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# Nonlinear Instabilities on a Granular Bed Sheared by a Turbulent Liquid Flow

*The granular media is of great importance in our quotidian, and their transport by a fluid flow is frequently found in nature and in industry. When the shear stresses exerted by the fluid flow on a granular bed are bounded to some limits, a mobile granular layer known as bed-load takes place in which the grains stay in contact with the fixed part of the granular bed. Under these conditions, a flat granular bed may be unstable, generating ripples and dunes. In a recent article (Franklin, 2010), the mechanisms of this instability were explained and a linear stability analysis was presented, in which a scaling between the fluid flow conditions and the typical length of the initial bed-forms was proposed. The present paper proposes a nonlinear stability analysis (weakly nonlinear approach) applicable to sheared granular beds, shedding light on the evolution of the bed-forms after their initial phase. The scope of the nonlinear analysis is the same as that of Franklin (2010): granular beds under turbulent liquid flows and in the presence of bed-load. It is shown here that, in this case, the initial instabilities saturate (supercritical bifurcation). Also, a discussion is made on some published experimental data.*

**Keywords:** two-phase flow, granular bed, bed-load, nonlinear instabilities, pattern formation

## Introduction

The granular media is of great importance in our quotidian. We can account for their importance by considering some figures: (i) arid regions (soil composed of sand and of other solid fragments) occupy 20% of Earth's emerged surfaces; (ii) the world annual production of grains and aggregates is approximately ten billion metric tons; and (iii) the processing of granular media consumes roughly 10% of all the energy produced worldwide (Duran, 1999). The transport of granular matter entrained by a fluid flow is frequently found in nature and in industry. It is present, for example, in the erosion of river banks, in the formation of dunes and in hydrocarbon pipelines conveying sand.

When shear stresses exerted by the fluid flow on the granular bed are able to move some grains, but are relatively small compared to the grains weight, the flow is not able to transport grains as a suspension. Instead, a mobile layer of grains known as bed-load takes place in which the grains stay in contact with the fixed part of the granular bed. Under water, the thickness of this mobile layer is a few grain diameters (Bagnold, 1941; Raudkivi, 1976).

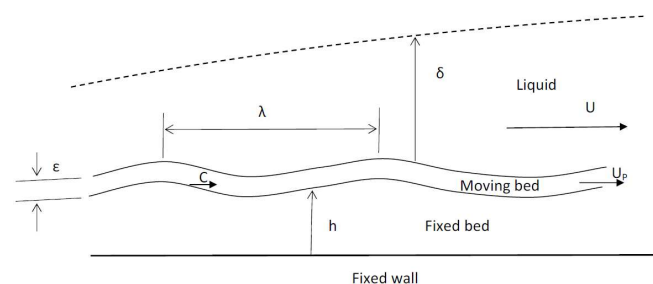
An initially flat granular bed may become unstable and give rise to bed-forms when submitted to a fluid flow. These forms, initially two-dimensional, may grow and generate patterns such as dunes. In nature, some examples affecting human activities are the aeolian and the aquatic dunes. The migrating aeolian dunes are one of the mechanisms of the expansion of deserts (Bagnold, 1941). The aquatic dunes observed on the bed of some rivers create a supplementary friction between the bed and the water, affecting the water depth and being related to flood problems. In cases where their size is comparable to the water depth, water flows can experiment strong depth variations, seriously affecting navigation (Engelund and Fredsoe, 1982). In industry, examples are mostly related to closed-conduit flows conveying grains, such as hydrocarbon pipelines conveying sand. In such cases, the bed-forms generate supplementary pressure loss, but also pressure and flow rate transients (Kuru et al., 1995; Franklin, 2008).

A balance between the local erosion and deposition of grains determines the stability of a granular bed. If there is erosion at the crests of the granular bed, the amplitude of initial bed undulations decreases and the bed is stable. On the contrary, the bed is unstable.

If there is neither erosion nor deposition at the crests, there is neutral stability (Franklin, 2010).

In a recent article (Franklin, 2010), the mechanisms of this instability were explained and a linear stability analysis was presented. It was seen that the basic mechanisms are three: the fluid flow perturbation by the shape of the bed, which is known to be the unstable mechanism (Jackson et al., 1975; Hunt et al., 1988; Weng et al., 1991), the relaxation effects related to the transport of grains and the gravity effects, which are the stable mechanisms (Valance and Langlois (2005) and Charru (2006) in the case of viscous flows, Franklin (2010) in the case of turbulent flows). The linear stability analysis of Franklin (2010) showed that the length-scale of the initial bed-forms varies with the fluid flow conditions.

