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Dynamic behaviour under moving concentrated masses of simply supported rectangular plates resting on variable Winkler elastic foundation

Abstract

The response of simply supported rectangular plates carrying moving masses and resting on variable Winkler elastic foundations is investigated in this work. The governing equation of the problem is a fourth order partial differential equation. In order to solve this problem, a technique based on separation of variables is used to reduce the governing fourth order partial differential equations with variable and singular coefficients to a sequence of second order ordinary differential equations. For the solutions of these equations, a modification of the Struble's technique and method of integral transformations are employed. Numerical results in plotted curves are then presented. The results show that response amplitudes of the plate decrease as the value of the rotatory inertia correction factor R_0 increases. Furthermore, for fixed value of R_0, the displacements of the simply supported rectangular plates resting on variable elastic foundations decrease as the foundation modulus F_0 increases. The results further show that, for fixed R_0 and F_0, the transverse deflections of the rectangular plates under the actions of moving masses are higher than those when only the force effects of the moving load are considered. Therefore, the moving force solution is not a safe approximation to the moving mass problem. Hence, safety is not guaranteed for a design based on the moving force solution. Also, the analyses show that the response amplitudes of both moving force and moving mass problems decrease both with increasing Foundation modulus and with increasing rotatory inertia correction factor. The results again show that the critical speed for the moving mass problem is reached prior to that of the moving force for the simply supported rectangular plates on variable Winkler elastic foundation.

Keywords

rectangular plates, winkler foundation, foundation modulus, rotatory inertia, resonance, moving force, moving mass.

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1 INTRODUCTION

The analyses of elastic structures (beams, plates and shells), resting on a subgrade, such as railway tracks, highway pavements, navigation locks and structural foundations, constitute an important part of the Civil Engineering and applied Mathematics literatures. In general, such analyses are mathematically complex due to the difficulty in modeling the mechanical response of the subgrade which is governed by many factors. When these structures are acted upon by moving loads, the dynamic analyses of the system become more cumbersome [7]. The crudest approximation known to the literatures to this problem is the so called "moving force" problem, in which the vehicle-track interaction is completely neglected, and the action of the vehicle is described as a concentrated force moving along the beam [12].

Several researchers have considered vehicle-track interaction in their analyses. These they commonly termed moving mass problems. These researchers include Stanisic et al [22], Milornir et al [11], Clastornic et al [3], Sadiku and Leipholz [19] and Gbadeyan and Oni [8]. Douglas et al [5] solved the problem of plate strip of varying thickness and the center of shear. In their work, they considered a free-vibrating strip with classical boundary conditions, precisely, they assumed the plate strip clamped at one end and free at the other end. Pesterev et al [18] came up with a series expansion method for calculating bending moment and shear force in the problem of vibration of a damped beam subject to an arbitrary number of moving loads. This kind of solution, though could be accurate, cannot account for vital information such as the phenomenon of resonance in the dynamical system.

Recently, several other researchers have made tremendous efforts in the study of dynamics of structures under moving loads, these include Oni [13], Oni and Omolofe [17], Oni and Awodola [14], Omer and Aitung [2], Adams [1], Savin [20], Jia-Jang [23]. In all of these, considerations have been limited to cases of one-dimensional (beam) problems. Where two-dimensional (plate) problems have been considered, the foundation moduli are taken to be constants. No considerations have been given to the class of dynamical problems in which the foundation is the type with stiffness variation. In an attempt to solve such two-dimensional problem, all the methods used in the above works break down due to the variation of the foundation model.

It is generally known that the dynamical problems of structures under moving load and resting on a foundation is generally complex, the complexity increases if the foundation stiffness varies along the structure. Aside the problem of singularity brought in by the inclusion of the inertia effects of the moving load, the coefficients of the governing fourth order partial differential equation are no longer constant but variable. Earlier researchers into beam member on variable elastic foundation include Franklin and Scott [6] who presented a closed-form solution to a linear variation of the foundation modulus using contour-integrals. Closely following this, Lentini [10] presented a finite difference method to solve the problem where the foundation stiffness varies along x as a power of x. Much later, Clastornik et al [4] presented a solution for the finite beams resting on a Winkler elastic foundation with stiffness variation that can be presented as a general polynomial of x. Though works in [4, 6, 10] are useful, the loads acting on the beams are not moving loads. In a recent development, Oni and Awodola [15] extended the works of these previous authors to investigate the dynamic response to moving concentrated masses of uniform Rayleigh beams resting on variable Winkler elastic foundation. Their work shows that for all variants of classical boundary conditions, the displacements of a uniform Rayleigh beam resting on variable elastic foundation and traversed by moving masses decrease both with increase in the foundation moduli and the rotatory inertial correction factor.

More recently, Oni and Awodola [16] considered the dynamic response under a moving load of an elastically supported non-prismatic Bernoulli-Euler beam on variable elastic foundation. The technique was based on the generalized Galerkin's method and integral transformations.

In all these previous investigations, extension of the theory to cover two-dimensional (plate) problem has not been effected, when the plate is on variable foundation. Therefore, this study concerns the response to moving concentrated masses of simply supported rectangular plate resting on Winkler elastic foundation with stiffness variation.

2 GOVERNING EQUATION

The equation governing the dynamic transverse displacement $W(x, y, t)$ of a rectangular plate when it is resting on a variable Winkler foundation and traversed by several moving concentrated masses is the fourth order partial differential equation given by

$$D\nabla^4 W(x,y,t) + \mu \frac{\partial^2 W(x,y,t)}{\partial t^2} + F(x)W(x,y,t) = \mu R_0 \frac{\partial}{\partial t^2} \nabla^2 W(x,y,t) + P(x,y,t) \tag{1}$$

where

$$D = \frac{Eh^2}{12(1-v)} \tag{2}$$

is the bending rigidity of the plate, ∇^2 is the two-dimensional Laplacian operator, $W(x, y, t)$ is the transverse displacement, h is the plate's thickness, E is the Young's Modulus, v is the Poisson's ratio ($v < 1$), μ is the mass per unit area of the plate, R_0 is the Rotatory inertia correction factor, $F(x)$ is the variable foundation's stiffness, $P(x, y, t)$ is the Moving load, x and y are respectively the spatial coordinates in x and y directions and t is the time coordinate.

When the effect of the mass of the moving load on the response of the plate is taken into consideration, the external moving surface load takes on the form

$$P(x,y,t) = P_f(x,y,t)\left[1 - \frac{\Delta^*}{g} W(x,y,t)\right] \tag{3}$$

where $P_f(x, y, t)$ is the continuous moving force, Δ^* is the substantive acceleration operator and g is the acceleration due to gravity.

The structure under consideration is assumed to be carrying an arbitrary number (say N) of concentrated masses M_i moving with constant velocities c_i, $i = 1, 2, 3, \ldots, N$ along a straight line parallel to the x-axis (no difficulty arises by assuming that masses travel in an arbitrary path) issuing from point $y = s$ on the y-axis. Thus, the moving force acting on the plate is defined as

$$P_f(x,y,t) = \sum_{i=1}^{N} M_i g \delta(x - c_i t)\,\delta(y - s) \tag{4}$$

where $\delta(.)$ is the Dirac-Delta function.

