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Vibration analysis of laminated functionally graded shallow shells with clamped cutout of the complex form by the Ritz method and the R-functions theory

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Abstract
The R-functions theory and Ritz approach are applied for analysis of free vibrations of laminated functionally graded shallow shells with different types of curvatures and complex planforms. Shallow shells are considered as sandwich shells of different types: a) face sheets of the shallow shells are made of a functionally graded material (FGM) and their cores are made of an isotropic material; b) face sheets of the shallow shells are isotropic, but the core is made of FGM. It is assumed that FGM layers are made of a mixture of metal and ceramics and effective material properties of layers are varied accordingly to Voigt's rule. Formulation of the problem is carried out using the first-order (Timoshenko's type) refined theory of shallow shells. Different types of boundary conditions, including clamped, simply supported, free edge and their combinations, are studied. The proposed method and the created computer code have been examined on test problems for shallow shells with rectangular planforms. In order to demonstrate the possibility of the developed approach, novel results for laminated FGM shallow shells with cut of the complex form are presented. Effects of different material distributions, mechanical properties of the constituent materials, lamination scheme, boundary conditions and geometrical parameters on natural frequencies are shown and analyzed.

Keywords:
functionally graded shallow shells; linear and nonlinear free vibrations; R-functions theory; method by Ritz

1. Introduction

The laminated functionally graded shallow shells play an important role in numerous engineering applications. According to the pioneering works of Koizumi [1] and Yamanouchi et al. [2], the functionally graded materials (FGMs) can be considered as a new class of composite materials used extensively for manufacturing of shell structural elements. The main advantages of these materials in comparison with conventional composite materials are the smoothness and continuous change of material properties along the thickness of an object. This allows to remove the appearance of stress concentration that is found in laminated composites. In addition, the

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graduation of material properties along the shell thickness allows for fabrication of the laminated composite shells with tailored properties of the shells.

In recent years, an extensive research aimed at static/dynamic analysis of these shells, including interaction of the mechanical, thermal, and electric fields, has been observed in the field of analysis of FGM shells. Since it is impossible to review all papers focused on free vibrations of FGM shells and plates, we analyze papers that are closely related to our investigations.

Liew et al. [3] have provided solutions to the thermal stress behavior of a functionally graded circular hollow cylinder,r where the temperature distribution has been assumed in the radial direction. Pelletier and Vel [4] have proposed an exact solution to the problem of steady-state thermoelastic response of FGM orthotropic cylindrical shells.

Arciniega and Reddy [5] have carried out large deformation analysis of FGM shells. The tensor-based finite element formulation and the first-order shear deformation theory (seven parameters) have been employed to derive the FG shell finite element. The validity of the presented approach has been illustrated by a few numerical examples.

Zhao and Liew [6] have studied a nonlinear response of FG ceramic-metal shell panels under mechanical and thermal fields. The geometric nonlinearity has been introduced in the von Kármán form and the material properties have been assumed to vary through the shell thickness. The full load-displacement path has been traced by employment of the arc-length method combined with the modified Newton-Rophson technique. Effects of the volume fraction exponent boundary conditions and material properties versus nonlinear shell response have been illustrated.

Iqbal et al. [7] have considered dynamic characteristics of FG cylindrical shells based on the wave propagation approach. Tornabene [8] and Tornabene et al. [9] have carried out an analysis of vibrations of FG conical, cylindrical, and annular shell structures. Yang et al. [10] have studied vibrations of curved shell using B-spline wavelet combined with the finite elements method.

Neves et al. [11] have considered free vibration problems of FG shells by employing the radial basis functions collocation. The used approach has been validated by numerical results dealing with the cylindrical and spherical shells with clamped/simply supported edges. Ebrahimi and Najafizadeh [12] have studied free vibrations of a 2D functionally graded cylindrical shell. Governing PDEs and boundary conditions have been discretized using the generalized differential/integral quadrative method. The Voigt and Mori-Tanaka models have been used to describe the material properties, and the obtained results have been validated with the data available in the literature. Free vibration analysis of the FGM truncated conical shells, circular cylindrical shells, and annular plates has been investigated by Ersoy et al. [13]. Authors have applied the method of discrete singular convolution and the method differential quadrature to solve problems in frame of higher-order shear deformation theory. Many researchers have been studying free vibrations of composite shell structures reinforced by carbon nanotubes (CNTs). One of the last papers devoted to this topic is the paper by Zgnarl et.al [14]. Authors of this paper have considered linear free vibration of the shells made of functionally graded carbon nanotube composites. The proposed refined model based on a discrete double directors shell element has been used.

The geometrically nonlinear analysis of functionally graded shells has been carried out by Daszkiewicz et al. [15] by employing the 6-parameter shell theory. The 2D Cosserat constitutive model yielded constitutive relation for the considered shells, and in particular, the influence of power-law exponent and micropolar material constants on the functionally graded shell properties have been investigated.

Mars et al. [16] have employed the geometrically nonlinear study of functionally graded shells by using Abaqus software. Static responses of several structural problems have been compared with reference solutions to validate the obtained results.

In general, various shell theories were developed for mathematical simulation of the shells made of functionally graded materials. Particularly, the classical theory (CST), the first-order refined theory (FSDT), and the higher-order shear deformation theory (HSDT) are the most commonly used for shallow shells. As it has been already mentioned, analyses of vibrations of the laminated and FGM shallow shells has been carried out by many investigator (see also [17-23]). Extensive literature reviews concerning the mentioned issues have been reported in references [24-29]. Recently, nonlinear free and forced vibrations of the FG shells have been extensively studied in addition to the linear vibrations [30-39]. Joint application of the FGM and pure metallic and ceramic is widely used for design of many elements of the modern constructions. However, the number of publications devoted to the study of multilayered FGM shallow shells is relatively small [40-43].

