra $sl_\star(2,R)$ gauge symmetry acting on phase space [18]. In this context, consider the coordinates q and p in the phase-space. The Poisson bracket

$$\{f,g\} = \frac{\partial f}{\partial q^a}\frac{\partial g}{\partial p_a} - \frac{\partial f}{\partial p_a}\frac{\partial g}{\partial q^a}, \quad (33)$$

can be written as

$$\{f,g\} = \varepsilon_{ij}\eta^{ab}\frac{\partial f}{\partial q_i^a}\frac{\partial g}{\partial q_j^b}. \quad (34)$$

where $q_1^a \equiv q^a$ and $q_2^a \equiv p^a$, with a and b running from 1 to n. It worth mentioning that the expression (34) is very similar to the the definition of a chirotope (see Ref. [8] and references therein).

Recently, a generalization of (34) was proposed [27], namely

$$\{f,g\}=(g_{ij}\Omega^{ab}+\varepsilon_{ij}\eta^{ab})\frac{\partial f}{\partial q_i^a}\frac{\partial g}{\partial q_j^b}. \tag{35}$$

Here, Ω^{ab} is skew-simplectic form defined in even dimensions. In particular, in four dimensions Ω_{ab} can be chosen as

$$\Omega_{ab}=\begin{pmatrix}0&-1&0&0\\1&0&0&0\\0&0&0&-1\\0&0&1&0\end{pmatrix}. \tag{36}$$

Here, by choosing $\eta_{ab}=diag(-1,1,-1,1)$ we make contact with $(2+2)$-dimensions which is the minimal 2t physics theory (see Refs. [28–31]).

Let us write the factor in (35) $g_{ij}\Omega^{ab}+\varepsilon_{ij}\eta^{ab}$ in the form

$$\mathbf{g}'_{ij}+\varepsilon_{ij}\eta, \tag{37}$$

with $\mathbf{g}'_{ij}=g_{ij}\Omega$. We recognize in (37) the typical form (18) for a complex structure. This proves that oriented matroid theory is also connected not only with $(2+2)$-physics but also with noncommutative geometry.

An alternative connection with 2t physics can be obtained by considering the signature $\eta_{ab}=diag(1,1,-1,-1)$, and its associated metric:

$$ds^2=(dx^1)^2+(dx^2)^2-(dx^3)^2-(dx^4)^2. \tag{38}$$

In fact, by defining the matrix

$$x^{ij}=\begin{pmatrix}x^1&x^3\\x^4&x^2\end{pmatrix}, \tag{39}$$

it can be seen that (38) can be expressed as

$$ds^2=dx^{ij}dx^{kl}\eta_{ik}\eta_{jl}, \tag{40}$$

where the indices i,j,k,l run from 1 to 2 as before, and η_{ij} stands for the third matrix defined in (7), namely $\eta_{ij}\equiv\begin{pmatrix}1&0\\0&-1\end{pmatrix}$. As before,noticing that in (38) the "spatial" coordinates x^1, x^2 are the elements of the main diagonal and the "time" coordinates x^3, x^4 corresponds to the main skew diagonal in (39), x^{ij} can be written in terms of the bases (7) as follows:

$$x^{ij}=X\delta^{ij}+S\varepsilon^{ij}+Y\eta^{ij}+T\lambda^{ij}; \tag{41}$$

where we used the definitions

$$\begin{aligned}X&=\tfrac{1}{2}(x^1+x^2),\qquad S=\tfrac{1}{2}(x^4-x^3),\\ Y&=\tfrac{1}{2}(x^1-x^2),\qquad T=-\tfrac{1}{2}(x^3+x^4),\end{aligned} \tag{42}$$

and considered the notation $\varepsilon^{ij}=\varepsilon_{kl}\eta^{ik}\eta^{jl}$ and $\lambda^{ij}=\lambda_{kl}\eta^{ik}\eta^{jl}$, where η^{ij} is the inverse flat 1+1 metric, and has the same components as η_{ij}.

Finally, consider the three index object η_{ijk} with components

$$\eta_{1ij}=\delta_{ij};\qquad \eta_{2ij}=\varepsilon_{ij}. \tag{43}$$

From these expressions and (7) it can be checked that η_{ijk} automatically satisfies

$$\eta_{ij1}=\eta_{ij};\qquad \eta_{ij2}=\lambda_{ij}. \tag{44}$$

Therefore η_{ijk} has the remarkable property of containing all the matrices in (7). This means that an arbitrary matrix Ω_{ij} can be written as

$$\Omega_{ij}=x^k\eta_{kij}+y^k\eta_{ijk}, \tag{45}$$

where $x^1=x$, $x^2=y$ and $y^1=r$, $y^2=s$. Here, x,y,r and s are defined in (5). Observe that $\eta_{ijk}=\eta_{jik}$, but $\eta_{kij}\neq\eta_{kji}$. It is worth mentioning that a similar structure was proposed in Ref. [14] in the context of nonsymmetric gravity [32].