The operator Δ^* used in equation (3) for masses traveling in an arbitrary path in the x-y plane is defined as

$$\begin{aligned}\Delta^* = {} & \frac{\partial^2}{\partial x^2}\left(\frac{dx}{dt}\right)^2 + \frac{\partial^2}{\partial y^2}\left(\frac{dy}{dt}\right)^2 + \frac{\partial^2}{\partial t^2} + 2\frac{\partial^2}{\partial x \partial y}\frac{dx}{dt}\frac{dy}{dt} \\ & + 2\frac{\partial^2}{\partial x \partial t}\frac{dx}{dt} + 2\frac{\partial^2}{\partial y \partial t}\frac{dy}{dt} + \frac{\partial}{\partial x}\frac{d^2x}{dt^2} + \frac{\partial}{\partial y}\frac{d^2y}{dt^2}\end{aligned} \tag{5}$$

On the assumptions of the paragraph above, this operator takes the form

$$\Delta^* = \frac{\partial^2}{\partial t^2} + 2c_i\frac{\partial^2}{\partial x \partial t} + c_i^2\frac{\partial^2}{\partial x^2} \tag{6}$$

As an example in this problem, a variable elastic foundation of the form

$$F(x) = F_0(4x - 3x^2 + x^3) \tag{7}$$

where F_0 is the foundation constant, is considered.

Thus, substituting (3), (4), (5) and (7) into (1), one obtains

$$\begin{aligned} D\,\nabla^4 W(x,y,t) + \mu\frac{\partial^2 W(x,y,t)}{\partial t^2} = {} & \mu R_0\left[\frac{\partial^4}{\partial t^2 \partial x^2} + \frac{\partial^4}{\partial t^2 \partial y^2}\right] W(x,y,t) \\ & - F_0\left[4x - 3x^2 + x^3\right] W(x,y,t) + \sum_{i=1}^{N}\left[M_i g \delta(x - c_i t)\delta(y - s)\right. \\ & \left. - M_i\left(\frac{\partial^2}{\partial t^2} + 2c_i\frac{\partial^2}{\partial t \partial x} + c_i^2\frac{\partial^2}{\partial x^2}\right) W(x,y,t)\delta(x - c_i t)\delta(y - s)\right]\end{aligned} \tag{8}$$

The initial conditions, without any loss of generality, is taken as

$$W(x,y,t) = 0 = \frac{\partial W(x,y,t)}{\partial t} \tag{9}$$

3 ANALYTICAL APPROXIMATE SOLUTION

The method of analysis involves expressing the Dirac-Delta function as a Fourier cosine series. Because of the variable foundation term, the elegant method of the generalized integral transform breaks down while the generalized Galerkin's method used in one-dimensional structural problems (Beam problems) could not handle the two-dimensional structural problem (Plate problems). Hence, the technique based on separation of variables is used to reduce the fourth order partial differential equation governing the motion of the plate to a set of coupled second order ordinary differential equations. Then, the modified asymptotic method of Struble is used to simplify these resulting equations. The method of integral transformation and convolution theory are then employed to obtain the closed form solution of the two-dimensional dynamical problems.

In order to solve equation (8), in the first instance, the deflection is written in the form [21]

$$W(x,y,t) = \sum_{n=1}^{\infty} \phi_n(x,y)T_n(t) \tag{10}$$

where ϕ_n are the known eigenfunctions of the plate with the same boundary conditions. The ϕ_n have the form of

$$\nabla^4\phi_n - \omega_n^4\phi_n = 0 \tag{11}$$

where

$$\omega_n^4 = \frac{\Omega_n^2\mu}{D} \tag{12}$$

Ω_n, $n = 1, 2, 3, \ldots$, are the natural frequencies of the dynamical system and $T_n(t)$ are amplitude functions which have to be calculated.

In order to solve the equation (8), it is rewritten as

$$\begin{aligned}\frac{D}{\mu}\nabla^4 W(x,y,t) + \frac{\partial^2 W(x,y,t)}{\partial t^2} &= R_0\left[\frac{\partial^4}{\partial t^2\partial x^2} + \frac{\partial^4}{\partial t^2\partial y^2}\right]W(x,y,t)\\ &-\frac{F_0}{\mu}\left[4x - 3x^2 + x^3\right]W(x,y,t) + \sum_{i=1}^{N}\left[\frac{M_i g}{\mu}\delta(x - c_i t)\delta(y - s)\right.\\ &\left.-\frac{M_i}{\mu}\left(\frac{\partial^2}{\partial t^2} + 2c_i\frac{\partial^2}{\partial t\partial x} + c_i^2\frac{\partial^2}{\partial x^2}\right)W(x,y,t)\delta(x - c_i t)\delta(y - s)\right]\end{aligned} \tag{13}$$

At this juncture, the right hand side of equation (13) is written in the form of a series, we have

$$\begin{aligned}&R_0\left[\frac{\partial^4}{\partial t^2\partial x^2} + \frac{\partial^4}{\partial t^2\partial y^2}\right]W(x,y,t) - \frac{F_0}{\mu}\left[4x - 3x^2 + x^3\right]W(x,y,t) + \sum_{i=1}^{N}\left[\frac{M_i g}{\mu}\delta(x - c_i t)\delta(y - s)\right.\\ &\left.-\frac{M_i}{\mu}\left(\frac{\partial^2}{\partial t^2} + 2c_i\frac{\partial^2}{\partial t\partial x} + c_i^2\frac{\partial^2}{\partial x^2}\right)W(x,y,t)\delta(x - c_i t)\delta(y - s)\right] = \sum_{n=1}^{\infty}\phi_n(x,y)B_n(t)\end{aligned} \tag{14}$$

Substituting equation (10) into equation (14) we have

$$\sum_{n=1}^{\infty} \{R_0 \left[\phi_{n,xx}(x,y)T_{n,tt}(t) + \phi_{n,yy}(x,y)T_{n,tt}(t)\right] - \frac{F_0}{\mu}\left[4x - 3x^2 + x^3\right]\phi_n(x,y)T_n(t)$$
$$+ \sum_{i=1}^{N} [\ \frac{M_i g}{\mu}\delta(x - c_i t)\delta(y - s) - \frac{M_i}{\mu}\left(\phi_n(x,y)T_{n,tt}(t) + 2c_i\phi_{n,x}(x,y)T_{n,t}(t)\right.$$
$$\left. + c_i^2\phi_{n,xx}(x,y)T_n(t)\ \right)\delta(x - c_i t)\delta(y - s)\ ]\ \} = \sum_{n=1}^{\infty}\phi_n(x,y)B_n(t) \tag{15}$$

where

$$\begin{array}{ll} \phi_{n,x}(x,y) \text{ implies } \frac{\partial\phi_n(x,y)}{\partial x}, & \phi_{n,xx}(x,y) \text{ implies } \frac{\partial^2\phi_n(x,y)}{\partial x^2}, \\ \phi_{n,y}(x,y) \text{ implies } \frac{\partial\phi_n(x,y)}{\partial y}, & \phi_{n,yy}(x,y) \text{ implies } \frac{\partial^2\phi_n(x,y)}{\partial y^2}, \\ T_{n,t}(t) \text{ implies } \frac{dT_n(t)}{dt}, \quad \text{and} & T_{n,tt}(t) \text{ implies } \frac{d^2T_n(t)}{dt^2} \end{array} \tag{16}$$