Especially, this applies to shells with a cutout, a complicated shape of the plan, and various kinds of boundary conditions. To study FGM shells with free cutouts, many researchers use the Ritz method. However, there are practically no papers in which multilayer FGM shells with fixed cutouts have been investigated, despite the fact that such objects occur quite often in practice. From our point of view, this is due to the fact that it is difficult to construct a system of coordinate functions satisfying the main boundary conditions without applying the R-functions theory. In addition, problems of graded shallow shells with complex shapes, different cutouts, holes, etc., and various boundary conditions have been rarely studied in the available literature.

The main aim of this paper is to present efficient and enough universal approach, which has been developed for laminated FG shallow shells and is based on the joint application of the R-functions theory and variational Ritz method. Formulation of the problem is carried out using the first-order refined shallow shells theory (FSDT).

So far, this approach has been used for multilayer shells and plates or structural members made only from FGM [44-47]. In this paper, this method is extended to multilayer shells, provided that some of the layers are made of FGM. Namely, this approach is applied to three-layer shallow shells like sandwich ones. Two types of lamination schemes are considered. Type 1-2 corresponds to sandwich shallow shells with FGM face sheets and an isotropic core. Type 2-2 describes sandwich shallow shells with isotropic face sheets (pure ceramics or metal), and a core made of FGM. It is assumed that FGM layers are made of a mixture of metal and ceramics and that effective material properties of layers are varied according to Voigt's rule. Analytical expressions for the mechanical characteristics of the shell are presented for different locations of isotropic and FGM layers obtained after integration over the total thickness of the shell.

The proposed method is validated by investigation of test problems for shallow shells with rectangular planforms and different boundary conditions. The current method is applied to novel vibration problems for double curved shallow shells with a complex form of the cutout.

It should be noticed that joint application of the R-functions theory and variational Ritz method yields relatively fast and reliable results even in the case of complex shapes of the graded shallow shells, which, on contrary to the widely used finite element method, allows one to control vibrations of the studied shells. One of the main advantages of the proposed approach is the presentation of the solution in an analytic form, which is an important factor in studying of nonlinear vibrations of the shells under consideration.

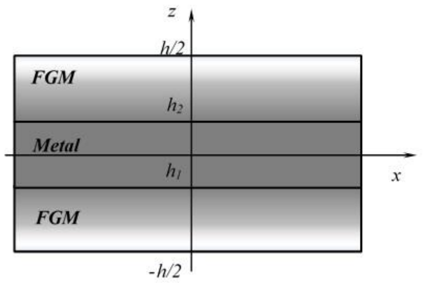
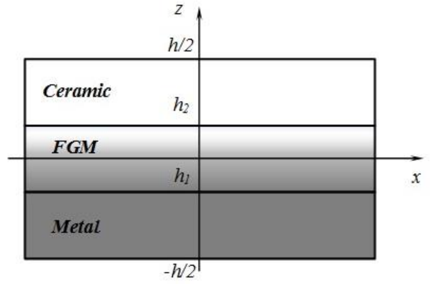
2. Mathematical formulation

Consider a three-layer functionally graded shallow shell with a uniform thickness *h.* It is assumed that the FGM layers are made of a mixture of ceramics and metals. A double curved shallow shell can have an arbitrary planform. The effective material properties of layers vary continuously and smoothly in thickness direction and may be estimated by the following Voigt's law:

$$E^{(r)}=\left(E_u^{(r)}-E_l^{(r)}\right)V_c^{(r)}+E_l^{(r)},\quad \nu^{(r)}=\left(\nu_u^{(r)}-\nu_l^{(r)}\right)V_c^{(r)}+\nu_l^{(r)},\quad \rho^{(r)}=\left(\rho_u^{(r)}-\rho_l^{(r)}\right)V_c^{(r)}+\rho_l^{(r)} \tag{1}$$

where $E_u^{(r)},\nu_u^{(r)},\rho_u^{(r)}$ and $E_l^{(r)},\nu_l^{(r)},\rho_l^{(r)}$ are Young's modulus, Poisson's ratio and mass density of the upper and lower surfaces of the r -layer, respectively; $V_c^{(r)}$ is the volume fraction of ceramic. As an example, the value $V_c^{(r)}$ is reported for the scheme lamination of types 1-2 and 2-2 in Table 1.

Table 1. Value of volume fraction $V_c^{(r)}$ for two types of laminated FGM shallow shells

Type 1-2	Type 2-2
	
$\begin{cases} V_c^{(1)} = \left(\dfrac{z - h_1}{-h_1 - h/2} \right)^{p_1}, & z \in [-h/2, h_1], \\ V_c^{(2)} = 0, & z \in [h_1, h_2], \\ V_c^{(3)} = \left(\dfrac{z - h_2}{h/2 - h} \right)^{p_3}, & z \in [h_2, h/2] \end{cases}$	$\begin{cases} V_c^{(1)} = 0, & z \in [-h/2, h_1], \\ V_c^{(2)} = \left(\dfrac{z - h_2}{h/2 - h} \right)^{p_2}, & z \in [h_1, h_2], \\ V_c^{(3)} = 1, & z \in [h_2, h/2] \end{cases}$

Shallow shells of type 1-2 correspond to sandwich shallow shells with FGM face sheets and isotropic (metal) core (Table 1). The shells of the type 2-2 correspond to sandwich shallow shells with FGM core and ceramics on the top face sheet and metal on the bottom face sheet (Table 1).

It should be emphasized that the values $p_1,\ p_2,\ p_3$ are the power law FGM exponents of the corresponding layer. The thickness of the layers may be varied. The ratio of thickness of layers from bottom to top is denoted by the combination of three numbers. For example, “1-2-1” denotes that ratio of thickness of the layers is defined as $h^{(1)} : h^{(2)} : h^{(3)} = 1:2:1,$ where: $h^{(1)} = h_1 + h/2,\ h^{(2)} = h_2 - h_1,\ h^{(3)} = h/2 - h_2$ (see Table 1).

