BOUNDARY LAYER STAGNATION-POINT FLOW OF A THIRD GRADE FLUID OVER AN EXPONENTIALLY STRETCHING SHEET

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Abstract - In this paper, the mixed convection steady boundary layer stagnation point flow and heat transfer of a third grade fluid over an exponentially stretching sheet is investigated. Both the analytical and numerical solutions are carried out. The analytical solutions are obtained through the homotopy analysis method (HAM) while the numerical solutions are computed by using the Keller box method (K-b). Comparison of the HAM and Keller-box methods is also given. The effects of important physical parameters are presented through graphs and the salient features are discussed.
Keywords: Boundary layer flow; Heat transfer; Third grade fluid; Exponential stretching/shrinking; Homotopy analysis method; Keller-box technique.

INTRODUCTION

The theory of non-Newtonian fluids offers mathematicians, engineers and numerical specialists varied challenges in developing analytical and numerical solutions for the highly nonlinear governing equations. However, due to the practical significance of these non-Newtonian fluids, many authors have presented various non-Newtonian fluid models like Buongiorno (2006), Nadeem and Ali (2009), Nadeem and Akbar (2009), Lukaszewics (2003), Nadeem *et al*. (2010a), Nadeem *et al*. (2010b), Elbashbeshy *et al*. (2011), Nadeem *et al*. (2010c). The third grade fluid model is one of the most significant fluid models that exhibits all the properties of shear thinning and shear thickening fluids. The effect of the variable magnetic field over the Couette flow of a third grade fluid was studied by Hayat and Kara (2005). Moreover, Hayat *et al*. (2007) have also presented the analytical solution of the problem of a third grade fluid in a porous half

space subject to sudden motion of a flat plate. Recently, Sahoo and Do (2010) analyzed slip effects over the flow and heat transfer of an electrically conducting third grade fluid past a stretching sheet. They concluded that slip causes a decrease in the momentum boundary layer thickness while producing an increase in the thermal boundary layer (thickness). Furthermore, Hayat *et al*. (2001) inspected the effects of fluctuating flow of a third grade fluid on a porous plate in a rotating medium. Later on, the problem of steady, laminar flow of a third grade fluid through a porous flat channel was encountered by Ariel (2003). He considered the case when the injection rate of the fluid at a boundary is the same as the suction rate of the fluid at the other boundary. Asghar *et al*. (2003) discussed the unsteady flow of a third grade fluid in the case of suction where fluid is assumed to be along an infinite permeable wall.

The present work studies the boundary layer stagnation point flow of a third grade fluid through an exponentially stretching sheet. The governing

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highly nonlinear equations of the third grade fluid model are simplified by using a boundary layer approximation and similarity transformation. The reduced nonlinear equations are solved analytically and numerically. The analytical solutions are obtained by using the homotopy analysis method. Details of HAM can be found in the works of Nadeem *et al.* (2009a), Nadeem *et al.* (2009b), Nadeem and Ali (2009), Liao (2003), Liao (2004), Nadeem and Awais (2008), Nadeem and Akbar (2009), Liao (1999), Abbasbandy (2004), Nadeem and Akbar (2010). The numerical solutions are calculated with the help of the Keller-box method described in Keller (1978), Cebeci and Bradshaw (1984) and Ali (1994). The comparisons of both the solutions are also presented through graphs and tables. The particular features of the parameters are discussed through graphs of the velocity and temperature profiles and also through tables.