Multiplying both sides of equation (15) by $\phi_p(x,y)$ and integrating on area A of the plate, we have

$$\sum_{n=1}^{\infty}\int_A \{R_0\left[\phi_{n,xx}(x,y)\phi_p(x,y)T_{n,tt}(t) + \phi_{n,yy}(x,y)\phi_p(x,y)T_{n,tt}(t)\right]$$
$$-\frac{F_0}{\mu}\left[4x - 3x^2 + x^3\right]\phi_n(x,y)\phi_p(x,y)T_n(t) + \sum_{i=1}^{N}[\ \frac{M_i g}{\mu}\phi_p(x,y)\delta(x - c_i t)\delta(y - s)$$
$$-\frac{M_i}{\mu}\left(\phi_n(x,y)\phi_p(x,y)T_{n,tt}(t) + 2c_i\phi_{n,x}(x,y)\phi_p(x,y)T_{n,t}(t)\right.$$
$$\left.+c_i^2\phi_{n,xx}(x,y)\phi_p(x,y)T_n(t)\ \right)\delta(x - c_i t)\delta(y - s)\ ]\ \}\ dA = \sum_{n=1}^{\infty}\int_A \phi_n(x,y)\phi_p(x,y)B_n(t)\,dA \tag{17}$$

Considering the orthogonality of $\phi_n(x,y)$

$$B_n(t) = \frac{1}{P^*}\sum_{n=1}^{\infty}\int_A \{R_0\left[\phi_{n,xx}(x,y)\phi_p(x,y)T_{n,tt}(t) + \phi_{n,yy}(x,y)\phi_p(x,y)T_{n,tt}(t)\right]$$
$$-\frac{F_0}{\mu}\left[4x - 3x^2 + x^3\right]\phi_n(x,y)\phi_p(x,y)T_n(t) + \sum_{i=1}^{N}[\ \frac{M_i g}{\mu}\phi_p(x,y)\delta(x - c_i t)\delta(y - s)$$
$$-\frac{M_i}{\mu}\left(\phi_n(x,y)\phi_p(x,y)T_{n,tt}(t) + 2c_i\phi_{n,x}(x,y)\phi_p(x,y)T_{n,t}(t)\right.$$
$$\left.+c_i^2\phi_{n,xx}(x,y)\phi_p(x,y)T_n(t)\ \right)\delta(x - c_i t)\delta(y - s)\ ]\ \}\ dA \tag{18}$$

where

$$P^* = \int_A \phi_p^2 dA$$

Using (18), equation (13), taking into account (10) and (11), can be written as

$$\phi_n(x,y)\left[\frac{D\omega_n^4}{\mu}T_n(t)+T_{n,tt}(t)\right]=\frac{\phi_n(x,y)}{P^*}\sum_{q=1}^{\infty}\int_A\{R_0\left[\phi_{q,xx}(x,y)\phi_p(x,y)T_{q,tt}(t)\right.$$
$$\left.+\phi_{q,yy}(x,y)\phi_p(x,y)T_{q,tt}(t)\right]-\frac{F_0}{\mu}\left[4x-3x^2+x^3\right]\phi_q(x,y)\phi_p(x,y)T_q(t)$$
$$+\sum_{i=1}^{N}\left[\ \frac{M_ig}{\mu}\phi_p(x,y)\delta(x-c_it)\delta(y-s)-\frac{M_i}{\mu}\left(\phi_q(x,y)\phi_p(x,y)T_{q,tt}(t)\right.\right.$$
$$\left.\left.+2c_i\phi_{q,x}(x,y)\phi_p(x,y)T_{q,t}(t)+c_i^2\phi_{q,xx}(x,y)\phi_p(x,y)T_q(t)\ \right)\delta(x-c_it)\delta(y-s)\ \right]\}\ dA \tag{19}$$

Equation (19) must be satisfied for arbitrary x, y (that is, each point of the plate) and this is possible only when

$$T_{n,tt}(t)+\frac{D\omega_n^4}{\mu}T_n(t)=\frac{1}{P^*}\sum_{q=1}^{\infty}\int_A\{R_0\left[\phi_{q,xx}(x,y)\phi_p(x,y)T_{q,tt}(t)\right.$$
$$\left.+\phi_{q,yy}(x,y)\phi_p(x,y)T_{q,tt}(t)\right]-\frac{F_0}{\mu}\left[4x-3x^2+x^3\right]\phi_q(x,y)\phi_p(x,y)T_q(t)$$
$$+\sum_{i=1}^{N}\left[\ \frac{M_ig}{\mu}\phi_p(x,y)\delta(x-c_it)\delta(y-s)-\frac{M_i}{\mu}\left(\phi_q(x,y)\phi_p(x,y)T_{q,tt}(t)\right.\right.$$
$$\left.\left.+2c_i\phi_{q,x}(x,y)\phi_p(x,y)T_{q,t}(t)+c_i^2\phi_{q,xx}(x,y)\phi_p(x,y)T_q(t)\ \right)\delta(x-c_it)\delta(y-s)\ \right]\}\ dA \tag{20}$$

The system in equation (20) is a set of coupled ordinary differential equations.

Considering the property of the Dirac-Delta function and expressing it in the Fourier cosine series as

$$\delta(x-c_it)=\frac{1}{L_X}\left[1+2\sum_{j=1}^{\infty}\cos\frac{j\pi c_it}{L_X}\cos\frac{j\pi x}{L_X}\right] \tag{21}$$

and

$$\delta(y-s)=\frac{1}{L_Y}\left[1+2\sum_{k=1}^{\infty}\cos\frac{k\pi s}{L_Y}\cos\frac{k\pi y}{L_Y}\right] \tag{22}$$

equation (20) becomes

$$
\begin{aligned}
&\frac{d^2T_n(t)}{dt^2} + \alpha_n^2 T_n(t) - \frac{1}{P^*}\sum_{q=1}^{\infty}\Bigg\{R_0 P_1^* \frac{d^2T_q(t)}{dt^2} - \frac{F_0}{\mu}P_2^* T_q(t) \\
&- \sum_{i=1}^{N}\frac{M_i}{L_X L_Y \mu}\Bigg[2\Bigg(\frac{P_3^*}{2} + \sum_{k=1}^{\infty}\cos\frac{k\pi s}{L_Y}\; P_3^{**}(k) + \sum_{j=1}^{\infty}\cos\frac{j\pi c_i t}{L_X}P_3^{***}(j) \\
&+2\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\cos\frac{j\pi c_i t}{L_X}\cos\frac{k\pi s}{L_Y}P_3^{****}(j,k)\Bigg)\frac{d^2T_q(t)}{dt^2} + 4c_i\Bigg(\frac{P_4^*}{2} + \sum_{k=1}^{\infty}\cos\frac{k\pi s}{L_Y}P_4^{**}(k) \\
&+\sum_{j=1}^{\infty}\cos\frac{j\pi c_i t}{L_X}P_4^{***}(j) + 2\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\cos\frac{j\pi c_i t}{L_X}\cos\frac{k\pi s}{L_Y}P_4^{****}(j,k)\Bigg)\frac{dT_q(t)}{dt} \\
&+2c_i^2\Bigg(\frac{P_5^*}{2} + \sum_{k=1}^{\infty}\cos\frac{k\pi s}{L_Y}P_5^{**}(k) + \sum_{j=1}^{\infty}\cos\frac{j\pi c_i t}{L_X}P_5^{***}(j) \\
&+\; 2\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\cos\frac{j\pi c_i t}{L_X}\cos\frac{k\pi s}{L_Y}P_5^{****}(j,k)\Bigg)T_q(t)\Bigg]\Bigg\} = \sum_{i=1}^{N}\frac{M_i g}{P^*\mu}\phi_p(c_i t, s)
\end{aligned}
\tag{23}
$$