This paper presents a nonlinear stability analysis in the same scope of Franklin (2010): the specific case of granular beds sheared by turbulent boundary-layers of liquids. The approach used here is the weakly nonlinear analysis (Landau and Lifchitz, 1994; Schmid and Henningson, 2001; Drazin and Reid, 2004; Charru, 2007). The main purpose of this analysis is to find if the initial instabilities saturate or not, explaining the length-scale of the aquatic dunes and ripples found in nature. Figure 1, reproduced from Franklin (2010), presents the dimensions involved in the studied problem.



**Figure 1.** Granular bed sheared by a fluid flow.  $U$  is the mean velocity of the fluid,  $U_p$  is the mean velocity of grains,  $h$  is the height of the fixed bed,  $\epsilon$  is thickness of the moving bed,  $c$  is the celerity of the bed-forms,  $\lambda$  is the wavelength of the bed-forms and  $\delta$  is the boundary-layer thickness. In the present paper,  $c \ll U_p \ll U$ ,  $\epsilon \ll h$ ,  $Re_\delta = \delta u/v > 10^2$  and, if  $\eta$  is the amplitude of the bed-forms,  $\eta/\lambda \ll 1$  and  $\eta/\delta \ll 1$ .  $u$  is the friction velocity (defined in the following sections) and  $v$  is the kinematic viscosity. Reproduced from Franklin (2010).

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The next two sections present a summary of the linear stability analysis of Franklin (2010) and a nonlinear analysis in the same scope of Franklin (2010), respectively. The following section discusses some previously published experimental data. A conclusion section follows.

## Nomenclature

$A$	= amplitude, $m$
$B$	= constant
$B_A$	= constant
$B_e$	= constant
$B_g$	= constant
$c$	= phase velocity, $m.s^{-1}$
$d$	= mean grain diameter, $m$
$g$	= acceleration of gravity, $m.s^{-2}$
$H$	= channel height, $m$
$h$	= local height of the granular bed, $m$
$i$	= imaginary number
$k$	= wave-number, $m^{-1}$
$L$	= length-scale, $m$
$Q$	= volumetric flow rate of grains by unit of width, in the basic state, $m^2.s^{-1}$
$q$	= local volumetric flow rate of grains by unit of width, $m^2.s^{-1}$
$u$	= fluid velocity, $m.s^{-1}$
$u_*$	= shear velocity, $m.s^{-1}$
$U_S$	= grain settling velocity, $m.s^{-1}$
$U_p$	= mean grain velocity, $m.s^{-1}$
$Re$	= Reynolds number ( $Re = HU/v$ )
$Re_\delta$	= friction Reynolds number ( $Re_\delta = \delta u_* / \nu$ )
$t$	= time, $s$
$x$	= horizontal (longitudinal) coordinate, $m$
$y$	= vertical coordinate, $m$
$y_0$	= rugosity height, $m$

## Greek Symbols

$\delta$	= boundary-layer thickness, $m$
$\varepsilon$	= thickness of the moving bed, $m$
$\varepsilon$	= small parameter used in gauge functions
$\eta$	= amplitude of the initial instabilities, $m$
$\kappa$	= von Kármán constant
$\kappa_L$	= Landau constant
$\lambda$	= wavelength of the initial instabilities, $m$
$\mu$	= dynamic viscosity, $Pa.s$
$\rho$	= density, $kg.m^{-3}$
$\sigma$	= growth rate, $s^{-1}$
$\tau$	= shear stress, $Pa$
$\nu$	= kinematic viscosity, $m^2.s^{-1}$
$\zeta$	= integration variable, $m$
$\omega$	= frequency, $s^{-1}$

## Subscripts

d	= relative to deposition
g	= relative to the acceleration of gravity
i	= relative to the imaginary part
max	= relative to the most unstable (amplified) mode
0	= relative to the basic state
p	= relative to grains
r	= relative to the real part
s	= relative to settling
sat	= relative to the saturated regime

## Superscripts

$\wedge$	= perturbation
*	= complex conjugate

## Summary of the Linear Stability Model

Franklin (2010) presented a linear stability analysis of the initial bed-forms on a granular bed sheared by a turbulent liquid flow, without free-surface effects. The analysis presented was two-dimensional, which is justified by taking into consideration the Squire's Theorem for parallel flows (Drazin and Reid, 2004): the most unstable modes in parallel flows are two-dimensional.