FORMULATION

Consider the stagnation point flow of a steady incompressible third grade fluid over an exponentially stretching sheet. The Cartesian coordinates (x, y) are used such that x is along the surface of the sheet, while y is taken as normal to it. The related boundary layer equations of motion in the presence of heat transfer are:

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0, \tag{1}$$

$$v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} =$$

$$U_\infty \frac{dU_\infty}{dx} + \upsilon \frac{\partial^2 v_x}{\partial y^2} + \frac{\alpha_1}{\rho} [v_x \frac{\partial^3 v_x}{\partial x \partial y^2} + \frac{\partial v_x}{\partial y} \frac{\partial^2 v_x}{\partial x \partial y}$$

$$+ v_y \frac{\partial^3 v_x}{\partial y^3} + \frac{\partial v_x}{\partial x} \frac{\partial^2 v_x}{\partial y^2}] \tag{2}$$

$$+ \frac{2\beta_3}{\rho} [4 \left(\frac{\partial v_x}{\partial x}\right)^2 \frac{\partial^2 v_x}{\partial y^2} - \frac{\partial^2 v_x}{\partial x^2} \left(\frac{\partial v_x}{\partial y}\right)^2$$

$$+ 12 \frac{\partial v_x}{\partial x} \frac{\partial v_x}{\partial y} \frac{\partial^2 v_x}{\partial x \partial y} + 6 \frac{\partial v_y}{\partial x} \frac{\partial v_x}{\partial y} \frac{\partial^2 v_x}{\partial y^2}],$$

$$v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2}. \tag{3}$$

Here (v_x, v_y) are the velocity components along the (x, y) axes, ρ is the fluid density, ν is the kinematic viscosity, α_1 and β_3 are the parameters of the third grade fluid, T is temperature, α is the thermal diffusivity, p is pressure and U_∞ is the free stream velocity. The corresponding boundary conditions for the problem are:

$$\text{at} \quad y = 0, \quad u = U_w, \quad v = 0, \quad T = T_w(x), \tag{4}$$

$$\text{as} \quad y \to \infty, \quad u \to U_\infty, \quad T \to T_\infty, \tag{5}$$

where the free stream velocity U_∞, the stretching velocity U_w, and the surface temperature T_w, are defined as:

$$U_\infty = ae^{x/L}, \quad U_w = be^{x/L}, \quad T_w = T_\infty + ce^{x/L}, \tag{6}$$

where a and b are constant velocities, c is constant temperature and L is the reference length.

Defining the following similarity transformations:

$$v_x = ae^{x/L} f'(\eta), \tag{7}$$

$$v_y = -\left(\frac{va}{2L}\right)^{\frac{1}{2}} e^{x/2L} (f(\eta) + \eta f'(\eta)),$$

$$\theta = \frac{T - T_\infty}{T_w - T_\infty}, \qquad \eta = \left(\frac{a}{2vL}\right)^{\frac{1}{2}} e^{x/2L} y. \tag{8}$$

With the help of the transformations in Eqs. (7) and (8), the governing equations take the form:

$$f''' + ff'' - 2f'^2 + 2$$

$$+ \beta \left(2\eta f' f''' + 5f' f''' + 3f''^2 - ff^{iv}\right) +$$

$$\Gamma \begin{pmatrix} 68 f' f''^2 + 22\eta f' f'' f''' + 31\eta f''^3 \\ + 21\eta^2 f''^2 f''' - 6ff'' f''' + 16 f'^2 f''' \end{pmatrix} = 0, \tag{9}$$

$$\theta'' + Pr(f\theta' - 2f'\theta) = 0, \tag{10}$$

in which $\beta = \alpha_1 U_\infty / 2\mu L$ is the second grade parameter, $\Gamma = \beta_3 U_\infty^2 / 2v\rho L^2$ is the third grade parameter and $Pr = v / \alpha$ is the Prandtl number.

The boundary conditions in nondimensional form can be written as:

$$f(0) = 0, \ f'(0) = \varepsilon, \quad f' \to 1, \text{ as } \eta \to \infty, \quad (11)$$

$$\theta(0) = 1, \quad \theta \to 0, \text{ as } \eta \to \infty, \quad (12)$$

where $\varepsilon = b/a$.

