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Wave propagation in a non-uniform string

(Propagação de ondas em uma corda não-uniforme)

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In this article I present the behavior of the reflection and transmission coefficients of a pulse at a joint between two strings with mass densities μ_1 and μ_2. The joint is made of a string segment with mass density varying linearly from μ_1 to μ_2. It will be shown that the reflection of the pulse at the joint depends largely on the ratio between the pulse width and the length of the joint. Analogies with other physical systems such as antireflection coatings and tsunamis will be considered briefly.
Keywords: wave, pulse, string, non-uniform, inhomogeneous, discontinuous, anti-reflection, anti-reflective, tsunamis, pulse energy.

Neste artigo mostro o comportamento dos coeficientes de reflexão e transmissão de um pulso propagando através de uma emenda entre cordas com densidades de massa μ_1 e μ_2. A emenda é um segmento de corda com densidade de massa variando linearmente desde μ_1 à μ_2. Será mostrado que a reflexão do pulso na emenda depende sensivelmente da razão entre a largura do pulso e o comprimento da emenda. Discutirei brevemente analogias com outros sistemas físicos como camadas anti-refletoras e tsunamis.
Palavras-chave: onda, pulso, corda, não-uniforme, não-homogêneo, descontínuo, anti-refletor, tsunamis, energia de pulso.

1. Introduction

The basic concepts of wave mechanics in physics courses and textbooks for undergraduate students are usually illustrated with sine functions of a vibrating string. In these functions, properties such as standing waves, wavelength, period/frequency, phase and amplitude are well defined and they can be easily calculated given the tension and the mass density of the string. The relevance of these introductory chapters on wave mechanics is its applicability in many areas of physics like acoustic, optics, electromagnetism and quantum mechanics. In the textbooks and in the internet one can find various animations of waves in uniform strings [1–5] and in strings with different mass densities [6] joined together. There are also various articles describing the behavior of waves in non-uniform strings, e.g. the effect of non-uniformity on natural tones in musical instruments [7], in quantum waves [8,9] and seismological wave propagation inside the Earth [10]. The more sophisticated the physical model, the more phenomena one can describe and explain. In this work I introduce a specific non-uniformity in the string, being a sophistication that is not covered in the introductory texts, but it allows understanding the working principle of the anti-reflection coatings.

In this article, the non-uniformity consists of two semi-infinite strings with mass densities μ_1 and μ_2 joined with a string segment with mass density varying linearly from μ_1 to μ_2. The objective is to show the behavior of the reflection of a pulse passing through the string joint. To simulate a pulse is preferred instead of a sinusoidal travelling wave because the former is a limited wave package and it is easier to visualize and compare the amplitudes or the pulse width variations as the reflection drops.

2. The wave equation

The propagation of a wave in a string is described by the wave equation

$$\frac{\partial^2 y(x,t)}{\partial t^2} = v^2 \frac{\partial^2 y(x,t)}{\partial x^2}, \tag{1}$$

where $y(x,t)$ is the transversal displacement of the wave at position x and time t and v is the velocity in the x direction or *group velocity*, to be distinguished from the transversal velocity of the string $\partial y/\partial t$. The v depends on the tension and mass density of the string according

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to

$$v\sqrt{\frac{B}{\mu}}, \quad (2)$$

where B is the tension in the string, expressed in N, and μ is the linear mass density, expressed in kg/m.

3. Physical system

Figure 1 shows a representation of non-uniform string fixed at two boundaries far from the joint. Since our analysis will be restricted to the central portion of the string the pulse interaction with the boundaries will be neglected; so, in the pictures hereafter these boundaries will not be indicated. A pulse with a Gaussian profile having width Δx and initial amplitude y_0 propagates to the right. In $x = 0$ there is a joint segment of length ΔL, where the mass density of the string varies linearly from μ_1 to μ_2. Later in this article it will be important to consider that the string is inextensible so the pulse is formed by mass accumulation (not by strain) and the tension of the string is provided by a hanging weight (not by elastic forces).

After interacting with the joint the pulse will be partially reflected and partially transmitted. Here I'll present a numerical analysis of the reflection R coefficient as a function of ΔL and Δx. R is defined by the ratio between the energy of the reflected pulse and the energy of the incident pulse

The initial wave with the characteristics shown in Fig. 1 can be written as a standard Gaussian function

$$y(x,0) = y_0 \exp\left[-\frac{(x-x_0)}{2\Delta x^2}^2\right], \quad (3)$$

where x_0 is the peak position and Δx is the width of the Gaussian. The mass density of the string varies according to the function

$$\mu(x) = \begin{cases} \mu_1 \text{ for } x \leq 0 \\ \mu_1 + \frac{x}{L}(\mu_2 - \mu_1) \text{ for } 0 < x \leq \Delta L \\ \mu_2 \text{ for } x > \Delta L \end{cases} . \quad (4)$$

For the description of R it is efficient first to consider the energy of the pulse. Figure 2 shows an example of a pulse propagating to the right in a referential system Q. The pulse interacts with the joint and is partially reflected. The frames drawn enclosing the incident, the reflected and the transmitted pulses are the local referential system named P_i (i = 0, 1 or 2) that moves together with the pulse. This referential coordinate system facilitates the evaluation of the energy of the pulse as was demonstrated by Juenker [11]. In this analysis it is important that the string in inextensible, so the pulse is made by the action of mass only and not by stretching the string. In the system P, the pulse doesn't move; however, the string is seen moving from the right to the left along the pulse like a train in a Gaussian shaped railroad with local velocity v ,which will be denoted by v_1 in the string with mass density μ_1 and v_2 for μ_2 (see Fig. 3).