The stability analysis of Franklin (2010) was based on four equations, and a brief description is given below. Please, refer to Franklin (2010) for more details concerning the linear stability analysis. The four basic equations employed in the analysis describe the fluid flow perturbation by the shape of the bed, Eq. (1), the gravity effects (modeled in the previous equation), the transport of granular matter by a fluid flow, Eq. (4), the relaxation effects related to the transport of grains, Eq. (5), and the mass conservation of granular matter, Eq. (6).

For a hill with a height  $h$ , a surface rugosity  $y_0$  and a length  $2L$  between the half-heights (total length  $\approx 4L$ ), the perturbation of the longitudinal shear stress (dimensionless) caused by the fluid on the bed can be written as (Jackson and Hunt, 1975; Hunt et al., 1988; Weng et al., 1991):

$$\hat{\tau} = B_A \left( \frac{1}{\pi} \int \frac{\partial_x h}{x - \xi} d\xi + B_e \partial_x h \right) \quad (1)$$

where  $\xi$  is an integration variable and  $B_e = B - B_g/B_A$  (the term  $B_g/B_A$  was included in Franklin (2010)).  $B_A$  and  $B$  come from the fluid flow perturbation and are considered as constants as they vary with the logarithm of  $L/y_0$  (the variable used in the Jackson and Hunt (1975) gauge functions): varying  $L/y_0$  in three orders of magnitude does not change the orders of magnitude of  $B$ ,  $B_A$  and  $B_e$ .  $B_g$  is a coefficient taking into account the weight of the grains (gravity effects) and the friction between them.

If the perturbation is supposed small compared to a basic flow, the fluid flow over the bed can be written as the basic flow, unperturbed, plus the flow perturbation. For the shear stress on the bed surface

$$\tau = \tau_0 (1 + \hat{\tau}) \quad (2)$$

where  $\tau_0$  is the shear stress caused by the basic flow on the bed. For a developed turbulent liquid flow over a granular bed, the basic flow is a rough turbulent boundary-layer, which near the bed has the well known logarithmic profile (and from which  $\tau_0$  can be obtained):

$$u = u_* / \kappa \log \left( \frac{y}{y_0} \right) \quad (3)$$

where  $\kappa$  is the Kármán constant,  $y_0$  is the rugosity height,  $u(y)$  is the unperturbed velocity profile and  $u_*$  is the friction velocity, defined as  $u_* = \tau_0^{1/2} \rho^{-1/2}$ , where  $\rho$  is the density of the fluid.

The flow rate of grains in equilibrium with the fluid flow is known as "saturated flow rate of grains". From Bagnold (1941) and the shear stress given by Eq. (2)

$$q_{sat} / Q_{sat} \sim (1 + \hat{\tau})^{3/2} \quad (4)$$

where  $q_{sat}$  is the saturated volumetric flow rate of grains by unit of width and  $Q_{sat}$  is the saturated volumetric flow rate of grains by unit of width over a flat surface (basic state). If the fluid flow over the bed

changes, the flow rate of grains will lag some distance (or time) with respect to the fluid flow (relaxation effect). Charru et al. (2004) propose for the local volumetric flow rate of grains, by unit of width.

$$\partial_x q = \frac{q_{sat} - q}{L_{sat}} \quad (5)$$

where  $L_{sat} \sim L_d = d \left( \frac{u_*}{U_s} \right)$  is a distance called “saturation length”,  $d$  is the mean grain diameter and  $U_s$  is the typical settling velocity of a grain.

Finally, the two-dimensional mass conservation of grains is

$$\partial_t h + \partial_x q = 0 \quad (6)$$

The insertion of the normal modes given by Eq. (7) in Eqs. (1), (4) (5) and (6):

$$h(x, t) = H e^{\sigma - i\omega t + ikx}; \quad \frac{q(x, t)}{Q_{sat}} = 1 + Q e^{\sigma - i\omega t + ikx} \quad (7)$$

where  $\sigma$  is the growth rate,  $\omega$  is the frequency and  $k$  is the longitudinal wave-number, gives the solutions

$$\sigma = \frac{3Q_{sat} k^2 (B_e - B_A |k| L_{sat})}{2(1 + (kL_{sat})^2)}; \quad c = \frac{3Q_{sat} |k| (B_A + B_e |k| L_{sat})}{2(1 + (kL_{sat})^2)} \quad (8)$$