where

$$\alpha_n^2 = \frac{D\omega_n^4}{\mu},$$

$$P_1^* = \int_0^{L_X}\int_0^{L_Y}\left[\phi_{n,xx}(x,y) + \phi_{n,yy}(x,y)\right]\phi_p(x,y)\,dy\,dx,$$

$$P_2^* = \int_0^{L_X}\int_0^{L_Y}\left[4x - 3x^2 + x^3\right]\phi_n(x,y)\phi_p(x,y)\,dy\,dx,$$

$$P_3^* = \int_0^{L_X}\int_0^{L_Y}\phi_n(x,y)\,\phi_p(x,y)\,dy\,dx,$$

$$P_3^{**}(k) = \int_0^{L_X}\int_0^{L_Y}\cos\frac{k\pi y}{L_Y}\phi_n(x,y)\phi_p(x,y)\,dy\,dx,$$

$$P_3^{***}(j) = \int_0^{L_X}\int_0^{L_Y}\cos\frac{j\pi x}{L_X}\phi_n(x,y)\phi_p(x,y)\,dy\,dx,$$

$$P_3^{****}(j,k) = \int_0^{L_X}\int_0^{L_Y}\cos\frac{j\pi x}{L_X}\cos\frac{k\pi y}{L_Y}\phi_n(x,y)\phi_p(x,y)\,dy\,dx,$$

$$P_4^* = \int_0^{L_X}\int_0^{L_Y}\phi_{n,x}(x,y)\,\phi_p(x,y)\,dy\,dx,$$

$$P_4^{**}(k) = \int_0^{L_X}\int_0^{L_Y}\cos\frac{k\pi y}{L_Y}\phi_{n,x}(x,y)\phi_p(x,y)\,dy\,dx,$$

$$P_4^{***}(j) = \int_0^{L_X}\int_0^{L_Y}\cos\frac{j\pi x}{L_X}\phi_{n,x}(x,y)\phi_p(x,y)\,dy\,dx,$$

$$P_4^{****}(j,k) = \int_0^{L_X}\int_0^{L_Y} \cos\frac{j\pi x}{L_X}\cos\frac{k\pi y}{L_Y}\phi_{n,x}(x,y)\phi_p(x,y)\,dy\,dx,$$

$$P_5^{*} = \int_0^{L_X}\int_0^{L_Y} \phi_{n,xx}(x,y)\phi_p(x,y)\,dy\,dx,$$

$$P_5^{**}(k) = \int_0^{L_X}\int_0^{L_Y} \cos\frac{k\pi y}{L_Y}\phi_{n,xx}(x,y)\phi_p(x,y)\,dy\,dx,$$

$$P_5^{***}(j) = \int_0^{L_X}\int_0^{L_Y} \cos\frac{j\pi x}{L_X}\phi_{n,xx}(x,y)\phi_p(x,y)\,dy\,dx \text{ and}$$

$$P_5^{****}(j,k) = \int_0^{L_X}\int_0^{L_Y} \cos\frac{j\pi x}{L_X}\cos\frac{k\pi y}{L_Y}\phi_{n,xx}(x,y)\phi_p(x,y)\,dy\,dx,$$

The second order coupled differential equation (23) is the transformed equation governing the problem of a rectangular plate on a variable Winkler elastic foundation. This differential equation holds for all variants of the classical boundary conditions.

In what follows, ϕ_n(x,y) are assumed to be the products of the functions ψ_{ni}(x) and ψ_{nj}(y) which are the beam functions in the directions of x and y axes respectively [9]. That is

$$\phi_n(x,y) = \psi_{ni}(x)\psi_{nj}(y) \tag{24}$$

Since each of these beam functions satisfies all the boundary conditions in its direction, the kernel (the product of these beam functions) in the above integrals satisfies all boundary conditions for any plate problem of practical interest. In particular, these beam functions can be defined respectively, as

$$\psi_{ni}(x) = \sin\frac{\Omega_{ni}x}{L_X} + A_{ni}\cos\frac{\Omega_{ni}x}{L_X} + B_{ni}\sinh\frac{\Omega_{ni}x}{L_X} + C_{ni}\cosh\frac{\Omega_{ni}x}{L_X} \tag{25}$$

and

$$\psi_{nj}(x) = \sin\frac{\Omega_{nj}y}{L_Y} + A_{nj}\cos\frac{\Omega_{nj}y}{L_Y} + B_{nj}\sinh\frac{\Omega_{nj}y}{L_Y} + C_{nj}\cosh\frac{\Omega_{nj}y}{L_Y} \tag{26}$$

where A_{ni}, A_{nj}, B_{ni}, B_{nj}, C_{ni} and C_{nj} are constants determined by the boundary conditions. Ω_{ni} and Ω_{nj} are called the mode frequencies.

In order to solve equation (23) we shall consider a mass M traveling with uniform velocity c along the line $y = s$. The solution for any arbitrary number of moving masses can be obtained by superposition of the individual solution since the governing differential equation is linear. Thus for the single mass M_1 equation (23) reduces to

$$\frac{d^2T_n(t)}{dt^2} + \alpha_n^2 T_n(t) - \frac{1}{P^*}\sum_{q=1}^{\infty}\left\{R_0 P_1^* \frac{d^2T_q(t)}{dt^2} - \frac{F_0}{\mu}P_2^* T_q(t)\right.$$
$$-\Gamma\left[2\left(\frac{P_3^*}{2} + \sum_{k=1}^{\infty}\cos\frac{k\pi s}{L_Y}\ P_3^{**}(k) + \sum_{j=1}^{\infty}\cos\frac{j\pi ct}{L_X}P_3^{***}(j)\right.\right.$$
$$\left.+2\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\cos\frac{j\pi ct}{L_X}\cos\frac{k\pi s}{L_Y}P_3^{****}(j,k)\right)\frac{d^2T_q(t)}{dt^2} + 4c\left(\frac{P_4^*}{2} + \sum_{k=1}^{\infty}\cos\frac{k\pi s}{L_Y}P_4^{**}(k)\right.$$
$$\left.+\sum_{j=1}^{\infty}\cos\frac{j\pi ct}{L_X}P_4^{***}(j) + 2\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\cos\frac{j\pi ct}{L_X}\cos\frac{k\pi s}{L_Y}P_4^{****}(j,k)\right)\frac{dT_q(t)}{dt}$$
$$+2c^2\left(\frac{P_5^*}{2} + \sum_{k=1}^{\infty}\cos\frac{k\pi s}{L_Y}P_5^{**}(k) + \sum_{j=1}^{\infty}\cos\frac{j\pi ct}{L_X}P_5^{***}(j)\right.$$
$$\left.\left.\left.+\ 2\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\cos\frac{j\pi ct}{L_X}\cos\frac{k\pi s}{L_Y}P_5^{****}(j,k)\right)T_q(t)\right]\right\} = \frac{Mg}{P^*\mu}\Psi_{pi}(ct)\Psi_{pj}(s) \tag{27}$$

where

$$\Gamma = \frac{M}{L_X L_Y \mu} \tag{28}$$

Equation (27) is now the fundamental equation of our problem when the rectangular plate has arbitrary end support conditions.