Owing to the first-order shear deformation theory for the shallow shell (FSDT), the displacements components u_1, u_2, u_3 at a point (x, y, z) are expressed as functions of the middle surface displacements u, v and w in the Ox, Oy and Oz directions and the independent rotations ψ_x, ψ_y of the transverse normal to the middle surface about the Oy and Ox axes, respectively [3-6], i.e. we have

$$u_1 = u + z\psi_x, \quad u_2 = v + z\psi_y, \quad u_3 = w \tag{2}$$

The paper is organized in the following way. The mathematical formulation of the considered problem is given in Section 2. The method of solution is presented in Section 3, whereas Section 4 contains the numerical results. The last Section 5 concludes the carried out research.

Strain components $\varepsilon = \{\varepsilon_{11};\ \varepsilon_{22};\ \varepsilon_{12}\}^T$, $\chi = \{\chi_{11};\ \chi_{22};\ \chi_{12}\}^T$ at an arbitrary point of the shallow shell are as follows:

$$\varepsilon_{11} = u,_x + w/R_x \quad \varepsilon_{22} = v,_y + w/R_y \quad \varepsilon_{12} = u,_y + v,_x\,, \tag{3}$$

$$\varepsilon_{13} = w,_x + \psi_x, \quad \varepsilon_{23} = w,_y + \psi_y\,, \chi_{11} = \psi_x,_x\,, \quad \chi_{22} = \psi_y,_y \quad \chi_{12} = \psi_x,_y + \psi_y,_x$$

In-plane force resultant vector $N = (N_{11}, N_{22}, N_{12})^T$, bending and twisting moments resultant vector $M = (M_{11}, M_{22}, M_{12})^T$ and transverse shear force resultant $Q = (Q_x, Q_y)^T$ are calculated by integration along Oz-axes. They are defined by the following relation

$$\begin{bmatrix} N_{11} \\ N_{22} \\ N_{12} \\ M_{11} \\ M_{22} \\ M_{12} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 & B_{11} & B_{12} & 0 \\ A_{21} & A_{22} & 0 & B_{21} & B_{22} & 0 \\ 0 & 0 & A_{66} & 0 & 0 & B_{66} \\ B_{11} & B_{12} & 0 & D_{11} & D_{12} & 0 \\ B_{21} & B_{22} & 0 & D_{21} & D_{22} & 0 \\ 0 & 0 & B_{66} & 0 & 0 & D_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{12} \\ \chi_{11} \\ \chi_{22} \\ \chi_{12} \end{bmatrix}. \tag{4}$$

The elements A_{ij}, B_{ij}, D_{ij} of the matrix (4) have the following explicit forms

$$A_{ij} = \sum_{r=1}^{3} \int_{z_r}^{z_{r+1}} Q_{ij}^{(r)} dz, \; B_{ij} = \sum_{r=1}^{3} \int_{z_r}^{z_{r+1}} Q_{ij}^{(r)} z dz \quad D_{ij} = \sum_{r=1}^{3} \int_{z_r}^{z_{r+1}} Q_{ij}^{(r)} z^2 dz \tag{5}$$

The values $Q_{ij}^{(r)} \left(i, j = 1, 2, 6\right)$ are defined by the following expressions

$$Q_{11}^{(r)} = Q_{22}^{(r)} = \frac{E^{(r)}}{1-\left(\nu^{(r)}\right)^2}, \; Q_{12}^{(r)} = \frac{\nu^{(r)} E^{(r)}}{1-\left(\nu^{(r)}\right)^2}, \; Q_{66}^{(r)} = \frac{E^{(r)}}{2\left(1+\nu^{(r)}\right)},$$

and the transverse shear force resultants Q_x, Q_y have the following form

$$Q_x = K_s^2 A_{33} \varepsilon_{13}, \quad Q_y = K_s^2 A_{33} \varepsilon_{23}, \tag{6}$$

where K_s^2 denotes the shear correction factor. In this paper, we take $K_s^2 = 5/6$.

Further we will consider materials with Poisson's ratio independent of temperature and the same for both ceramics and metal, i.e. $\nu_m = \nu_c$. This assumption allows to compute the coefficients A_{ij}, B_{ij}, D_{ij}. Analytical expressions of these coefficients for shells of Types 1-2 and 2-2 are presented below provided that the following notation is employed:

$$as1 = \left(\frac{h}{2} + h_1\right), \; as2 = h_2 - \frac{h}{2}, \quad bs1 = \frac{1}{2as1}, \quad bs2 = \frac{1}{2as2}, \quad E_{cm} = E_c - E_m. \tag{7}$$

Type 1-2:

$$A_{11} = \frac{1}{1-\nu^2}\left(E_{cm}\left(\frac{as1}{p_1+1} - \frac{as2}{p_3+1} + \right) + E_m h\right), \tag{8}$$

$$\mathbf{B}_{11} = \frac{E_{cm}}{1-\nu^2}\left(as1\left(\frac{h_1}{p_1+1} - \frac{as1}{p_1+2}\right) - as2\left(\frac{h_2}{p_3+1} - \frac{as2}{p_3+2}\right)\right), \tag{9}$$

$$D_{11} = \frac{1}{1-\nu^2}\left(E_{cm}\left(as1\left(\frac{h_1^2}{p_1+1} - \frac{2as1}{p_1+2}h_1 + \frac{as1^2}{p_1+3}\right) - as2\left(\frac{as2^2}{p_3+3} - 2h_2\frac{as2}{p_3+2}h + \frac{h_2^2}{p_3+1}\right)\right) + \frac{E_m}{12}h^3\right) \tag{10}$$

Type 2-2:

$$A_{11}=\frac{1}{1-\nu^{2}}\left(E_{cm}\left(\frac{h_2-h_1}{p_2+1}-h_2\right)+\frac{h}{2}\left(E_c+E_m\right)\right), \tag{11}$$

$$\mathbf{B}_{11}=\frac{E_{cm}}{1-\nu^{2}}\left(\frac{as1}{p_2+2}\left(h_2+\frac{h_1}{p_2+1}\right)+\frac{1}{2}\left(\frac{h^2}{4}-h_2^2\right)\right), \tag{12}$$

$$D_{11}=\frac{1}{1-\nu^{2}}\left(E_{cm}\left(as1\left(\frac{h_1^2}{p_2+1}+\frac{2as1}{p_2+2}h_1+\frac{as1^2}{p_2+3}\right)-\frac{h_2^3}{3}\right)+\frac{\left(E_m+E_c\right)}{24}h^3\right). \tag{13}$$

Notice that values $A_{12}, A_{66}, .B_{12}, B_{66}, D_{12}, D_{66}$ for all types of the lamination schemes are defined in the following way:

$$R_{12}=\nu R_{11},\quad R_{22}=R_{11},\quad R_{66}=\frac{1-\nu}{2}R_{11}, \tag{14}$$

where symbol *R* is common for letters *A,B*, and *D*.