The skinfriction coefficient and the local Nusselt numbers are obtained in dimensionless form as:

$$(\varepsilon \mathrm{Re})^{1/2} C_f = \frac{1}{\varepsilon^2}\left(\varepsilon + 7\beta + \frac{16\Gamma}{\varepsilon}\right)$$
$$f''(0) + \frac{4\mathrm{Re}\,\Gamma}{\varepsilon^6}, \quad (13)$$

$$\mathrm{Nu}/\mathrm{Re}_x^{1/2} = -\theta'(0). \quad (14)$$

SOLUTION OF THE PROBLEM

Analytical Solution

The analytical solution of the above boundary value problem is obtained with the help of HAM. For the HAM solution, we choose the initial guesses as

$$f_0(\eta) = (\varepsilon - 1) + \eta - (\varepsilon - 1)e^{-\eta}, \quad (15)$$

$$\theta_0(\eta) = e^{-\eta}. \quad (16)$$

The corresponding auxiliary linear operators are:

$$L_f = \frac{d^3}{d\eta^3} + \frac{d^2}{d\eta^2}, \quad L_\theta = \frac{d^2}{d\eta^2} + \frac{d}{d\eta}. \quad (17)$$

They satisfy:

$$L_f[c_1 + c_2\eta + c_3 e^{-\eta}] = 0, \quad L_\theta[c_4 + c_5 e^{-\eta}] = 0, \quad (18)$$

where $c_i (i=1,...,5)$ are arbitrary constants. The 0th-order deformation equations are defined as:

$$(1-q)L_f[\hat{f}(\eta;q) - f_0(\eta)] = q\hbar_1 N_f[\hat{f}(\eta;q)], \quad (19)$$

$$(1-q)L_\theta[\hat{\theta}(\eta;q) - \theta_0(\eta)] = q\hbar_2 N_\theta[\hat{\theta}(\eta;q)], \quad (20)$$

in which:

$$N_f[\hat{f}(\eta;q)] = \hat{f}''' + \hat{f}\hat{f}'' - 2\hat{f}'^2 + 2 +$$
$$\beta(2\eta\hat{f}''\hat{f}''' + 5\hat{f}'\hat{f}''' + 3\hat{f}''^2 - \hat{f}\hat{f}^{iv})$$
$$+\Gamma(68\hat{f}'\hat{f}''^2 + 22\eta\hat{f}'\hat{f}''\hat{f}''' + 31\eta\hat{f}''^3 \quad (21)$$
$$+21\eta^2\hat{f}''^2\hat{f}''' - 6\hat{f}\hat{f}''\hat{f}''' + 16\hat{f}'^2\hat{f}'''),$$

$$N_\theta[\hat{\theta}(\eta;q)] = \hat{\theta}'' + \Pr(\hat{f}\hat{\theta}' - 2\hat{f}'\hat{\theta}). \quad (22)$$

The appropriate boundary conditions for the 0th-order system are:

$$\hat{f}(0;q) = 0, \quad \hat{f}'(0;q) = \varepsilon,$$
$$\hat{f}'(\eta;q) \to 1, \text{ as } \eta \to \infty, \quad (23)$$

$$\hat{\theta}(0;q) = 1, \quad \hat{\theta}(\eta;q) \to 0, \text{ as } \eta \to \infty. \quad (24)$$

With the help of the software MATHEMATICA, the solution can be written as:

$$f(\eta) = \lim_{Q\to\infty} \sum_{m=1}^{Q}\left[\sum_{n=1}^{2k-2}\sum_{k=1}^{m} a_{nk}^m e^{-2n\eta}\eta^k\right], \quad (25)$$

$$\theta(\eta) = \lim_{Q\to\infty} \sum_{m=1}^{Q}\left[\sum_{n=1}^{2k-2}\sum_{k=1}^{m} c_{nk}^m e^{-2n\eta}\eta^k\right], \quad (26)$$

where a_{nk}^m and c_{nk}^m are arbitrary constants.