In what follows, we shall solve the equation (27) when the plate has simple supports at all its edges.

An elastic rectangular plate resting on a variable Winkler elastic foundation and having simple supports at all its edges has the boundary conditions given by

$$W(0,y,t) = 0, \qquad W(L_X,y,t) = 0 \tag{29a}$$

$$W(x,0,t) = 0, \qquad W(x,L_Y,t) = 0 \tag{29b}$$

$$\frac{\partial^2 W(0,y,t)}{\partial x^2} = 0, \qquad \frac{\partial^2 W(L_X,y,t)}{\partial x^2} = 0 \tag{30a}$$

$$\frac{\partial^2 W(x,0,t)}{\partial y^2} = 0, \qquad \frac{\partial^2 W(x,L_Y,t)}{\partial y^2} = 0 \tag{30b}$$

Hence for the normal modes

$$\Psi_{ni}(0) = 0, \qquad \Psi_{ni}(L_X) = 0 \tag{31a}$$

$$\Psi_{nj}(0) = 0, \qquad \Psi_{nj}(L_Y) = 0 \tag{31b}$$

$$\frac{\partial^2 \Psi_{ni}(0)}{\partial x^2} = 0, \qquad \frac{\partial^2 \Psi_{ni}(L_X)}{\partial x^2} = 0 \tag{32a}$$

$$\frac{\partial^2 \Psi_{nj}(0)}{\partial y^2} = 0, \qquad \frac{\partial^2 \Psi_{nj}(L_Y)}{\partial y^2} = 0 \tag{32b}$$

When use is made of equations (31a), (31b), (32a), (32b) and the initial conditions given by equation (9), it can be shown that

$$A_{ni} = 0, B_{ni} = 0, C_{ni} = 0, \text{ and } \Omega_{ni} = n_i\pi \tag{33}$$

$$A_{nj} = 0, B_{nj} = 0, C_{nj} = 0 \text{ and } \Omega_{nj} = n_j\pi \tag{34}$$

Similarly,

$$A_{pi} = 0, B_{pi} = 0, C_{pi} = 0, \text{ apd } \Omega_{pi} = p_i\pi \tag{35}$$

$$A_{pj} = 0, B_{pj} = 0, C_{pj} = 0 \text{ apd } \Omega_{pj} = p_j\pi \tag{36}$$

Thus, we substitute equations (33), (34), (35) and (36) into the transformed equation (27) to obtain the transformed equation for a rectangular plate, resting on a variable Winkler elastic foundation and having simple supports at all its edges. That is

$$\begin{aligned}
&\frac{d^2T_n(t)}{dt^2} + \alpha_n^2 T_n(t) - \frac{1}{P^*}\sum_{q=1}^{\infty}\Bigg\{-R_0\left[\left(\frac{q\pi}{L_X}\right)^2 + \left(\frac{q\pi}{L_Y}\right)^2\right] I_{1a}(x)I_{1a}(y)\frac{d^2T_q(t)}{dt^2} \\
&-\frac{F_0F^*}{\mu}I_{1a}(y)T_q(t) - \frac{M}{L_XL_Y\mu}\Bigg[\Bigg(I_{1a}(x)I_{1a}(y) + 2\sum_{k=1}^{\infty}\cos\frac{k\pi s}{L_Y}I_{1a}(x)I_{1a}^k(y) \\
&+2\sum_{j=1}^{\infty}\cos\frac{j\pi ct}{L_X}I_{1a}^j(x)I_{1a}(y) + 4\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\cos\frac{j\pi ct}{L_X}\cos\frac{k\pi s}{L_Y}I_{1a}^j(x)I_{1a}^k(y)\Bigg)\frac{d^2T_q(t)}{dt^2} \\
&2c\Bigg(\frac{q\pi}{L_X}I_{2b}(x)I_{1a}(y) + 2\sum_{k=1}^{\infty}\frac{q\pi}{L_X}\cos\frac{k\pi s}{L_Y}I_{2b}(x)I_{1a}^k(y) + 2\sum_{j=1}^{\infty}\frac{q\pi}{L_X}\cos\frac{j\pi ct}{L_X}I_{2b}^j(x)I_{1a}(y) \\
&+4\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{q\pi}{L_X}\cos\frac{j\pi ct}{L_X}\cos\frac{k\pi s}{L_Y}I_{2b}^j(x)I_{1a}^k(y)\Bigg)\frac{dT_q(t)}{dt} - c^2\Bigg(\left(\frac{q\pi}{L_X}\right)^2 I_{1a}(x)I_{1a}(y) \\
&+2\sum_{k=1}^{\infty}\left(\frac{q\pi}{L_X}\right)^2\cos\frac{k\pi s}{L_Y}I_{1a}(x)I_{1a}^k(y) + 2\sum_{j=1}^{\infty}\left(\frac{q\pi}{L_X}\right)^2\cos\frac{j\pi ct}{L_X}I_{1a}^j(x)I_{1a}(y) \\
&+4\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\left(\frac{q\pi}{L_X}\right)^2\cos\frac{j\pi ct}{L_X}\cos\frac{k\pi s}{L_Y}I_{1a}^j(x)I_{1a}^k(y)\Bigg)T_q(t)\Bigg]\Bigg\} = \frac{Mg}{\mu P^*}\sin\frac{p_j\pi s}{L_Y}\sin\frac{p_i\pi ct}{L_X}
\end{aligned} \tag{37}$$

where

$$
\begin{aligned}
F^* &= 4I_{1a}^*(x) - 3I_{1a}^{**}(x) + I_{1a}^{***}(x),\\
I_{1a}^*(x) &= \int_0^{L_X} x \sin\theta_{ni}x \sin\theta_{pi}x\,dx,\\
I_{1a}^{**}(x) &= \int_0^{L_X} x^2 \sin\theta_{ni}x \sin\theta_{pi}x\,dx,\\
I_{1a}^{***}(x) &= \int_0^{L_X} x^3 \sin\theta_{ni}x \sin\theta_{pi}x\,dx,\\
I_{1a}(x) &= \int_0^{L_X} \sin\theta_{ni}x \sin\theta_{pi}x\,dx,\\
I_{1a}(y) &= \int_0^{L_Y} \sin\theta_{nj}y \sin\theta_{pj}y\,dy,\\
I_{1a}^j(x) &= \int_0^{L_X} \cos\frac{j\pi x}{L_X} \sin\theta_{ni}x \sin\theta_{pi}x\,dx,\\
I_{1a}^k(y) &= \int_0^{L_Y} \cos\frac{k\pi y}{L_Y} \sin\theta_{nj}y \sin\theta_{pj}y\,dy,\\
I_{2b}(x) &= \int_0^{L_X} \cos\theta_{ni}x \sin\theta_{pi}x\,dx,\\
I_{2b}^j(x) &= \int_0^{L_X} \cos\frac{j\pi x}{L_X} \cos\theta_{ni}x \sin\theta_{pi}x\,dx,\\
\theta_{ni} = \frac{\Omega_{ni}}{L_X},\ \ \theta_{nj} &= \frac{\Omega_{nj}}{L_Y},\ \ \theta_{pi} = \frac{\Omega_{pi}}{L_X}\ \text{ and }\ \theta_{pj} = \frac{\Omega_{pj}}{L_Y}
\end{aligned}
\tag{38}
$$