3. Method of solution

In order to solve the free vibration problem, let us present a vector of unknown functions as

$$\begin{aligned}&\vec{U}(\bar{u}(x,y,t),\bar{v}(x,y,t),\bar{w}(x,y,t),\overline{\psi_x}(x,y,t),\overline{\psi_y}(x,y,t))=\\&=\vec{U}(u(x,y),v(x,y),w(x,y),\psi_x(x,y),\psi_y(x,y))\sin\lambda t,\end{aligned} \tag{15}$$

where λ is the vibration frequency. Applying the Ostrogradskiy-Hamilton principle, we get the variational equation in the form

$\partial I=0$, (16)

where

$$I=U(u,v,w,\psi_x,\psi_y)-\lambda^2 T\left(u,v,w,\psi_x,\psi_y\right).$$

Strain U and kinetic energy T are defined by the following relations:

$$U=\frac{1}{2}\iint_{\Omega}\left(N_{11}\varepsilon_{11}+N_{22}\varepsilon_{22}+N_{12}\varepsilon_{12}+M_{11}\chi_{11}+M_{22}\chi_{22}+M_{12}\chi_{12}+Q_x\varepsilon_{13}+Q_y\varepsilon_{23}\right)dxdy, \tag{17}$$

$$T=\frac{1}{2}\iint_{\Omega}I_0\left(u^2+v^2+w^2\right)+2I_1\left(u\psi_x+v\psi_y\right)+I_2\left(\psi_x{}^2+\psi_y{}^2\right)dxdy, \tag{18}$$

where I_0, I_1, I_2 are defined as follows

$$\left(I_0, I_1, I_2\right)=\sum_{r=1}^{3}\int_{z_r}^{z_{r+1}}\left(\rho^{(r)}\right)\left(1, z, z^2\right)dz$$

and stands $\rho^{(r)}$ for mass density of r-th layer.

Below, analytical expressions for these integrals are presented provided that $\nu_m=\nu_c$.

Type 1-2:

$$I_0=\rho_{cm}\left(\frac{as1}{p_1+1}-\frac{as2}{p_3+1}+\right)+\rho_m h\frac{1}{1-\nu^2},\quad \rho_{cm}=\rho_c-\rho_m\,, \tag{19}$$

$$I_1=\rho_{cm}\left(as1\left(\frac{h_1}{p_1+1}-\frac{as1}{p_1+2}\right)-as2\left(\frac{h_2}{p_3+1}-\frac{as2}{p_3+2}\right)\right), \tag{20}$$

$$I_2=\left(\rho_{cm}\left(as1\left(\frac{h_1^2}{p_2+1}+\frac{2as1}{p_2+2}h_1+\frac{as1^2}{p_2+3}\right)-\frac{h_2^3}{3}\right)+\frac{\left(\rho_m+\rho_c\right)}{24}h^3\right); \tag{21}$$

Type 2-2:

$$I_0=\rho_{cm}\left(\frac{h_2-h_1}{p_2+1}-h_2\right)+\frac{h}{2}\left(\rho_c+\rho_m\right), \tag{22}$$

$$I_1=\rho_{cm}\left(\frac{as1}{p_2+2}\left(h_2+\frac{h_1}{p_2+1}\right)+\frac{1}{2}\left(\frac{h^2}{4}-h_2^2\right)\right), \tag{23}$$

$$I_2=\rho_{cm}\left(as1\left(\frac{h_1^2}{p_2+1}+\frac{2as1}{p_2+2}h_1+\frac{as1^2}{p_2+3}\right)-\frac{h_2^3}{3}\right)+\frac{\left(\rho_m+\rho_c\right)}{24}h^3. \tag{24}$$

Minimization of the functional (16) will be performed using Ritz's method. On the other hand, the necessary sequence of coordinate functions will be constructed by the R-functions theory [42].

4. Numerical results

In order to verify the accuracy of the results obtained by the proposed approach, abbreviated to RFM (R-functions method) [44-47], we consider the solution of several test problems. Solving presented problems is carried out by created software in framework of the computer system POLE-Rl [48]

4.1. Validation of the presented results

Task 1. Natural frequencies of the laminated FGM square shallow shells of Type 1-2 and 2-2 with various boundary conditions and geometrical parameters: $h/a=0.1$; $b/a=1$; $a/R_x=0.2$ are analyzed. The power-law exponent for each FGM layer is taken to be $p_1=p_2=p_3=p$. The material constituents M_1 and M_2 are assumed to be aluminum and alumina [18,19,21,43]. The material properties of the FG mixture used in the present study are shown in Table 2.

The boundary conditions are defined as follows:

CCCC – the shell is clamped on sides $x=\pm\frac{a}{2}$, $y=\pm\frac{b}{2}$;

SSSS – the shell is simply supported on sides $x=\pm\frac{a}{2}$, $y=\pm\frac{b}{2}$;

SFSF – the shell is free on sides $x=\pm\frac{a}{2}$ and simply-supported on sides $y=\pm\frac{b}{2}$;

SCSC – the shell is simply supported on sides $x=\pm\frac{a}{2}$ and clamped on sides $y=\pm\frac{b}{2}$.