The inverse η^{ijk} of η_{ijk} can be defined by the relation

$$\eta^{ijk}\eta_{ijl}=2\delta^k_l, \tag{46}$$

or

$$\eta^{kij}\eta_{lij}=2\delta^k_l. \tag{47}$$

Explicity, we obtain the components

$$\eta^{1ij}=\delta^{ij};\quad \eta^{2ij}=-\varepsilon^{ij};\quad \eta^{ij1}=\eta^{ij};\quad \eta^{ij2}=-\lambda^{ij}. \tag{48}$$

Traditionally, starting with a flat space described by the metric η_{ij}, one may introduce a curved metric $g_{\mu\nu}=e^i_\mu e^j_\nu\eta_{ij}$ via the *zweibeins* e^i_μ. So, this motivate us to introduce the three-index curved metric

$$g_{\mu\nu\lambda}=e^i_\mu e^j_\nu e^k_\lambda\eta_{ijk}. \tag{49}$$

It seems very interesting to try to develop a gravitational theory based in $g_{\mu\nu\lambda}$, for at least two reasons. First, the η_{ijk} contains the four basic matrices (7), which we proved are linked to matroid theory. Therefore this establishes a bridge between matroids and $g_{\mu\nu\lambda}$. Thus, a gravitational theory based in $g_{\mu\nu\lambda}$ may provide an alternative gravitoid theory (see Ref. [4]). Secondly, since the matrices (7) are also linked to Clifford algebras, such a gravitational theory may determine spin structures, which are necessary for supersymmetric scenarios. These and another related developments will be reported elsewhere [33].

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[1] A. Björner, M. Las Vergnas, B. Sturmfels, N. White and G. Ziegler, *Oriented Matroids*, (Cambridge University Press, Cambridge, 1993).

[2] J.G. Oxley, *Matroid Theory*, (Oxford Science Publications. Oxford University Press, New York, 1992).

[3] J.A. Nieto, Rev. Mex. Fis. **44**, 358 (1998); e-Print: hep-th/9807107.

[4] J.A. Nieto and M.C. Marín, J. Math. Phys. **41**, 7997 (2000); hep-th/0005117.

[5] J.A. Nieto, J. Math. Phys. **45**, 285 (2004); hep-th/0212100.

[6] J.A. Nieto and M.C. Marín, Int. J. Mod. Phys. A **18**, 5261 (2003); hep-th/0302193.

[7] J.A. Nieto, Adv. Theor. Math. Phys. **8**, 177 (2004); hep-th/0310071.

[8] J.A. Nieto, Rev. Mex. Fis. E **51**, 5 (2005); hep-th/0407093.

[9] J.A. Nieto, "Toward a connection between the oriented matroid theory and supersymmetry"; hep-th/0510185.

[10] J.A. Nieto, "Maximal supersymmetry in eleven-dimensional supergravity revisited and chirotopes", hep-th/0603139.

[11] P. Guha, Int. J. Geom. Meth. Mod. Phys. **1**, 795 (2004).

[12] J.A. Nieto, Adv. Theor. Math. Phys. **10**, 747 (2006); hep-th/0506106.

[13] E. Witten, "Mirror Symmetry, Hitchin's Equations, And Langlands Duality", arXiv:0802.0999.

[14] J. A. Nieto and J. Socorro, "Is nonsymmetric gravity related to string theory?", hep-th/9610160.

[15] C. Castro, Phys. Lett. B **668**, 442 (2008).

[16] J. B. Conway, *Functions of One Complex Variable I*, (Graduate Texts in Mathematics 1978 Springer-Verlag New.York)

[17] N. Levinson and R. M. Redheffer, *Functions Complex Variables*, (Holden Day, 1970).

[18] I. Bars, Phys. Rev. D **64**, 126001 (2001); hep-th/0106013.

[19] I. Bars, and S. J. Rey, Phys. Rev. D **64**, 046005 (2001); hep-th/0104135.

[20] I. Bars, Class. Quant. Grav. **18** (2001) 3113; hep-th/0008164.

[21] M. O. Katanaev and I. V. Volovich, Ann. Phys. **197**, 1 (1990).

[22] C. Nash, *Differential Topology and Quantum Field Theory*, (Academic Pr, San Diego 1991).

[23] G. Springer, *Introduction to Riemann Surfaces*, (Addison-Wesley, Publishing Company, Inc., reading, Massachusetts USA 1957).

[24] L. Anderson, New Perspectives in Geom. Com. **38**, 1 (1999).

[25] L. Anderson, J. F. Davis, "Mod 2 cohomology of combinatorial Grassmannians", arXiv:math/9911158.

[26] V. M. Villanueva, J. A. Nieto, L. Ruiz, J. Silvas, J. Phys. A **38**, 7183 (2005); hep-th/0503093.

[27] J. A. Nieto, "SL(2,R)-Symmetry and Noncommutative Phase Space in (2+2) Dimensions", arXiv:0809.3429.

[28] J. A. Nieto, Nuovo Cim. B **120**, 135 (2005); hep-th/0410003.

[29] J. A. Nieto, Mod. Phys. Lett. A **22**, 2453 (2007); hep-th/0606219.

[30] M. Carvalho and M. W. de Oliveira Phys. Rev. D **55**, 7574 (1997); hep-th/9612074.

[31] M. A. De Andrade, O. M. Del Cima and L. P. Colatto, Phys. Lett. B **370**, 59 (1996); hep-th/9506146.

[32] J. W. Moffat, Phys. Lett. B **355**, 447 (1995).

[33] E. A. Leon and J. A. Nieto, work in progress, DIFUS, Universidad de Sonora (2009).