Further simplification and rearrangement of (37), taking into account (33), (34), (35) and (36), yields

$$
\begin{aligned}
&\frac{d^2T_n(t)}{dt^2} + \alpha_n^2 T_n(t) - \frac{1}{P^*}\sum_{q=1}^{\infty}\Bigg\{\frac{-R_0L_XL_Y\pi^2}{4}\left(\frac{q^2}{L_X^2}+\frac{q^2}{L_Y^2}\right)\frac{d^2T_q(t)}{dt^2}\\
&-\frac{F_0F^*L_Y}{2\mu}T_q(t) - \Gamma\Bigg[\frac{L_XL_Y}{4}\frac{d^2T_q(t)}{dt^2} + 2cL_Y\left(\frac{qp_i}{p_i^2-q^2}+\sum_{j=1}^{\infty}\frac{q\pi}{L_X}\tau(j)\cos\frac{j\pi ct}{L_X}\right)\frac{dT_q(t)}{dt}\\
&-\frac{(cq\pi)^2L_Y}{4L_X}T_q(t)\Bigg]\Bigg\} = \frac{Mg}{P^*\mu}\sin\frac{p_j\pi s}{L_Y}\sin\frac{p_i\pi ct}{L_X}
\end{aligned}
\tag{39}
$$

where

$$
\Gamma = \frac{M}{L_XL_Y\mu}
\tag{40}
$$

and

$$
\tau(j) = \frac{8p_i[p_i^2-j^2-q^2]}{j^4+q^4+p_i^4-2[j^2p_i^2+j^2q^2+p_i^2q^2]}
\tag{41}
$$

Equation (39) is now the fundamental equation of our problem when the rectangular plate resting on variable Winkler foundation has simple support at all its edges. In what follows, we shall discuss two cases of the equation.

Case I: Simply supported plate traversed by moving force

An approximate model of the system, when the inertia effect of the moving mass M is neglected, that is, when $\Gamma = 0$ in equation (39), is the moving force problem associated with the system. Thus the differential equation is given by

$$\frac{d^2T_n(t)}{dt^2} + \alpha_n^2 T_n(t) - \frac{1}{P^*}\sum_{q=1}^{\infty}\left\{\frac{-R_0 L_X L_Y \pi^2}{4}\left(\frac{q^2}{L_X^2} + \frac{q^2}{L_Y^2}\right)\frac{d^2T_q(t)}{dt^2} - \frac{F_0 F^* L_Y}{2\mu}T_q(t)\right\} = \frac{Mg}{P^*\mu}\sin\frac{p_j\pi s}{L_Y}\sin\frac{p_i\pi ct}{L_X} \tag{42}$$

To solve equation (42) using the Struble's asymptotic technique [16], first, we neglect the rotatory inertial term and rearrange the equation to take the form

$$\frac{d^2T_n(t)}{dt^2} + \left[\alpha_n^2 + \Gamma^*\frac{F^*L_Y}{2}\right]T_n(t) + \Gamma^*\frac{F^*L_Y}{2}\sum_{\substack{q=1\\ q\neq n}}^{\infty} T_q(t) = K_0\sin\frac{p_j\pi s}{L_Y}\sin\frac{p_i\pi ct}{L_X} \tag{43}$$

where

$$\Gamma^* = \frac{F_0}{P^*\mu} \ \ textand \ \ K_0 = \frac{Mg}{P^*\mu} \tag{44}$$

Consider a parameter $\lambda < 1$ for any arbitrary mass ratio Γ^*, defined as

$$\lambda = \frac{\Gamma^*}{1+\Gamma^*} \tag{45}$$

It can be shown that

$$\Gamma^* = \lambda + o(\lambda^2) \tag{46}$$

Thus, the homogeneous part of equation (43) can be replaced with

$$\frac{d^2T_n(t)}{dt^2} + \gamma_m^2 T_n(t) = 0 \tag{47}$$

where

$$\gamma_m = \alpha_n + \frac{\lambda F^* L_Y}{4\alpha_n} \tag{48}$$

represents the modified frequency due to the effect of foundation.

Thus using (47), equation (42) can be written as

$$\frac{d^2T_n(t)}{dt^2} + \gamma_m^2 T_n(t) + \frac{\lambda_0 L_X L_Y \pi^2}{4}\sum_{q=1}^{\infty}\left(\frac{q^2}{L_X^2} + \frac{q^2}{L_Y^2}\right)\frac{d^2T_q(t)}{dt^2} = K_0 \sin\frac{p_j \pi s}{L_Y}\sin\frac{p_i \pi ct}{L_X} \tag{49}$$

where

$$\lambda_0 = \frac{R_0}{P^*} \tag{50}$$

The homogeneous part of equation (49) can be written as

$$\frac{d^2T_n(t)}{dt^2} + \frac{\gamma_m^2}{1 + \frac{\lambda_0 L_X L_Y \pi^2}{4}\left(\frac{n_i^2}{L_X^2} + \frac{n_j^2}{L_Y^2}\right)} T_n(t)$$
$$+ \frac{\frac{\lambda_0 L_X L_Y \pi^2}{4}}{1 + \frac{\lambda_0 L_X L_Y \pi^2}{4}\left(\frac{n_i^2}{L_X^2} + \frac{n_j^2}{L_Y^2}\right)} \sum_{\substack{q=1 \\ q\neq n}}^{\infty}\left(\frac{q^2}{L_X^2} + \frac{q^2}{L_Y^2}\right)\frac{d^2T_q(t)}{dt^2} = 0 \tag{51}$$

Consider the parameter $\epsilon_0 < 1$ for any arbitrary mass ratio λ_0 defined as

$$\varepsilon_0 = \frac{\lambda_0}{1 + \lambda_0} \tag{52}$$

which implies

$$\lambda_0 = \varepsilon_0 + o(\varepsilon_0^2) \tag{53}$$

Following the same argument, (51) can be replaced with

$$\frac{d^2T_n(t)}{dt^2} + \gamma_{mf}^2 T_n(t) = 0 \tag{54}$$

where

$$\gamma_{mf} = \gamma_m\left[1 - \frac{\varepsilon_0 L_X L_Y \pi^2}{8}\left(\frac{n_i^2}{L_X^2} + \frac{n_j^2}{L_Y^2}\right)\right] \tag{55}$$

Therefore, the moving force problem (42) for the simply supported rectangular plate is reduced to the non-homogeneous ordinary differential equation given as

$$\frac{d^2T_n(t)}{dt^2} + \gamma_{mf}^2 T_n(t) = K_0 \sin\frac{p_j \pi s}{L_Y}\sin\frac{p_i \pi ct}{L_X} \tag{56}$$

When equation (56) is solved in conjunction with the initial conditions, one obtains expression for $T_n(t)$. Thus in view of equation (10), one obtains

$$W(x,y,t) = \sum_{ni=1}^{\infty}\sum_{nj=1}^{\infty}\frac{K_0 \sin\frac{p_j \pi s}{L_Y}}{\gamma_{mf}[\gamma_{mf}^2 - (p_i \pi c/L_X)^2]}\left[\gamma_{mf}\sin\frac{p_i \pi ct}{L_X} - \frac{p_i \pi c}{L_X}\sin\gamma_{mf} t\right]\sin\frac{n_i \pi x}{L_X}\sin\frac{n_j \pi y}{L_Y} \tag{57}$$

as the transverse-displacement response to a moving force of a simply supported rectangular plate on a variable Winkler elastic foundation.

