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Notas e Discussões

A little subtlety on an electrostatic problem

(Uma pequena sutileza em um problema eletrostático)

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This paper outlines some mathematical considerations regarding a classical problem in elementary electrostatics: the value of electric field at the surface of a conducting sphere with uniform distribution of charge. It is emphasized a consequence following from an elementary but general proof of the, so-called, Gauss' law.
Keywords: electric field, conducting charged sphere, Gauss's law.

O artigo resume algumas considerações matemáticas em relação a um problema clássico em eletrostática: o valor do campo elétrico na superfície de uma esfera condutora carregada uniformemente. Enfatiza-se uma conseqüência que decorre de uma prova elementar, porém geral, da lei de Gauss.
Palavras-chave: campo elétrico, esfera carregada condutora; lei de Gauss.

This short paper outlines some mathematical considerations on a question asked by a few students of an Italian College-level physics course (Liceo Scientifico). The matter was the classical problem of evaluating the electric field inside and outside a uniformly distributed charge on a surface of a sphere, a well known problem explicitly found in elementary physics textbooks [1, 2]. Even if it is clear that any *real* physical problem in electrostatic *must* give continuous field everywhere (and one value function), discontinuities in electric field often arise in simple problem solving involving surface charges.

It is known that the field inside any conductor is zero because of the *inverse square law* as pointed out in the celebrated P.S.S.C. film *The Coulomb Law.* In the special case of a spherical conductor of radius a, an alternative motivation, *usually* emphasized that at the centre $O$the field is zero because of the spherical symmetry; then applying Gauss's law through a spherical surface of radius r (being $r < a$) and because no *sources of field* are is in the interior from $\Phi s(\boldsymbol{E}) = 0$ follows $\boldsymbol{E} = \boldsymbol{0}$ being $\Phi s(\boldsymbol{E})$ the electric field flux through the spherical surface. Equally applying the Gauss law for $r > a$ the well known result of Eq. (1) follows

$$\begin{cases} \mathbf{E} = \mathbf{E}(r) = \mathbf{0}, \quad 0 < r < a \\ \mathbf{E} = \mathbf{E}(r) = \dfrac{1}{4\pi\varepsilon_o}\dfrac{Q}{r^2}\dfrac{\mathbf{r}}{r}, \quad r > a. \end{cases} \tag{1}$$

Usually it is noticed as the field is *discontinuous* at $r = a$ being

$$\lim_{r\to a^-} \mathbf{E}(r) = 0; \quad \lim_{r\to a^+} \mathbf{E}(r) = \frac{1}{4\pi\varepsilon_o}\frac{Q}{a^2}\frac{\mathbf{r}}{r}. \tag{2}$$

This is problem solving final position.

Because in Italy the physics and mathematics teacher is the same, a subtle question was arisen by a few students: "because the concept of *continuity* (or *discontinuity)* refers (at least *properly*) to a one-valued real function $E(r)$ in a point a *internal* to its domain, what is the electric field value at $x = a$?". This is a well posed question at least in the context of a simple "problem solving" involving a pure surface charge, because an electric field *must* exist in all space including $r = a$. Hence Eqs. (2) *need* a requirement of field existence in $r = a$.

This question is considered in Ref. [3] from a physical point of view and with a correct final answer, but the limit case shown in Fig. 2.12 of this book is misleading because shows an *infinite-value* function in $r = a$ (a vertical step in Fig. 2.12 of this book). A correct answer to this question (in the framework of a classically posed *problem solving*) may simply follow from Gauss's law. It is noticed that in some textbooks Gauss's law is simply quoted [1-3], and/or proved in the special case of a spherical surface surrounding a charge located at the centre [4]; other textbooks [3-6] demonstrate Gauss law using a spherical surface (with a charge in the centre) and then extending to the generic surface using the inverse square law. Namely as in Ref. [6]

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$$\int_{any\ surface} E_n da = \begin{cases} 0;\ q \text{ outside } S \\ \dfrac{q}{\varepsilon_o};\ q \text{ inside } S \end{cases} \tag{3}$$

where E_n is the field component on the outward oriented infinitesimal surface $\boldsymbol{dS}$, q is a charge and S any closed surface. *None* of the textbooks quoted, consider the special case of the electric charge *at* the surface.

An elementary *direct* proof of Gauss' law appears on a very old Italian textbook [7] but is always referring to an internal or external charge to a closed surface in free space. The extension to the special case to a charge on the surface follows from an obvious observation on integration domain.

If the charge is in interior, the proof requires an integration of the infinitesimal flux $d\Phi(\boldsymbol{E})$

$$d\Phi\left(\mathbf{E}\right) = \frac{Q}{4\pi\varepsilon_o}\, d\omega, \tag{4}$$

on the *whole* solid angle (4π steradians); being Q the charge, ε_o free space permittivity and $d\omega$ an infinitesimal solid angle. If the charge is in exterior, from the inverse square law and the opposite signs of elementary surface orientations dS it follows a zero flux. In the end, if the charge is located *exactly* at a point of the surface (and the surface has locally a tangent plane at each point of S), the flux evaluation requires in this special case an integration on *half* of the whole solid angle (2π steradians).

Therefore, at $r = a$ the electric field does exists, and its value is

$$\mathbf{E}(a) = \frac{1}{8\pi\varepsilon_o}\frac{Q}{a^2}\frac{\mathbf{r}}{r}. \tag{5}$$

Namely the field can be put in compact form valid everywhere for all distances r

$$\mathbf{E}(r) = \frac{1 + sgn(r-a)}{2}\frac{1}{4\pi\varepsilon_o}\frac{Q}{r^2}\frac{\mathbf{r}}{r}, \tag{6}$$

as the graphic of $\boldsymbol{E}$ *vs.* r of the Fig. 1 suggests and a simple inspection of Eq. (6) gives a function expressed in a compact form, which is existing *everywhere* and *one-valued* function as expected for an electric field. Eq. (6) obviously gives the full correct solution to the feature of *problem solving example* posed. The "problem solving" example is surely in disagreement with a real physical situation where a continuous function in the neighborhood of $r = a$ is expected, but this overcomes the aim of an elementary physics course.

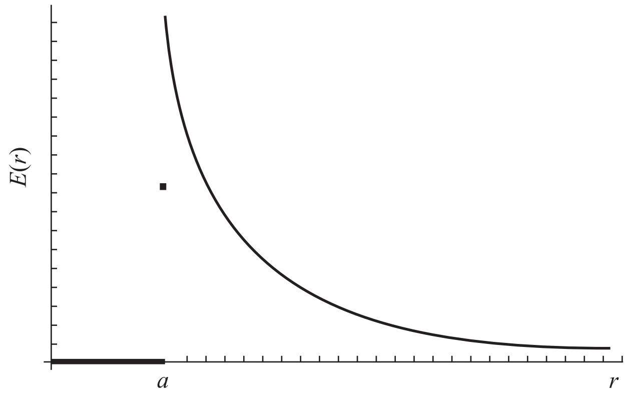

Figura 1 - Electric field $E(r)$ *vs.* distance r for an uniform charge distribution having spherical symmetry.

Surprisingly, the *one-valued* requirement for a central field of force also appear unnoticed in a classical textbook of potential theory [8].

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