where  $c = \omega/k$  is the phase velocity. The most unstable (or amplified) mode is the one for which instabilities grow faster, corresponding to  $\partial\sigma/\partial k = 0$ . This gives the following results for the most unstable modes:

$$\lambda_{max} \approx \frac{3B_A}{2B_e} L_{sat} \quad (9)$$

$$\sigma_{max} \approx \frac{2B^3}{9B_A^2} (B_A - 2) Q_{sat} \frac{1}{L_{sat}^2} \quad (10)$$

$$c_{max} \approx \frac{B}{B_A} Q_{sat} \frac{1}{L_{sat}} \quad (11)$$

Based on these results, Franklin (2010) performed a stability analysis and compared it to some published experimental data. The stability analysis showed the existence of long-wave instability, with the fluid flow conditions, the relaxation effects and the gravity effects playing an important role. The saturation length-scale  $L_{sat}$ , related to the relaxation effects, was seen to be the major responsible for the stabilization of small waves, also playing a role in the growth rate, that varies as  $\sigma_{max} \sim L_{sat}^{-2}$ . On the other hand, gravity was seen to play a smaller role in the stabilization of small waves, but to strongly affect the growth rate. Changes in the fluid flow were seen to cause variations in the growth rate proportional to the shear velocity:  $\sigma_{max} \sim u_*$ . Concerning the wavelength of the most unstable mode, it was seen to scale with the fluid flow as  $\lambda_{max} \sim u_*$ . The scaling  $\lambda_{max} \sim u_*$  is in agreement with the experimental results obtained by Kuru et al. (1995) and Franklin (2008).

Different from previous stability analysis for turbulent regime, it was proposed in Franklin (2010) that the initial wavelength varies

with the flow conditions of the carrier liquid. This explains, for the first time, some previous experimental results.

### Nonlinear Analysis

In a stability analysis, it is considered a basic state (stationary solutions of order  $O(1)$ ), submitted to small perturbations (of order  $O(\varepsilon)$ , with  $\varepsilon \ll 1$ ). If the analysis is linear, like the one in Franklin (2010), the products of perturbations and of their derivatives (of order  $O(\varepsilon^2)$ ) are neglected, conducting to linear equations. The obtained linear equations admit solutions of the kind of Eq. (7), called normal modes.

The linear analysis can determine the most amplified mode in case of instability, but it is only valid as long as the perturbations remain small, so that the terms of order  $O(\varepsilon^2)$  can be neglected. Nevertheless, the prediction of an exponential growth (see Eq. (7)) during the linear phase of the instability means that the domain of validity of the linear analysis is bounded to the very early stages of the instability growth.

In some cases, the comparison of linear stability analyses with experimental data shows good agreement, even when the measured instabilities are no longer small. In those cases, how can we explain the agreement, if the linear analysis is out of its domain of validity? This question may only be answered by performing a nonlinear stability analysis.

A nonlinear stability analysis is presented here, using a weakly nonlinear approach (Landau and Lifchitz, 1994; Schmid and Henningson, 2001; Drazin and Reid, 2004; Charru, 2007). It is first presented a description of this approach, followed by its application to the granular bed instability.

#### The weakly nonlinear approach

The linear analysis admits plane waves as solutions, which can be written as

$$h(x, t) = \frac{1}{2} \hat{h} e^{\sigma - i\omega t + ikx} + c.c. \quad (12)$$

where  $\hat{h} = |\hat{h}| e^{i\phi}$  is a complex amplitude ( $\phi$  is its phase) and  $c.c.$  stands for complex conjugate. Taking the real part of Eq. (12), it can be seen that  $\sigma > 0$  corresponds to amplification of perturbations (instability) and  $\sigma < 0$  to their dumping (stability).

It can be shown (Drazin and Reid, 2004; Charru, 2007) that the perturbation (Eq. (12)) can be written as

$$h(x, t) = \frac{1}{2} (A(t)f(\bar{x}) + A^*(t)f^*(\bar{x})) \quad (13)$$

where  $f(\bar{x})$  describes the spatial structure of the mode and  $A(t)$  its temporal evolution ( $A(t)$  corresponds to the temporal evolution of the amplitude,  $A \sim e^{(i\phi + \sigma)t}$ ). The symbol  $*$  corresponds to the complex conjugate.

In a linear approach, the amplitude of a normal mode obeys

$$\frac{dA}{dt} = \sigma A \quad (14)$$

because  $A \sim e^{(i\phi + \sigma)t}$  and, as there is symmetry in time (time origin is arbitrary),  $A \sim e^{\sigma t}$  is also solution.