Numerical Solution

The numerical solution of Equations (9) and (10) subject to the boundary conditions (11) and (12) is obtained through the Keller-box scheme. For this scheme we first reduce these equations to a first order system; the system obtained is then approximated by central differences. Further, these difference equations are linearized by Newton's method. The resulting tri-diagonal system is then solved using the block-elimination technique. Results obtained from Keller-box are discussed and compare with HAM in the next section.

RESULTS AND DISCUSSION

The behavior of the different parameters involved for the stagnation point flow of a third grade fluid

over an exponentially stretching sheet has been obtained numerically by Keller-box and analytically through HAM, which are discussed in this section. Figs. 1 and 2 show the $\hbar$-curves for velocity and the temperature field for different values of the involved parameters. From these figures, we observe that the convergence region is sufficiently large for smaller values of the respective parameters but decreases very rapidly with an increase in these parameters. A comparison of the numerical and analytical solution of f for different values of ε, β and Γ is sketched in Figs. 3 – 5 respectively; the results are in good agreement. Figs. 6 and 7 are plotted to compare the HAM and Keller box solution obtained for θ for different values of Pr and ε respectively. These figures also guarantee the fact that our numerical and analytical solutions are convergent. This holds only for small values of the parameters involved. For larger values like $Pr = 70$ and beyond the HAM solution may not converge. A similar observation

holds for the other parameters as well. Fig. 8 shows the behavior of f' for different values of ε The velocity field changes its behavior at $\varepsilon = 1$, i.e., for $\varepsilon < 1$, the velocity field is increasing, for $\varepsilon = 1$, it becomes constant and for $\varepsilon < 1$, it is decreasing, meanings that a higher stretching velocity corresponds to a decrease in the velocity profile. Fig. 9 graphs the velocity profile for different values of β. It is observed that, for $\varepsilon < 1$, an increase in the second grade parameter β causes the velocity field to increase, whereas for $\varepsilon < 1$; an increase in β corresponds to a decrease in the velocity field. Fig. 10 plots the velocity profile f' for different values of Γ. The behavior of Γ is similar to that of β. Figs. 10 and 12 plot the temperature profile for different values of the parameters Pr and ε. We note that in both cases increasing the parameters corresponds to a decrease in the temperature profile and the thermal boundary layer thickness.
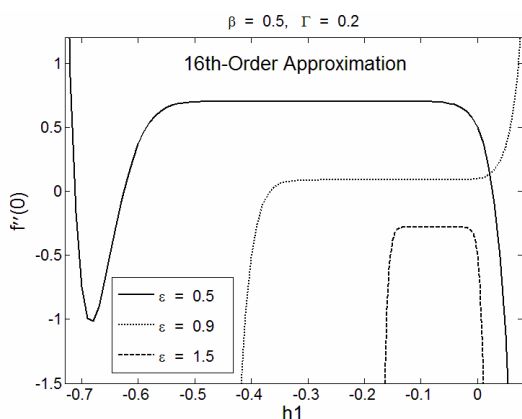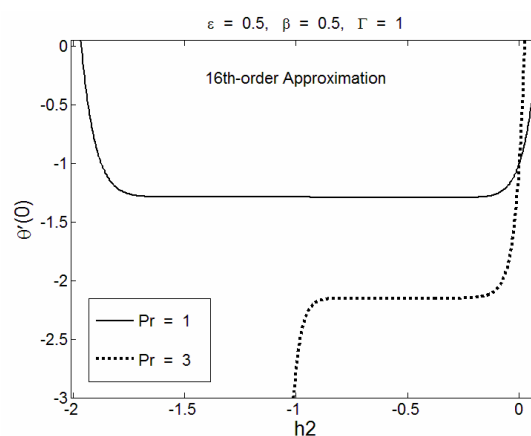

Figure 1: $\hbar$-curves for f for different values of ε.

Figure 2: $\hbar$-curves for θ for different values of Pr.

