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Notas e Discussões

The noninertial origin of the reduced mass

(A origem não-inercial da massa reduzida)

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A different way to obtain the equation of motion that governs the relative motion in a two-particle system is presented. It provides the physical interpretation of the reduced mass as a noninertial effect.

Keywords: two-particle system, noninertial frame of reference, reduced mass.

Uma maneira diferente de se obter a equação de movimento que governa o movimento relativo em um sistema de duas partículas é apresentada. Com ela, podemos interpretar fisicamente a massa reduzida como sendo uma quantidade oriunda de efeito não-inercial.

Palavras-chave: sistema de duas partículas, sistema de referência não-inercial, massa reduzida.

The reduced mass is a quantity defined when we study the two-particle system problem in mechanics. It appears when the equations of motion which govern the motions of the system constituents are transformed into two others, one for the center of mass and other for the relative motion between the two particles, playing the role of the inertia for the relative motion. In elementary textbooks on classical mechanics (see, for example, Refs. [1] and [2]) and also in more advanced ones (see, for example, Refs. [3] to [6]), the equation of motion for the relative motion between the two particles is obtained in such ways that it is not possible to understand the physical origin of the reduced mass. The objective of this note is to present an alternative way to achieve the equation of this relative motion which permits a clear identification of the physical origin of the reduced mass.

Let us, then, consider the two-particle system shown in Fig. 1. The mass of these particles are denoted by m_1 and m_2 while their position vectors, relative to an observer O attached to an inertial frame of reference, are $\boldsymbol{r_1}$ and $\boldsymbol{r_2}$, respectively. Just for the sake of simplicity, we are supposing that there are no external forces acting on this system so that its particles are driven by their mutual interaction which gives rise to the forces $\boldsymbol{F_{1(2)}}$ (acting on particle 1 due to its interaction with particle 2) and $\boldsymbol{F_{2(1)}}$ (vice versa). Again to simplify the discussion, let us suppose that m_1 and m_2 are constants so that, in the case we are considering, the classical time evolutions of $\boldsymbol{r_1}$ and $\boldsymbol{r_2}$ obey the Newton's second law written as:

$$m_1\boldsymbol{a_1} = \boldsymbol{F_{1(2)}} \tag{1}$$

$$m_2\boldsymbol{a_2} = \boldsymbol{F_{2(1)}}\ , \tag{2}$$

where $\boldsymbol{a_1}$ and $\boldsymbol{a_2}$ stand for the acceleration of each particle.

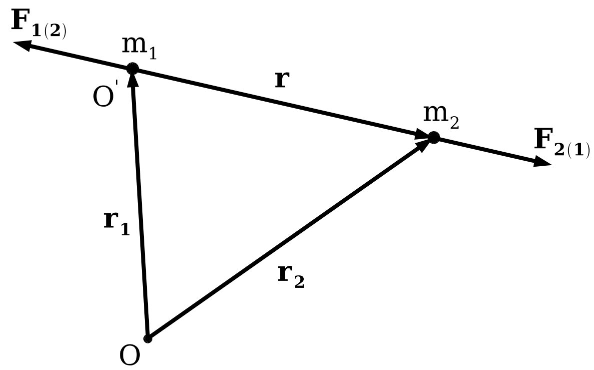

Figura 1 - Schematic representation of a two-particle system showing some of the quantities used in the text.

Suppose now we want to describe the motion of particle 2 with respect to an observer O' attached to particle 1, which constitutes a noninertial frame of reference once particle 1 is accelerated (see Eq. 1). As we are supposing m_2 to be constant, the position vector $\boldsymbol{r}$ of particle 2 with respect to particle 1 (see Fig. 1) has its time evolution governed by the following equation of

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motion:

$$m_2\boldsymbol{a} = \boldsymbol{F_{2(1)}} + \boldsymbol{F_{in}} \ , \tag{3}$$

where $\boldsymbol{a}$ is the acceleration of particle 2 relative to particle 1 and $\boldsymbol{F_{in}}$ is the inertial force given by:

$$\boldsymbol{F_{in}} = -m_2\boldsymbol{a_1} \ . \tag{4}$$

Determining the acceleration $\boldsymbol{a_1}$ from Eq. (1) and remembering that $\boldsymbol{F_{1(2)}} = -\boldsymbol{F_{2(1)}}$ due to the Newton's third law, it follows from Eq. (4) that:

$$\boldsymbol{F_{in}} = \frac{m_2}{m_1}\boldsymbol{F_{2(1)}} \ . \tag{5}$$

Substitution of the above equation into Eq. (3) leads to the following equation of motion for particle 2 with respect to the observer O':

$$m_2\boldsymbol{a} = \left(1 + \frac{m_2}{m_1}\right)\boldsymbol{F_{2(1)}} \ , \tag{6}$$

which can also be written as:

$$\frac{m_2}{1 + m_2/m_1}\boldsymbol{a} = \boldsymbol{F_{2(1)}} \ . \tag{7}$$

Then, due to its noninertial nature, the observer O' describes the motion of particle 2 as a particle of mass m_2 which is driven by the force $\boldsymbol{F_{2(1)}}$ amplified by a factor of $1 + m_2/m_1$ (see Eq. 6). Equivalently, this observer can describe the motion of particle 2 as if it had its mass m_2 reduced by the same factor but driven by the force $\boldsymbol{F_{2(1)}}$ (see Eq. 7). The quantity obtained by dividing the mass m_2 by the factor $1 + m_2/m_1$ is the so called reduced mass (μ) of the two-particle system considered, that is,

$$\mu = \frac{m_1 m_2}{m_1 + m_2} \ , \tag{8}$$

and brings Eq.(7) to its final form:

$$\mu\boldsymbol{a} = \boldsymbol{F_{2(1)}} \ . \tag{9}$$

To conclude this note, we would like to call attention to the point that the reduced mass of a two-particle system has its origin in the combination of two distinct things. The first is the noninertial nature of the observer O' attached to particle 1, which leads to the inertial force $\boldsymbol{F_{in}}$ in Eq. (3). The second is that the force which drives O' ($\boldsymbol{F_{1(2)}}$) is intimately related to the force applied to particle 2 ($\boldsymbol{F_{2(1)}}$) through the Newton's third law, as they arise from the mutual interaction of the particles that constitute the system, which permits to write $\boldsymbol{F_{in}}$ in terms of $\boldsymbol{F_{2(1)}}$ as in Eq. (5).

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