The argument of Landau and Lifchitz (1994) is that, for  $A$  small, Eq. (14) can be seen as a power series truncated at  $O(1)$ . In order to

capture nonlinear effects, they proposed the expansion of this equation in higher order powers, that are  $A^2, AA^*, A^3, A^2A^*, A(A^*)^2, + O(A^4)$ , but keeping only the terms that resonate with the linear one.

An analysis of the resonances with the linear term, or an analysis of the symmetries, allows the exclusion of all the listed terms, except  $A^2A^* = |A|^2A$  (resonant term): this term will interact with the linear one, and evolve much faster than the others (which can then be neglected). The amplitude is then governed by the Landau Equation

$$\frac{dA}{dt} = \sigma A - \kappa_L |A|^2 A + O(A^4) \tag{15}$$

where  $\kappa_L$  is complex and is known as Landau constant. Separating the real part from the imaginary part (for  $|A| = a, A = ae^{i\phi}$  and  $\kappa_L = \kappa_r + i\kappa_i$ ), we obtain for the real part

$$\frac{da}{dt} = \sigma a - \kappa_r a^3 \tag{16}$$

Equation (16) is the model equation for some typical bifurcations of the dynamics systems theory. When  $\kappa_r > 0$ , there is a supercritical bifurcation, with saturation of the amplitude modulus. When  $\kappa_r < 0$ , there is a subcritical bifurcation, without saturation. In this case, an analysis with higher order terms is necessary.

**Nonlinear stability of a granular bed**

The same approach is now applied to the case of a granular bed sheared by a turbulent liquid flow. It is worth to note here that, in cases where the partial differential equations governing the system give rise to waves with limited wave-number, the number of Fourier components is limited (for instance, when the large scales are limited by the finite dimensions of the system or by periodicity, when the small scales are limited by dissipative processes, etc.). This means that the normal modes of Eq. (13) may be written as

$$h(x,t) = \frac{1}{2} \sum_{n=-\infty}^{\infty} A_n(t) e^{inx} \tag{17}$$

where  $A_{-n} = ae^{-i\phi} = A_n^*$ .

Before inserting the form described by Eq. (17) in the model, Eq. (4) needs to be expanded, differently from Franklin (2010), at least until the first nonlinear term (in order to be coherent with a nonlinear model). The expansion of this equation in a Taylor series gives:

$$\frac{q_{sat}}{Q_{sat}} = 1 + \frac{3}{2}(\hat{\tau}) + \frac{3}{8}(\hat{\tau}^2) + O(\epsilon^3) \tag{18}$$

Equation (1), for the perturbation of the shear stress, is too complicated to be employed in a non-linear analysis. However, it can be simplified if we note that its first term, the convolution product, is a non-local term that varies with the shape of the bed (Andreotti et al., 2002). This term may then be replaced by a bed dimensionless shape  $h/L$  (where  $L$  is a characteristic length of the bed-form) multiplied by a constant

$$\hat{\tau} \sim \frac{D}{L} h + E \partial_x h \tag{19}$$

where  $D$  corresponds to  $B_A$  multiplied by a constant and  $E$  corresponds to the product of  $B_A$  and  $B_e$ , so that  $D$  and  $E$  are considered as constants. Inserting Eq. (19) in Eq. (18), combining it with Eqs. (4), (5) and (6) and noting that  $q \sim c h$  (Franklin, 2008), the following equation can be found:

$$\partial_t h + B_1(h)^2 + B_2(\partial_x h)^2 + B_3 h \partial_x h + B_4 h + B_5 \partial_x h + B_6 = 0 \tag{20}$$

where  $B_1$  to  $B_5$  involve  $Q_{sat}, L_{sat}, E, D$  and  $L$ , so that  $B_1$  to  $B_5$  are only functions of  $u_*$  and  $d$ , and they may be treated as constants in an analysis of a given granular bed submitted to a given fluid flow.  $B_6$  is a constant, obtained from  $c \sim h^{-1}$  (Franklin, 2010).