Table 2. Material properties of the used FGMs shallow shells

Material	Properties		
	E(GPa)	ν	ρ $\left(kg/m^3\right)$
Aluminum (Al)	70	0.3	2707
Alumina (Al_2O_3)	380	0.3	3800

Table 3. Comparison of fundamental frequency parameter $\Omega_L^{(1)}=\lambda_1 a^2/h\sqrt{\rho_0/E_0}$ of cylindrical and spherical shallow shells with square planform and various boundary conditions (Type 1-2).

scheme	p	Methods	Cylindrical shell k_1=0.2, k_2=0				Spherical shell k_1=k_2=0.2			
			SFSF	SSSS	CCCC	SCSC	SFSF	SSSS	CCCC	SCSC
1-0-1	0.6	[Jin G.et al (2015)]	0.8843	1.8023	3.0433	2.4855	0.8924	1.8643	3.1027	2.5465
		RFM	0.8856	1.8070	3.0691	2.5032	0.8937	1.8689	3.1278	2.5636
	5	[Jin G.et al (2015)]	0.7185	1.4566	2.4252	1.9894	0.7237	1.4982	2.4657	2.0308
		RFM	0.7198	1.4613	2.4493	2.0061	0.7250	1.5028	2.4894	2.0471
	20	[Jin G.et al (2015)]	0.5681	1.1566	1.947	1.5919	0.5730	1.1948	1.9840	1.6296
		RFM	0.5689	1.1598	1.9644	1.6036	0.5739	1.1979	2.0007	1.6409
1-1-1	0.6	[Jin G.et al (2015)]	0.8656	1.7561	2.9305	2.4023	0.8722	1.8071	2.9807	2.4535
		RFM	0.8672	1.7617	2.9590	2.4220	0.8737	1.8131	3.0085	2.4726
	5	[Jin G.et al (2015)]	0.6635	1.3462	2.2461	1.8414	0.6685	1.3857	2.2845	1.8806
		RFM	0.6647	1.3505	1.2680	1.8565	0.6697	1.3899	2.3059	1.8953
	20	[Jin G.et al (2015)]	0.5369	1.0948	1.8506	1.5109	0.5419	1.1331	1.8871	1.5485
		RFM	0.5376	1.0976	1.8660	1.5215	0.5426	1.1358	1.9022	1.5587
1-2-1	0.6	[Jin G.et al (2015)]	0.8326	1.6862	2/8005	2.2990,	0.8384	1.7330	2.8462	2.3455
		RFM	0.8342	1.6919	2.8291	2.3189	0.8400	1.7385	2.8742	2.3649
	5	[Jin G.et al (2015)]	0.6274	1.2742	2.1318	1.7462	0.6323	1.3129	2.1693	1.7845
		RFM	0.6285	1.2781	2.1519	1.7601	0.6334	1.3167	2.3189	1.7980
	20	[Jin G.et al (2015)]	0.5195	1.0605	1.7969	1.4659	0.5246	1.0989	1.8335	1.5036
		RFM	0.5202	1.0631	1.8115	1.4759	0.5253	1.1014	1.8476	1.5132

Comparison of the results obtained by developed computer code which realizes proposed approach is carried out for double-curved shallow shells and is presented in Tables 3,4,5. Fundamental frequency parameters $\Omega_L^{(1)}=\lambda_1 a^2/h\sqrt{\rho_0/E_0}$ $(\rho_0=1\ \ kg/m^2, E_0=1\ \ GPa)$ of laminated FGM spherical and cylindrical panels of Type 1-2 and different thickness scheme are shown in Table 3.

The values of the fundamental linear frequency parameters $\Omega_L^{(1)}=\lambda_1 a^2/h\sqrt{\rho_0/E_0}$ of laminated FGM cylindrical and spherical shells of Types 2-2 for thickness scheme 1-2-1 are reported in Table 4.

Table 4. Comparison of fundamental frequency parameter $\Omega_L^{(1)} = \lambda_1 a^2 / h \sqrt{\rho_0 / E_0}$ of cylindrical and spherical shallow shells with square planform and various boundary conditions (thickness scheme 1-2-1).

Type of the shell	p	Methods	Cylindrical shell k_1=0.2, k_2=0				Spherical shell k_1=k_2=0.2			
			SFSF	SSSS	CCCC	SCSC	SFSF	SSSS	CCCC	SCSC
2-2	0.6	[Jin G.et al (2015)]	0.6459	1.3372	2.3283	1.8784	0.6571	1.4134	2.3989	1.9254
		RFM	0.6464	1.3389	2.3402	1.8864	0.6576	1.4150	2.4103	1.9599
	5	[Jin G.et al (2015)]	0.6042	1.2457	2.1569	1.7440	0.6133	1.3074	2.2147	1.8042
		RFM	0.6047	1.2476	2.1693	1.7523	0.6139	1.3091	2.2266	1.8121
	20	[Jin G.et al (2015)]	0.6072	1.2483	2.1511	1.7426	0.6154	1.3043	2.2039	1.7974
		RFM	0.6078	1.2505	2.1646	1.7517	0.6160	1.3061	2.2169	1.8061

Comparison of the obtained results for hyperbolic paraboloidal shallow shells of Type 1-2 and 2-2 ($k_1 = 0.2; \quad k_2 = -0.2$) with different ratio of thickness of each layer and for different boundary conditions are shown in Table 5.

These results were obtained using 28 admissible functions to approximate each of the functions u, v, ψ_x, ψ_y, and 36 admissible functions to approximate deflection *w*.

Due to the doubly-symmetric nature of the shell, at numerical implementation of the developed software, the integration is performed only above one-quarter domain. It can be observed that presented results are in excellent agreement with those reported in reference [43].

Comprehensive comparison of the obtained results with available ones presented in Tables 3-5, shows the accuracy and reliability of the proposed approach and developed software.

Table 5. Comparison of fundamental frequency parameter $\Omega_L^{(1)} = \lambda_1 a^2 / h \sqrt{\rho_0 / E_0}$ of hyperbolic paraboloidal shallow shells with square planform and various boundary conditions ($k_1 = 0.2; \quad k_2 = -0.2$).