Figure 3: Comparison of the numerical and the HAM solutions for f' for different values of ε.
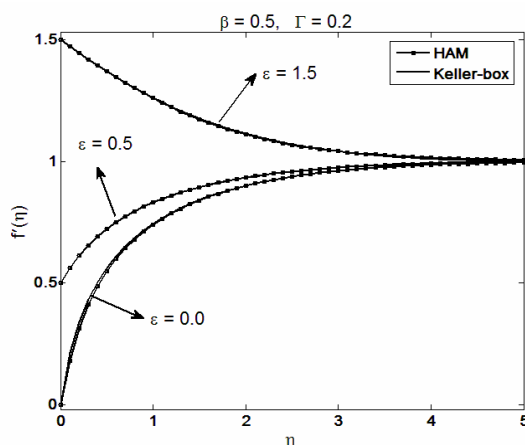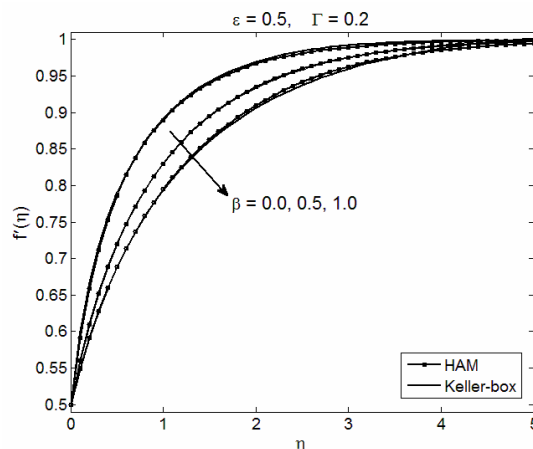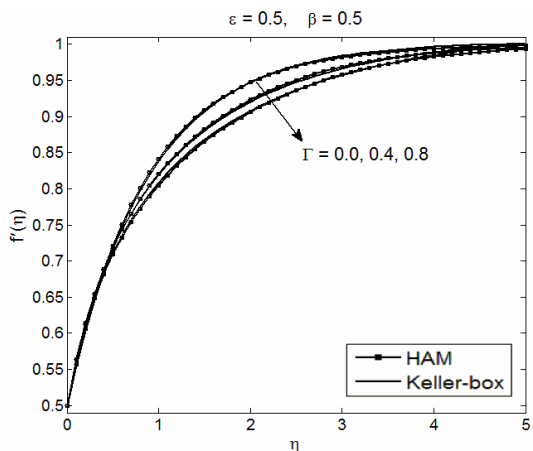

Figure 4: Comparison of the numerical and HAM solutions for f' for different values of β.

Figure 5: Comparison of the numerical and HAM solutions for f' for different values of Γ.
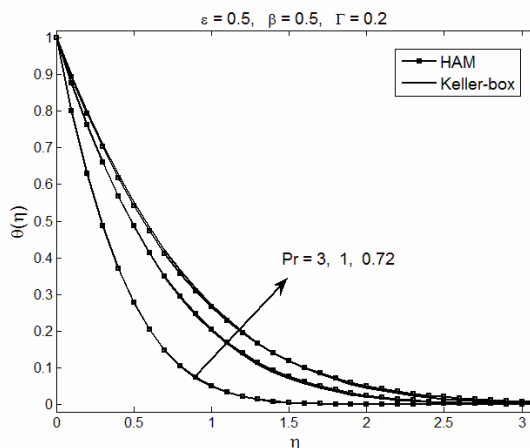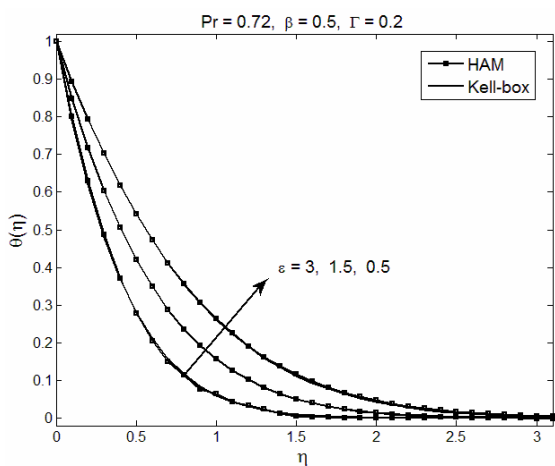
Figure 6: Comparison of the numerical and HAM solutions for θ for different values of Pr.