Normalizing the problem by its characteristic length ( $k^{-1}$ ), and inserting the normal modes of the form of Eq. (17) in Eq. (20) give

$$\begin{aligned} & \frac{1}{2} \sum_{n=-\infty}^{\infty} \left[ \frac{dA_n}{dt} + A_n B_4 + i B_3 n A_n \right] e^{inx} + \\ & + \frac{1}{2} \sum_{n=-\infty}^{\infty} [A_n^2 B_1 + B_2 (i n A_n)^2] e^{2inx} + \\ & + \frac{1}{2} \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} [B_3 A_p i q A_q] e^{i(q+p)x} + B_6 = 0 \end{aligned} \tag{21}$$

By inspecting Eq. (21), we can see that it is the third term in the equation that can resonate with the linear part (in the first term). This resonance will only occur if  $q + p = n$ . In this case, the third term in Eq. (21) can be written as

$$-\frac{i B_3}{2} \sum_{n=-\infty}^{\infty} \sum_{p=-\infty}^{\infty} [p A_{p+n} A_p^*] e^{inx} \tag{22}$$

and, keeping in Eq. (21) only the terms that resonate with the linear part, we find

$$\frac{dA_n}{dt} = \sigma_n A_n + i B_3 \sum_{p=-\infty}^{\infty} [p A_{p+n} A_p^*] \tag{23}$$

where  $\sigma_n = -(B_4 + i n B_5)$ .

Comparing Eq. (23) with the linear analysis (Eq. (14)), it can be seen that the non-linearities are in the third term. If this term is neglected, we find that the solution is stable for  $\sigma_n < 0$  and unstable for  $\sigma_n > 0$ . Once the initial (linear) instability takes place, the perturbations grow in an exponential way and, after a time-scale equal to  $\sigma_n^{-1}$ , they can no-longer be analyzed by a linear approach: in the nonlinear phase, the nonlinear terms are no-longer negligible and they must be taken into account. However, if some of them resonate with the initial (linear) modes, they are expected to grow much faster than the other nonlinear terms, so that they are the only ones to be taken into account (this is the same idea developed by Landau and Lifchitz (1994)). The third term in Eq. (23) is the one that contains the nonlinear resonant part of the problem.

In order to better understand the behavior of the nonlinear part of Eq. (23), we can analyze only the first three modes:

$$\frac{dA_1}{dt} = \sigma_1 A_1 - B_3 i A_2 A_1^* + O(A_1^5) \tag{24}$$

$$\frac{dA_2}{dt} = \sigma_2 A_2 - B_3 i A_1^2 + O(A_1^4) \tag{25}$$

$$\frac{dA_3}{dt} = \sigma_3 A_3 - 3B_3 i A_1 A_2 + O(A_1^5) \tag{26}$$

Now, observing that in the neighborhood of the instability only the fundamental mode is unstable ( $\sigma_1 > 0$  and  $\sigma_n < 0$  for  $n > 1$ , where  $|\sigma_n| \gg |\sigma_1|$ ), we can expect a characteristic time for the fundamental ( $|\sigma_1|^{-1}$ ) much greater than that for the other modes ( $|\sigma_n|^{-1}$ ): there is a dominant effect of the fundamental on the onset of the instability. For  $n > 1$ , on the onset of the instability, the temporal derivatives vary with  $\sigma_1 A_n \ll |\sigma_n| A_n$  (due to the dominant effect of the fundamental). The time derivatives may then be neglected for  $n > 1$

$$A_2 = \frac{B_3 i}{\sigma_2} A_1^2 + O(A_1^4) \tag{27}$$

$$A_3 = \frac{3B_3 i}{\sigma_3} A_1 A_2 + O(A_1^5) \tag{28}$$

which shows that  $A_n \sim \epsilon^n$ . Inserting Eq. (27) into Eq. (24), we can find an equation for the fundamental similar to the Landau Equation (Eq. (15)):

$$\frac{dA_1}{dt} = \sigma_1 A_1 - \kappa_L A_1 |A_1|^2 + O(A_1^5) \tag{29}$$

where  $\kappa_L = -\frac{B_3^2}{\sigma_2} > 0$ . This corresponds to a supercritical bifurcation (Glendinning, 1999; Charru, 2007): the nonlinear term resonating with the linear one will saturate the instability, so that, after the initial exponential growth, the instability attenuates, reaching a finite value for the amplitude and maintaining the same wavelength.

A bifurcation diagram can be drawn in order to visualize the saturation of the fundamental mode  $A_1$  as a function of a control parameter (Glendinning, 1999). Considering the Eq. (24) and that there is a dominant effect of the fundamental mode on the onset of the instability, the control parameter here is  $\sigma_1$ .