Case II: Simply supported rectangular plate resting on variable foundation and traversed by a moving mass

In this section we seek the solution to the entire equation (39) when no term of the equation is neglected. To solve this problem, we use the modified asymptotic method of Struble already alluded to [16]. To this end, we rearrange equation (39) to take the form

$$\frac{d^2T_n(t)}{dt^2} - \frac{2cL_Y\eta_0\left(\frac{n_ip_i}{p_i^2-n_i^2}+\sum_{j=1}^{\infty}\frac{n_i\pi}{L_X}\tau(j)\cos\frac{j\pi ct}{L_X}\right)}{1-\eta_0\left(\frac{L_XL_Y}{4}\right)}\frac{dT_n(t)}{dt} + \frac{G_f^2+\frac{\eta_0(cn_i\pi)^2L_Y}{4L_X}}{1-\eta_0\left(\frac{L_XL_Y}{4}\right)}T_n(t)$$
$$-\frac{\eta_0}{1-\eta_0\left(\frac{L_XL_Y}{4}\right)}\sum_{\substack{q=1\\ q\neq n}}^{\infty}\left[\frac{L_XL_Y}{4}\frac{d^2T_q(t)}{dt^2}+2cL_Y\left(\frac{qp_i}{p_i^2-q^2}+\sum_{j=1}^{\infty}\frac{q\pi}{L_X}\tau(j)\cos\frac{j\pi ct}{L_X}\right)\frac{dT_q(t)}{dt}\right.$$
$$\left.-\frac{(cq\pi)^2L_Y}{4L_X}T_q(t)\right] = \frac{\eta_0 gL_XL_Y}{P^*\left[1-\eta_0\left(\frac{L_XL_Y}{4}\right)\right]}\sin\frac{p_j\pi s}{L_Y}\sin\frac{p_i\pi ct}{L_X} \tag{58}$$

where Γ has been written as a function of the mass ratio η_0.

Thus, considering the homogeneous part of the equation (58) and going through the same arguments and analysis as the previous case, the modified frequency corresponding to the frequency of the free system due to the presence of the moving mass is

$$\beta_f = \gamma_{mf}\left[1-\frac{\eta_0}{2}\left(1+\frac{(cn_i\pi)^2}{\gamma_{mf}^2L_X^2}\right)\right] \tag{59}$$

retaining terms to o(η_0) only.

Therefore, to solve the non-homogeneous equation (58), the differential operator which acts on $T_n(t)$ and $T_q(t)$ is replaced by the equivalent free system operator defined by the modified frequency β_f. That is

$$\frac{d^2T_n(t)}{dt^2}+\beta_f^2T_n(t) = G_0\sin\frac{n_j\pi s}{L_Y}\sin\frac{n_i\pi ct}{L_X} \tag{60}$$

where

$$G_0 = \frac{\eta_0 gL_XL_Y}{P^*} \tag{61}$$

Clearly, equation (60) is directly analogous to equation (56). Hence when equation (60) is solved in conjunction with the initial conditions, one obtains expression for $T_n(t)$. Thus in view of equation (10), we have

$$W(x,y,t) = \sum_{ni=1}^{\infty}\sum_{nj=1}^{\infty}\frac{G_0\sin\frac{p_j\pi s}{L_Y}}{\beta_f[\beta_f^2-(p_i\pi c/L_X)^2]}\left[\beta_f\sin\frac{p_i\pi ct}{L_X}-\frac{p_i\pi c}{L_X}\sin\beta_f t\right]\sin\frac{n_i\pi x}{L_X}\sin\frac{n_j\pi y}{L_Y} \tag{62}$$

Equation (62) represents the transverse-displacement response to a moving mass of a simply supported rectangular plate on a variable Winkler elastic foundation.

4 DISCUSSION OF THE ANALYTICAL SOLUTIONS

In studying undamped system such as this, it is desirable to examine the phenomenon of resonance. Equation (57) clearly shows that the simply supported rectangular plate on a variable Winkler elastic foundation and traversed by a moving force reaches a state of resonance whenever

$$\gamma_{mf} = \frac{p_i \pi c}{L_X} \tag{63}$$

while equation (62) shows that the same plate under the action of a moving mass experiences resonance when

$$\beta_f = \frac{p_i \pi c}{L_X} \tag{64}$$

where

$$\beta_f = \gamma_{mf}\left[1 - \frac{\eta_0}{2}\left(1 + \frac{(cn_i\pi)^2}{\gamma_{mf}^2 L_X^2}\right)\right] \tag{65}$$

Equations (64) and (65) imply that

$$\gamma_{mf}\left[1 - \frac{\eta_0}{2}\left(1 + \frac{(cn_i\pi)^2}{G_f^2 L_X^2}\right)\right] = \frac{p_i \pi c}{L_X} \tag{66}$$

Since $\left[1 - \frac{\eta_0}{2}\left(1 + \frac{(cn_i\pi)^2}{G_f^2 L_X^2}\right)\right] < 1$ for all n_i, it can be deduced from equation (66) that, for the same natural frequency, the critical speed (and the natural frequency) for the system of a simply supported rectangular plate traversed by a moving mass is smaller than that of the same system traversed by a moving force. Thus, for the same natural frequency of the plate, resonance is reached earlier when we consider the moving mass system than when we consider the moving force system.

5 NUMERICAL CALCULATIONS AND DISCUSSION OF RESULTS

In order to carry out the calculations of practical interests in dynamics of structures and engineering design for the simply supported plate resting on variable Winkler elastic foundation, a rectangular plate of length $L_Y = 0.914$m and breadth $L_X = 0.457$m is considered. It is assumed that the mass travels at the constant velocity 0.8123m/s. Furthermore, values for E, S and Γ are chosen to be 2.109×10^9kg/m^2, 0.4m and 0.2 respectively. For various values of the foundation modulus F_0 and the rotatory inertia correction factor R_0, the deflections of the simply supported plate are calculated and plotted against time t.

Figures 1 and 2 display the effect of foundation modulus (F_0) on the transverse deflection of the simply supported rectangular plate in both cases of moving force and moving mass respectively. The graphs show that the response amplitude decreases as the value of the foundation modulus increases. Values of F_0 between 0 N/m^3 and 3000 N/m^3 are used.