Scheme	p	Methods	Type of the shell 1-2				Scheme	Type of the shell 2-2			
			SFSF	SSSS	CCCC	SCSC		SFSF	SSSS	CCCC	SCSC
1-0-1	0.6	[Jin G.et al (2015)]	0.8997	1,7761	3.0634	2.5193	0-1-1	0.8059	1.5873	2.8180	2.3047
		RFM	0.9010	1.7809	3.0890	2.5366		0.8067	1.5901	2.8349	2.3160
	5	[Jin G.et al (2015)]	0.7281	1.4384	2.4389	2.0125		0.6516	1.2781	2.3107	1.8839
		RFM	0.7299	1.4431	2.4629	2.0290		0.6521	1.2796	2.3215	1.8910
	20	[Jin G.et al (2015)]	05775	1.1404	1.9597	1.6128		0.6283	1.2320	2.2298	1.8176
		RFM	0.5784	1.1436	1.9767	1.6243		0.6287	1.2335	2.2401	1.8243
1-2-1	0.6	[Jin G.et al (2015)]	0.8440	1.6656	2.8159	2.3250	1-2-1	0.6660	1.3082	2.3543	1.9207
		RFM	0.8455	1.6713	2.8443	2.3446		0.6665	1.3100	2.3660	1.9284
	5	[Jin G.et al (2015)]	0.6369	1.2575	2.1445	1.7675		0.6206	1.2225	2.1791	1.7797
		RFM	0.6379	1.2614	2.1645	1.7812		0.6211	1.2245	2.1914	1.7878
	20	[Jin G.et al (2015)]	0.5291	1.0445	1.8093	1.4867		0.6220	1.2273	2.1716	1.7755
		RFM	0.5298	1.0471	1.8237	1.4964		0.6226	1.2295	2.1850	1.7844

4.2. Free vibrations of the functionally graded shells with clamped cutout of the complex form

As practice shows, special attention should be paid to the study of plates and shells with holes and cutouts. Cutouts are often required in the shell elements due to practical necessity, for instance in order to facilitate structure, provide access and compound with other parts, for venting, and other reasons. Cutouts can be both free and fixed on their border. Their form can also be arbitrary (not only circle). There are practically no works about vibrations of the laminated shallow shells with clamped or simply supported cutouts. However, such boundary conditions can be found quite often in practice. To contribute to new results and illustrate the versatility and efficiency of the proposed method and developed computer code, let us consider the shallow shell with a shape of the plan presented in the Fig. 1.

Suppose that the shell is clamped at the internal border of the region. On the outer boundary of the region, the shell can be either clamped or simply supported or have the mixed boundary conditions like boundary conditions in Task 1 (CCCC, SSSS, SFSF and SCSC).

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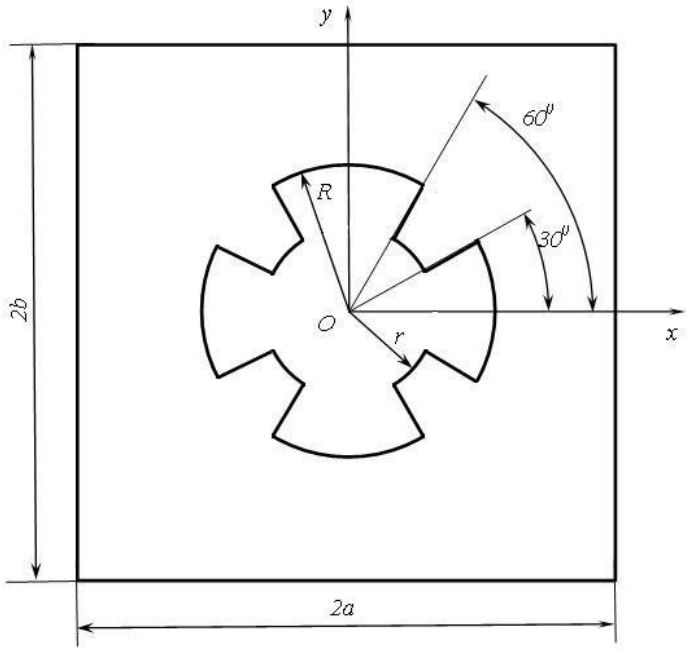
Figure 1. Shape of the plan of the laminated FGM shallow shell

The following geometric parameters are fixed:



$b/a=1, k_1=R_x/2a=0.2, k_2=R_y/2a=(0,0.2,-0.2),$ $r/2a=0.125, R/2a=0.25, h/2a=0.1$.

The solution structure for shells with complete clamped on inside and outside borders is assumed as follows:

$$w=\omega\Phi_1,\quad u=\omega\Phi_2,\quad v=\omega\Phi_3,\quad ,\quad \psi_x=\omega\Phi_4,\quad \psi_y=\omega\Phi_5 \tag{25}$$

For another type of the boundary conditions, we propose to take solution structure satisfying kinematic boundary conditions in the following form

$$w=\omega^{(w)}\Phi_1,\quad u=\omega^{(u)}\Phi_2,\quad v=\omega^{(v)}\Phi_3,\quad ,\quad \psi_x=\omega^{(\psi_x)}\Phi_4,\quad \psi_y=\omega^{\psi_y}\Phi_5, \tag{26}$$

where: $\Phi_i, i=1,...,5,$ are indefinite components of the structure [42-45] presented as an expansion in a series of some complete system (power polynomials, trigonometric polynomials, splines etc.); $\omega=0$ is the equation of the whole border of the shell planform. The functions $\omega^{(u)}, \omega^{(v)}, \omega^{(w)}, \omega^{(\psi_x)}, \omega^{(\psi_y)}$ are constructed by the R-functions theory in such a way that they vanish on those parts of the boundary where the functions u, v, w, ψ_x, ψ_y are zero.