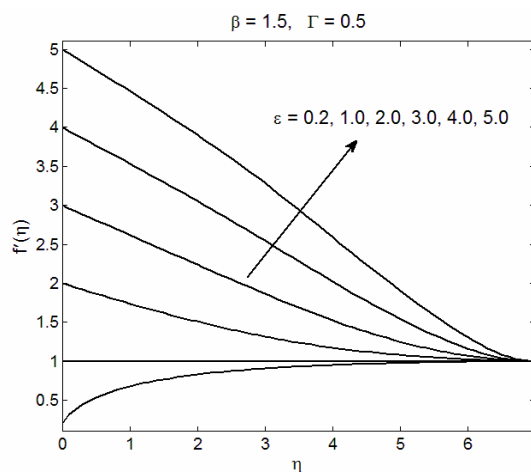**Figure 7:** Comparison of the numerical and HAM solutions for θ for different values of ε.

Figure 8: Influence of ε on f'.
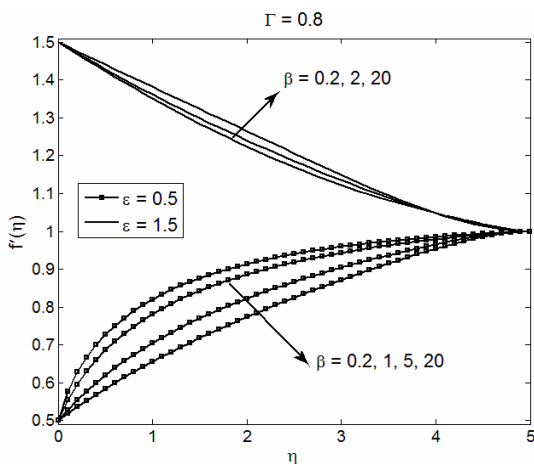
Figure 9: Influence of β on f' for different values of ε.
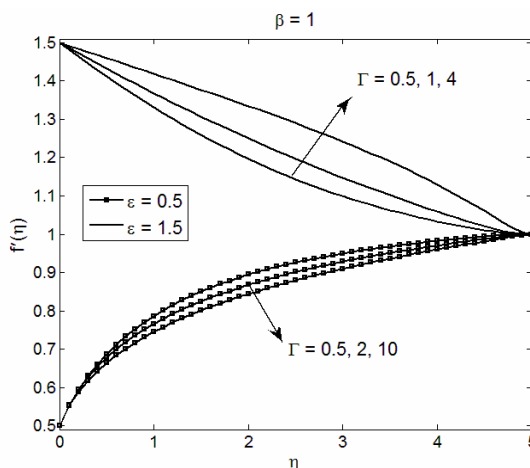
Figure 10: Influence of Γ on f' for different values of ε.
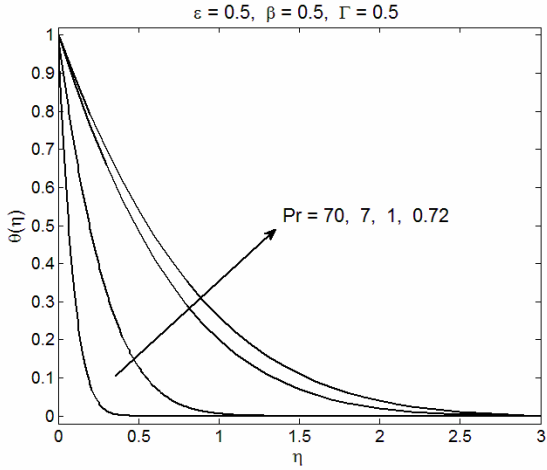
Figure 11: Influence of Pr on θ.