From Eq. (29), it can be seen that, once instability has been triggered, the amplitude of the stationary points are  $\pm \sqrt{\frac{\sigma_1}{\kappa_L}}$ .

Figure 2 shows the bifurcation diagram for the fundamental mode (dimensionless amplitude modulus  $|A_1|$  versus the dimensionless linear growth rate  $\sigma_1$ ) for a fixed value of the Landau constant ( $\kappa_L = 1$ ). The continuous curves correspond to stable states (attractors) and the dashed curve corresponds to the unstable states.

From Fig. 2, we can see that this is a supercritical bifurcation, the diagram corresponding to the well known supercritical pitchfork bifurcation. So, after the initial exponential growth (linear phase), the granular bed instabilities saturate, with their amplitude following the bifurcation diagram of Fig. 2, but keeping the same wavelength.

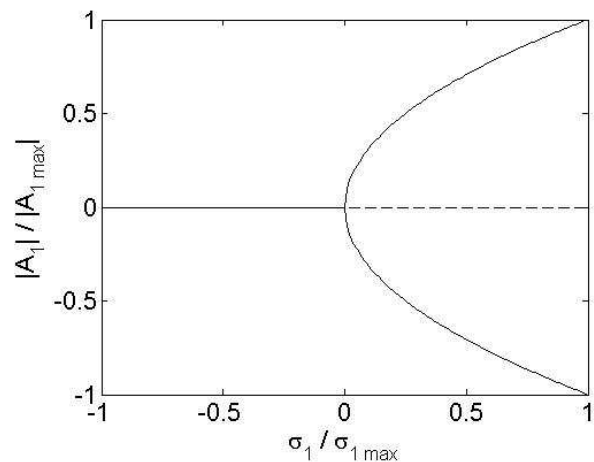


Figure 2. Bifurcation diagram for the fundamental mode (amplitude modulus  $|A_1|$  versus the linear growth rate  $\sigma_1$ , normalized by their maximum values, respectively). The continuous curves correspond to the stable states (attractors) and the dashed curve corresponds to the unstable states. This diagram corresponds to the well-known pitchfork bifurcation.

### Discussion of Some Available Experimental Data

Some published experimental data concerning the development of ripples are examined here. The objective is to verify if the saturation predicted by the present nonlinear analysis is experimentally observed.

As the subject of this paper is the evolution of the bed-forms just after their initial growth (linear phase), the experimental data to be examined must correspond to the formation and development of ripples in their early stages, i.e., after their initial growth but before any ripple coalescence has taken place. So, whenever the experimental data concern the evolution of ripples in long time-scales, care must be taken not to use the data at the end time, where coalescence has already occurred.

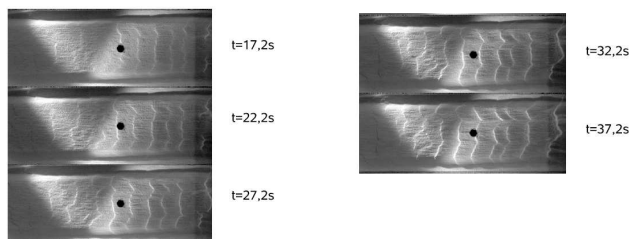
Kuru et al. (1995) presented a theoretical and experimental study of the initial instabilities on a granular bed on a horizontal pipe flow, which is a case without free-surface effects. Their experimental test section was a 31.1 mm diameter pipe, 7 m long, and they employed a mixture of water and glycerin as the fluid media and glass beads as the granular media. More details can be found in Kuru et al. (1995). In each experimental test, an initially plane granular bed was submitted to a specific flow of liquid and, when the ripples were visible, the tests were stopped. The wavelength of these ripples was then measured and associated to the initial instabilities. However, they reported that the amplitudes of the ripples were 2-3 mm (10 to 20 times the grain diameters), so that they correspond in fact to the early stages of the nonlinear phase. The fact that they didn't notice any length-scale variation from the time when the ripples were first visualized to the complete stop of the experiment (this time interval is not negligible compared to the time-scale for ripples formation, of only a few seconds) means that these forms saturate after their initial growth, agreeing with the nonlinear analysis developed in this paper.

Coleman et al. (2003) experimentally studied the granular bed instabilities in a closed-conduit turbulent liquid flow (without free-surface effects). Their experimental test section was a 6 m long horizontal closed-conduit of rectangular cross-section (300 mm wide by 100 mm high), and they employed water as the fluid media and glass beads as the granular media. The fluid flow was in the range  $26000 < Re < 70000$  ( $Re = \frac{UH}{\nu}$ ,  $H$  is the channel height).