In order figure out the solution structure (13) and (14), we should construct the equation of the whole border and functions $\omega^{(u)}, \omega^{(v)}, \omega^{(w)}, \omega^{(\psi_x)}, \omega^{(\psi_y)}$. Using the R-operations [42], we build the equation on the border in the following form:

$$\omega = \omega_{inside} \wedge_0 \omega_{outside}, \tag{27}$$

where

$$\omega_{inside} = \left(-\left(\left(\left(\left(f_1 \wedge_0 f_2\right) \vee_0 \left(\overline{f_1} \wedge_0 \overline{f_2}\right) \vee_0 \left(\left(f_3 \wedge_0 f_4\right) \vee_0 \left(\overline{f_3} \wedge_0 \overline{f_4}\right)\right)\right) \vee_0 f_5\right) \wedge_0 f_6\right)\right)$$

$$\omega_{outside} = f_7 \wedge_0 f_8$$

The functions $f_i, i = 1,\ldots,8$ are defined as follows

$$f_1 = \left(y + \frac{1}{\sqrt{3}}x\right) \geq 0, \quad f_2 = \left(-y + \frac{1}{\sqrt{3}}x\right) \geq 0, \quad f_3 = \left(y - \sqrt{3}x\right) \geq 0, \quad f_4 = \left(y + \sqrt{3}x\right) \geq 0$$

$$f_5 = \left(r_1^2 - x^2 - y^2\right) \geq 0, \quad f_6 = \left(r_2^2 - x^2 - y^2\right) \geq 0, \quad f_7 = \left(a^2 - x^2\right) \geq 0, \quad f_8 = \left(b^2 - y^2\right) \geq 0$$

Below we write down expressions for functions $\omega^{(u)}, \omega^{(v)}, \omega^{(w)}, \omega^{(\psi_x)}, \omega^{(\psi_y)}$ for different boundary conditions on the outside part of the region border provided that a cut of the shell is clamped. We have, respectively:

CCCC:

$$\omega^{(u)} = \omega^{(v)} = \omega^{(w)} = \omega^{(\psi_x)} = \omega^{(\psi_y)} = \omega; \tag{28}$$

SSSS:

$$\omega^{(u)} = \omega^{(v)} = \omega^{(w)} = \omega^{(\psi_x)} = \omega^{(\psi_y)} = \omega; \tag{29}$$

SFSF:

$$\omega^{(w)} = \omega^{(u)} = \omega^{(\psi_x)} = \omega_{inside} \wedge_0 f_8, \quad \omega^{(v)} = \omega^{(\psi_y)} = \omega_{inside}; \tag{30}$$

SCSC:

$$\omega^{(w)} = \omega^{(u)} = \omega^{(\psi_x)} = \omega, \quad \omega^{(v)} = \omega^{(\psi_y)} = \omega_{inside} \wedge_0 = f_7. \tag{31}$$

Indefinite components $\Phi_i, i = 1,\ldots,5$ in solution structures (25)-(26) are approximated by a system of power polynomials taking into account the double-symmetric nature of the problem.

Consequently, sequences of polynomials are chosen in the following way:

$$\Phi_1: 1, x^2, y^2, x^4, x^2y^2, y^4, x^6, x^4y^2, x^2y^4, y^6, \cdots$$

$$\Phi_2, \Phi_4: x, x^3, xy^2, x^5, x^3y^2, xy^4, x^7, x^5y^2, x^3y^4, xy^6, \cdots$$

$$\Phi_3, \Phi_5: y, x^2y, y^3, x^4y, x^2y^3, y^5, x^6y, x^4y^3, x^2y^5, y^7, \cdots$$

Integration is performed over one-quarter domain. In Table 6, the fundamental frequency parameters $\Omega_L^{(1)} = \lambda_1 a^2 \sqrt{\rho_c / E_c} / h$ for cylindrical, spherical and hyperbolic paraboloidal shells of Type 2-2 and two thickness schemes (2-1-2) and (2-2-1) are presented.

Notice that for the considered shells with general thickness *h/*2*a*=0.1, the fundamental frequencies parameters are close for cylindrical, spherical and hyperbolic paraboloidal shells. If a shell is clamped on the whole border (CCCC), then the spherical shell has the largest frequency and cylindrical panel has the smallest one. However, if a shell is simply supported on its outside boundary and its cut is clamped, then this regularity is broken for a given ratio of layers thickness. The hyperbolic paraboloidal shell has the greatest frequency, and the frequencies of the spherical panels are smaller than corresponding ones of the cylindrical panels. This example shows the effect of boundary conditions for different schemes of thickness. It means that every case requires individual analysis.

Table 6. Fundamental frequency parameters $\Omega_L^{(1)} = \lambda_1 a^2 \sqrt{\rho_c / E_c} / h$ for shells of Type 2-2 with clamped cutout and simply supported or clamped outside contour of the domain (*Fig. 1*)

Scheme	p	k_1=0.2, k_2=0		k_1=0.2, k_2= 0.2		k_1=0.2, k_2= -0.2	
		SSSS	CCCC	SSSS	CCCC	SSSS	CCCC
2-1-2	0.1	24.104	31.497	24.104	31.539	24.127	31.528
	0.5	23.662	30.905	23.659	30.945	23.688	30.936
	1	23.362	30.499	23.357	30.537	23.390	30.531
	5	22.846	29.770	22.836	29.805	22.877	29.802
	10	22.760	29.636	22.749	29.670	22.792	29.668
	20	22.718	29.567	22.706	29.635	22.750	29.599
2-2-1	0.1	24.027	31.373	24.026	31.415	24.051	31.404
	0.5	23.375	30.433	23.369	30.470	23.403	30.464
	1	22.946	29.785	22.934	29.818	22.975	29.816
	5	22.242	28.571	22.222	28.598	22.273	28.600
	10	22.099	28.286	22.078	28.313	22.129	28.314
	20	22.007	28.106	21.985	28.132	22.036	28.133

Effects yielded by the gradient index $p = p_1 = p_2 = p_3$ on the fundamental frequency parameter $\Omega_L^{(1)} = \lambda_1 a^2 \sqrt{\rho_c / E_c} / h$ for cylindrical, spherical and hyperbolic paraboloidal shells of Type 1-2 and 2-2 with different boundary conditions are shown in Figures 2,3,4. Different thickness schemes are taken for the considered shallow shells. The obtained results for the cylindrical shells with thickness scheme (1-2-1) are presented in Fig. 2.