Figure 12: Influence of ε on θ.
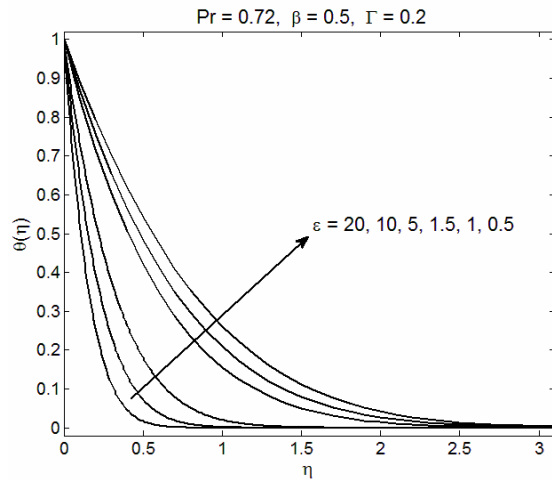
The coefficient of skin friction is graphed in Fig. 13 under the influence of the third grade parameter Γ against Re for different ε. From Fig. 13 it is noted that both Γ and ε cause an increase in the skin friction coefficient. Fig. 14 show the behavior of Nusselt numbers Nu for different values of the Reynolds number Re. It is noted that Re cause an increase in the Nusselt numbers.

Tables 1 and 2 and compare the HAM and Keller-box solutions at the boundary for f'' and θ', corresponding to the skin friction coefficient and local Nusselt number, respectively, for different values of the parameters involved. From Table 1, we note that ε and β have decreasing effects on the skin friction coefficient and that the rate of decrease is also decreasing, while Γ introduces an increase in the skin friction coefficient. Table 2 shows the variation in the Nusselt number for different values of the parameters. From Table 2 it is observed that Pr and ε increase the Nusselt number, while Γ decreases the Nusselt number and the heat transfer rate at the surface.
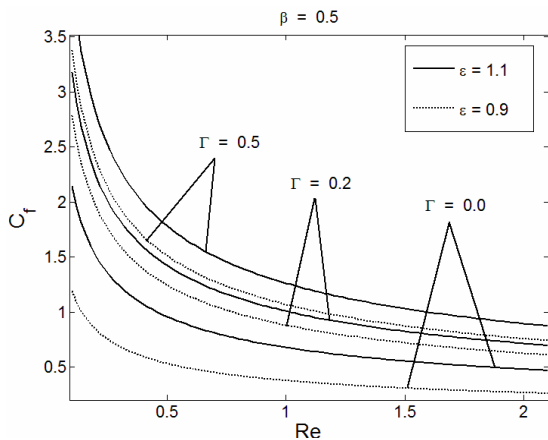
Figure 13: Skin friction coefficient against Re for different values of Γ and ε.
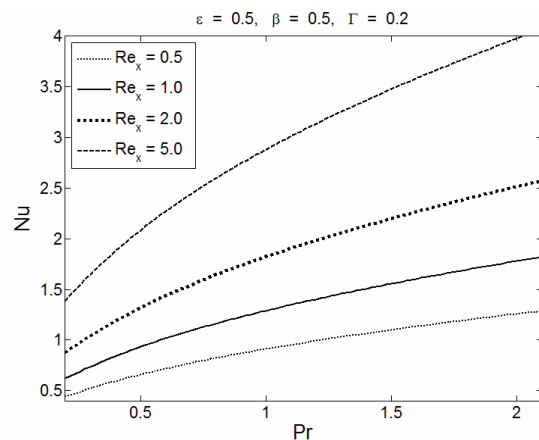

Figure 14: Nusselt numbers against Pr for different values of Re.