More details can be found in Coleman et al. (2003). Contrary to Kuru (1995) and Franklin (2008), they found that the initial instabilities scale with the grains diameter, but not with the fluid flow. However, analyzing their data, it can be observed that after the initial growth, and before any coalescence takes place, the wavelength of these forms saturates. Again, this agrees with the nonlinear analysis developed in this paper.

Franklin (2008) experimentally studied the initial instabilities on different granular beds under turbulent water flows. His experimental test section was a 6 m long horizontal closed-conduit of rectangular cross-section (120 mm wide by 60 mm high), made of transparent material. He employed water as the fluid media and glass and zirconium beads as the granular media. The fluid flow, in the range  $13000 < Re < 24000$ , was measured by PIV (Particle Image Velocimetry) and the granular bed evolution was measured by a high definition camera. More details can be found in Franklin and Charru (2007), Franklin (2008) and Franklin and Charru (2009).

Franklin (2008) measurements showed that the initial bed-forms are two-dimensional, as predicted by the linear stability analysis of Franklin (2010), and that, after the initial two-dimensional phase, bed-forms evolve to three-dimensional forms, as seen in Fig. 3. The wavelength of the three-dimensional forms, developed during the nonlinear phase of the instability, is the same as that of the linear phase (two-dimensional ripples). This is an experimental evidence of the saturation of the instabilities after the linear phase, corroborating the nonlinear analysis developed in the preceding section.



**Figure 3. Evolution of the wavelength  $\lambda$  of initial ripples on a granular bed sheared by a turbulent water flow (top view). Flow direction is from right to left,  $Re = 19900$  and the granular bed is composed of zirconium beads with  $d = 180 \mu\text{m}$ . The initial two-dimensional ripples evolve to three-dimensional forms (during the nonlinear phase) which keep the same wavelength. Figure extracted from Franklin (2008).**

## Conclusions

The transport of solid particles entrained by a fluid flow is frequent in nature and in industry. Under some fluid flow conditions, a mobile granular layer known as bed-load takes place in which the grains stay in contact with the fixed part of the granular bed. In some situations, an initially flat granular bed may become unstable, giving rise to ripples or dunes. The formation of dunes in deserts, in river beds and in petroleum pipelines conveying sand are some examples. A better knowledge of the instabilities on a granular bed and of their evolution is of great importance to understand nature as well as to improve grains-related industrial processes.

This paper presents a theoretical investigation of the nonlinear phase of the instabilities on granular beds sheared by turbulent liquid flows, without free-surface effects, such as those in which the liquid depth is many times greater than the typical height of the bed-forms, or flows in pipes and closed-conduits. The approach adopted here is the weakly nonlinear analysis (Landau and Lifchitz, 1994; Schmid and Henningson, 2001; Drazin and Reid, 2004; Charru, 2007), useful whenever a dominant mode can be proved to exist. This means that the modes resonating with this dominant one will

grow much faster than the others, which can be neglected. The analysis is then made on a bounded number of modes.

For the specific case studied in this paper, it was shown that there is a fundamental mode that dominates the dynamics of the instability and, on the instability onset, this mode comes from the linear phase (initial phase). It was also shown that, considering only the resonating terms, the instability is well described by the Landau Equation and that it corresponds to a supercritical bifurcation of the pitchfork type. So, after the initial exponential growth (linear phase), the granular bed instabilities saturate, i.e., they attenuate their growth rate and maintain the same wavelength.

The results from the nonlinear analysis were compared to some published experimental data concerning the formation and development of ripples in closed-conduit flows, for liquids in turbulent regime. The works of Kuru et al. (1995), Coleman et al. (2003) and Franklin (2008) showed that there are evidences of wavelength saturation during the nonlinear phase of the ripples formation. Nevertheless, even after saturation, in the long time-scales, the wavelengths of ripples may grow due to another mechanism: the coalescence between them (Coleman et al., 2003). This is a mechanism that is not related to hydrodynamic effects, and that usually happens after the saturation described here has been achieved, so that it is not treated in this paper.

In summary, for the specific case studied, it was theoretically shown here that the granular bed instabilities saturate with the same wavelength of the initial (linear) phase, after their initial growth. To the author knowledge up to now there is no theoretical treatment of this kind, proving theoretically the saturation of those forms under the conditions studied here. This explains the experimental observations of saturation of the granular bed-forms under turbulent flows.

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