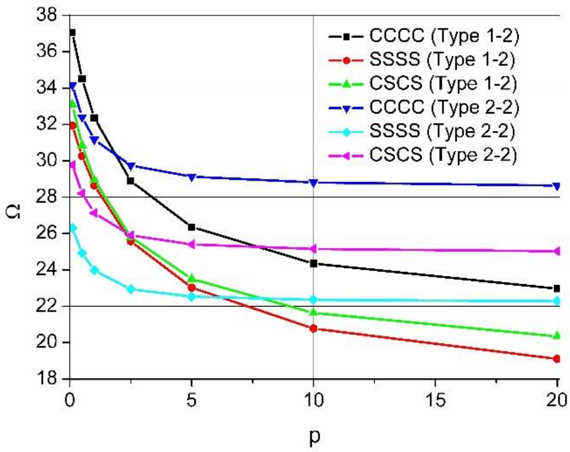


Figure 2. Variation of the fundamental frequency parameter $\Omega_L^{(1)} = \lambda_1 a^2 \sqrt{\rho_c / E_c} / h$ of cylindrical shells with increasing gradient index p (thickness scheme 1-2-1).

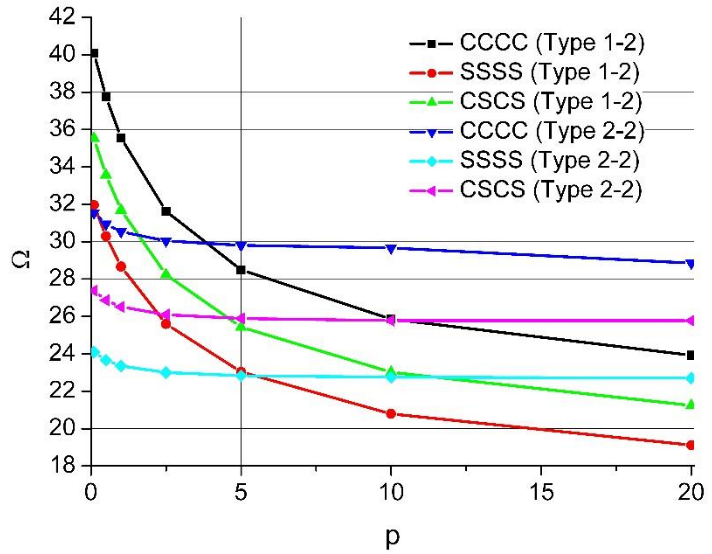


Figure 3. Variation of the fundamental frequency parameter $\Omega_L^{(1)} = \lambda_1 a^2 \sqrt{\rho_c / E_c} / h$ of the spherical shells with increasing gradient index p (thickness scheme 2-1-2)

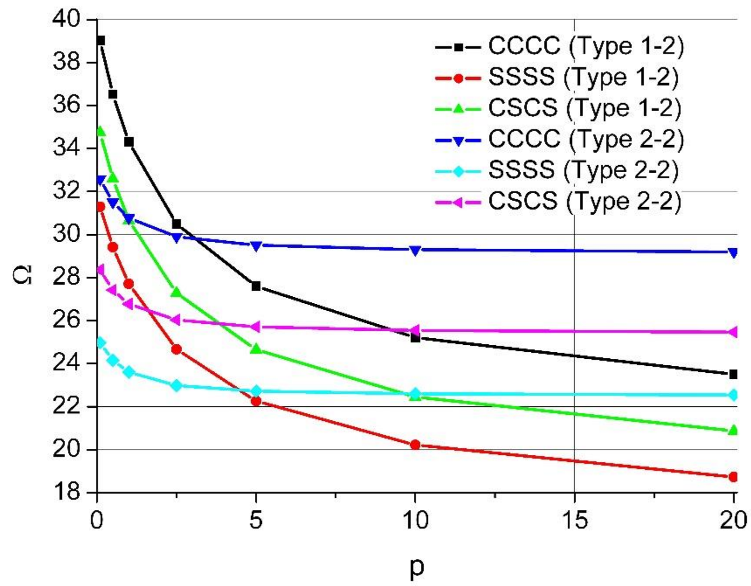


Figure 4. Variation of the fundamental frequency parameter $\Omega_L^{(1)} = \lambda_1 a^2 \sqrt{\rho_c / E_c} / h$ of Type 1-2 and 2-2 with thickness scheme (1-1-1) of hyperbolic paraboloidal shells with increasing gradient index p.

The effects of material types and power law exponents on the frequency parameter of spherical shells with (2-1-2) thickness scheme are presented in Fig 3. Similar results for hyperbolic paraboloid shells with (1-1-1) thickness scheme are shown in Fig. 4.

As follows from Figures 2-4, the value of fundamental frequency parameters essentially depends on the material type, thickness schemes, and boundary conditions. Obviously, the fundamental frequencies parameters for all considered cases decrease with increasing power-law exponent. For the shells of type 1-2, the decrease is more essential than for the shells of Type 2-2.

5. Concluding remarks

This paper proposes a method of investigation of free vibrations of laminated functionally graded shallow shells with complex shape of the planform. The method is based on the theory of R-functions and Ritz variational method. Comparison of the obtained results for shallow shells of the doubly-curved and square planform confirms the validation of developed software. New solution structures are proposed for shallow shells with clamped cutout of the complex form. In addition, novel results are obtained for cylindrical, spherical and hyperbolic paraboloidal shallow shells of FGM sandwich type with cutout of the complex shape. Effects of power law exponents, thickness schemes, and different boundary conditions are studied for shells with clamped cutout of the complex shape.

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