Table 1: Comparison of the behavior of the skin friction coefficient for the different parameters.

		f''(0)					
		HAM	K-b	HAM	K-b	HAM	K-b
	β\ε	0.0		0.5		1.0	
Γ = 0.0	0.0	1.6737	1.6737	0.9475	0.9474	0.0207	0.0207
	0.5	1.5651	1.5652	0.6141	0.6147	0.0111	0.0111
	1.0	1.5084	1.5084	0.4955	0.4955	0.0086	0.0086
	10	1.3470	1.3470	0.2710	0.2707	0.0046	0.0046
	20	1.3263	1.3263	0.2505	0.2505	0.0010	0.0011
Γ = 0.5	0.0	5.1395	5.1394	1.2386	1.2386	0.0074	0.0074
	0.5	4.3816	4.3816	0.80472	0.8048	0.0065	0.0066
	1.0	4.3726	4.3726	0.6325	0.6325	0.0062	0.0062
	10	1.6310	1.6310	0.2957	0.2957	0.0044	0.0044
	20	1.4740	1.4739	0.2640	0.2639	0.0042	0.0042
Γ = 1.0	0.0	6.8915	6.8915	1.1819	1.1819	0.0057	0.0057
	0.5	6.4037	6.4037	0.8699	0.8703	0.0055	0.0055
	1.0	6.1075	6.1081	0.7110	0.7111	0.00525	0.0053
	10	1.9600	1.9599	0.3193	0.3193	0.0043	0.0043
	20	1.6319	1.6319	0.2768	0.2768	0.0041	0.0041

Table 2: Comparison of the behavior of Nusselt number for different parameters.

			$-\theta'(0)$					
			HAM	K-b	HAM	K-b	HAM	K-b
		ε\Pr	0.72		1.0		10	
Γ = 0.0	β = 0	0.5	1.1663	1.1663	1.3475	1.3475	3.9196	3.9196
		1.0	1.3605	1.3605	1.5960	1.5959	5.0597	5.0597
		2.0	1.7040	1.7040	2.0311	2.0311	6.9145	6.9145
	β = .5	0.5	1.1271	1.1272	1.3014	1.3014	3.8242	3.8242
		1.0	1.3595	1.3595	1.5946	1.5946	5.0551	5.0551
		2.0	1.7986	1.7986	2.1367	2.1367	7.0789	7.0789
	β = 1	0.5	1.1149	1.1149	1.2870	1.2869	3.7937	3.7937
		1.0	1.3592	1.3592	1.5943	1.5943	5.0540	5.0540
		2.0	1.8174	1.8174	2.1571	2.1571	7.1072	7.1072
Γ = 0.5	β = 0	0.5	1.1596	1.1596	1.3413	1.3413	3.9460	3.9459
		1.0	1.3590	1.3590	1.5940	1.5940	5.0532	5.0532
		2.0	1.8727	1.8727	1.9835	1.9835	7.1226	7.1226
	β = .5	0.5	1.1399	1.1399	1.3173	1.3173	3.8760	3.8759
		1.0	1.3590	1.3590	1.5939	1.5939	5.0530	5.0529
		2.0	1.8727	1.8727	2.2148	2.2148	7.1771	7.1771
	β = 1	0.5	1.1292	1.1292	1.3044	1.3044	3.8412	3.8412
		1.0	1.3589	1.3589	1.5939	1.5939	5.0527	5.0527
		2.0	1.8707	1.8708	2.2131	2.2131	7.1756	7.1756

CONCLUSION

The following are the main conclusions from the above study:

1. With an increase in β and Γ the velocity profile f′ increases for ε < 1, while it decreases for ε > 1.

2. With an increase in Prandtl number Pr, the temperature profile θ decreases.

3. With an increase in the stretching ratio ε, the temperature profile θ decreases.

4. The skin-friction coefficient for a third grade fluid is greater than that for a viscous fluid.

5. Both numeric and HAM solutions are in excellent agreement.

ACKNOWLEDGMENT

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