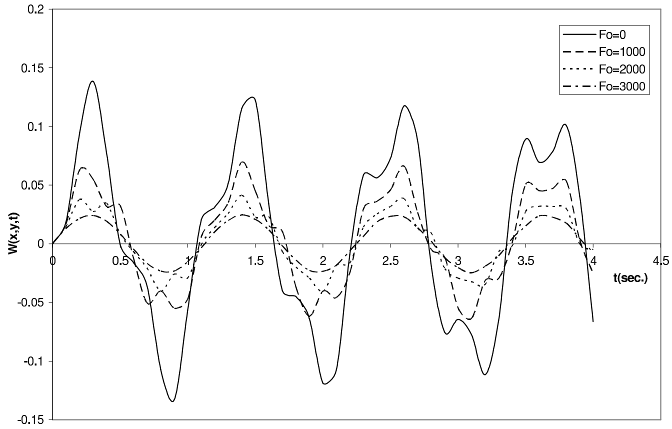

Figure 1 Deflection of moving force for simply supported rectangular plate on variable foundation for various values of foundation moduli F_0.

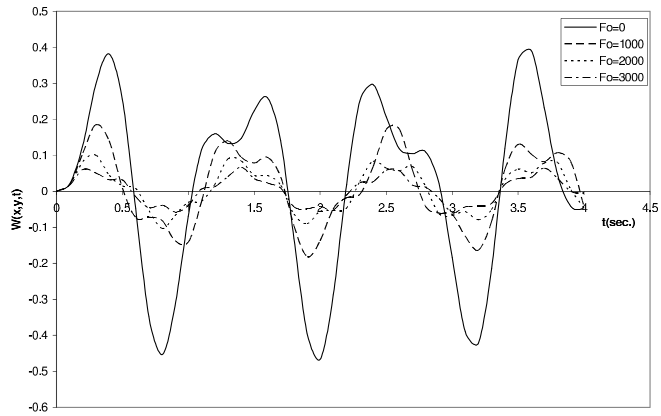

Figure 2 Deflection profile of moving mass for simply supported rectangular plate on variable foundation for various values of foundation moduli F_0.

The effect of rotatory inertia correction factor (R_0) on the transverse deflection in both cases of moving force and moving mass displayed in figures 3 and 4 respectively show that an increase in the value of the rotatory inertia correction factor decreases the deflection of the simply supported rectangular plate resting on variable Winkler elastic foundation. Here, values of R_0 between 0m and 0.2m are used.

Figure 5 compares the displacement curves of the moving force and moving mass for a simply supported rectangular plate with $F_0 = 1000$ N/m^3 and $R_0 = 1$m. Clearly, the response amplitude of a moving mass is greater than that of a moving force problem. However, this result holds for other choices of the values of F_0 and R_0.

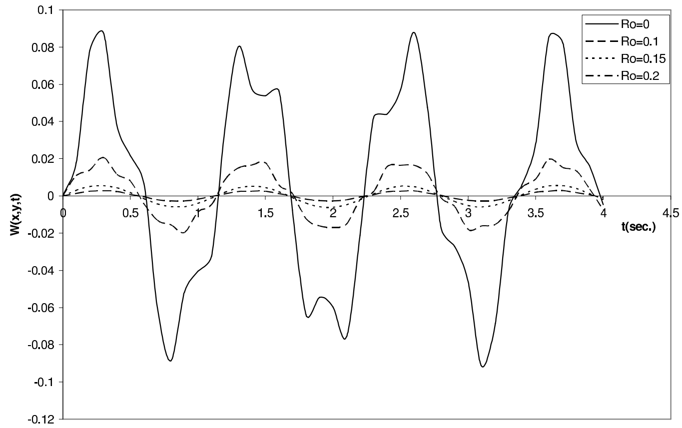

Figure 3 Deflection profile of moving force for simply supported rectangular plate on variable foundation for various values of rotatory inertia correction factor R_0.

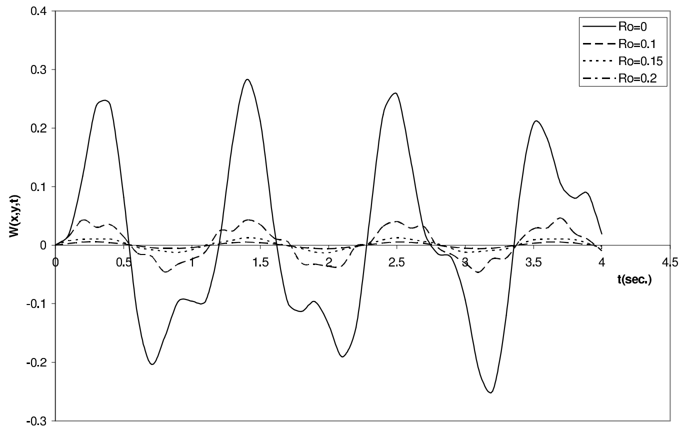

Figure 4 Deflection of moving mass for simply supported rectangular plate on variable foundation for various values of rotatory inertia correction factor R_0.

6 CONCLUSION

The problem of the dynamic behaviour under moving concentrated masses of rectangular plates resting on variable elastic foundation is considered in this work. The governing fourth order partial differential equation is a non-homogenous equation with variable and singular coefficients. The objective of the work has been to study the problem of the dynamic response to moving concentrated masses of rectangular plates on variable Winkler elastic foundations. In particular, the closed form solutions of the fourth order partial differential equations with variable and singular coefficients of the rectangular plate is obtained for both cases of moving force and moving mass. The method is based on (i) Separation of variables (ii) The modified Struble's technique and (iii) The method of integral transformations.

These solutions are analyzed and resonance conditions are obtained for the problem. The numerical analysis for both moving force and moving mass problems carried out show that

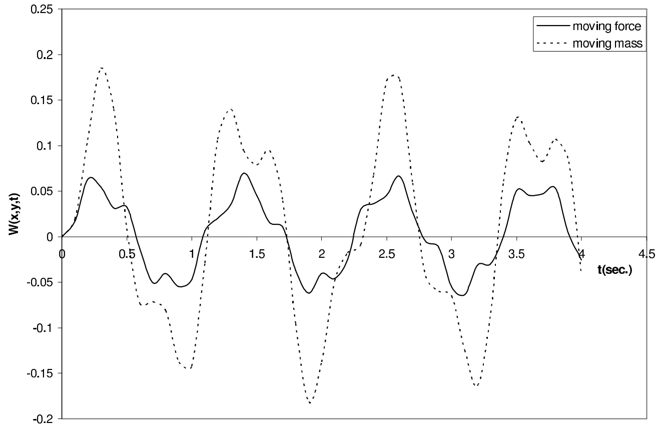

Figure 5 Comparison of the deflections of moving force and moving mass cases for simply supported rectangular plate on variable foundation with $F_0 = 1000$ and $R_0 = 1$.

the moving force solution is not an upper bound for the accurate solution of the moving mass solution and that as the rotatory inertia correction factor increases, the response amplitudes of the plates decrease for both cases of moving force and moving mass problem. When the rotatory inertia correction factor is fixed, the displacements of the simply supported rectangular plates resting on variable Winkler elastic foundations decrease as the foundation modulus increases.

Furthermore, for fixed values of rotatory inertia correction factor and foundation modulus, the response amplitude for the moving mass problem is greater than that of the moving force problem implying that resonance is reached earlier in moving mass problem than in moving force problem of the simply supported rectangular plate resting on variable elastic foundation. Hence, it is dangerous to rely on the moving force solutions.

Finally, for the simply supported rectangular plate resting on Winkler elastic foundation with stiffness variation, for the same natural frequency, the critical speed for moving mass problem is smaller than that of the moving force problem, and as rotatory inertia correction factor and the foundation modulus increase, the critical speeds increase showing that risk is reduced.

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