The energy of the propagating pulse can easily be obtained from Juenker's procedure as follows: In P any mass element, dm, has a velocity v parallel to the string. From P one can conclude that the velocity, v^*, of dm in Q is the vector sum of its velocity in P plus the velocity of P in Q as indicated in Fig. 3. Using the law of cosines, v^* results in

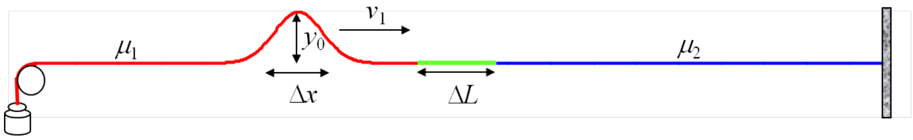

Figure 1 - (color online): Representation of the system at $t = 0$.

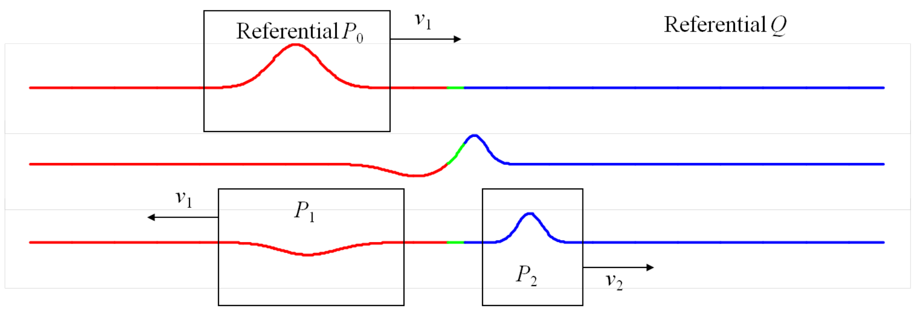

Figure 2 - (color online). Representation of pulse propagation across a joint: the energies of the initial, reflected and transmitted pulses are entirely enclosed in the frames of the referential systems P_0, P_1 and P_2 respectively.

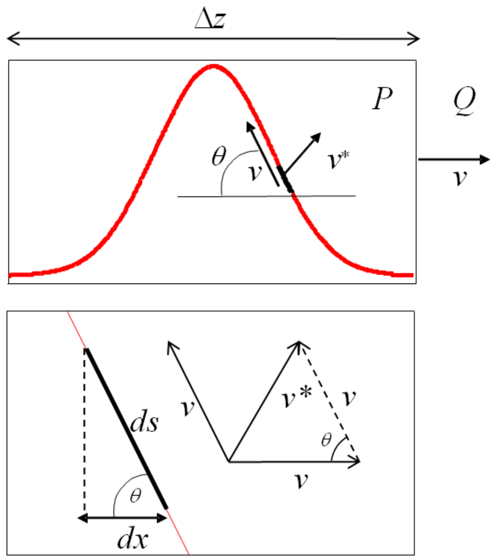

Figure 3 - (color online). In a referential P the string is seen moving from the right to the left throughout the wave with velocity v. As P moves with velocity v in Q the resulting velocity v^* is obtained from the law of cosines.

$$(v*)^2 = 2v^2(1 - \cos\theta). \tag{5}$$

The kinetic energy of a mass element is

$$dE = \frac{dm(v*)^2}{2} = \frac{\mu ds(v*)^2}{2}\mu v^2(1 - \cos\theta)ds, \tag{6}$$

where v^* was replaced by the expression in Eq. (5). Note from Fig. 3 that $\cos\theta = dx/ds$ so the energy of a mass element is given by

$$dE = \mu v^2 \left(1 - \frac{dx}{ds}\right) ds = \mu v^2(ds - dx). \tag{7}$$

The total energy of the pulse is the integral of Eq. (7)

$$E = \int_{x_a}^{x_b} \mu v^2(ds - dx) = \mu v^2(\Delta s - \Delta z), \tag{8}$$

where $\Delta z = x_b - x_a$ is the length of the frame that completely encloses the pulse and Δs is the integral in ds which gives the total length of the Gaussian curve in the interval Δz. Usually Eqs. (7) and (8) cannot be solved analytically when the pulse interacts with a non-uniform segment. Notice that to evaluate the energy in Eq. (8) it doesn't matter if Δz is larger than necessary to enclose the pulse, because the difference $\Delta s - \Delta z$ will tend to a constant. $\Delta s - \Delta z$ is the string accumulated to create the pulse.

The initial, reflected and transmitted pulse energies, E_0, E_1 and E_2 respectively, are

$$E_0 = \mu_1 v_1^2(\Delta s_0 - \Delta z_0), \tag{9}$$

$$E_1 = \mu_1 v_1^2(\Delta s_1 - \Delta z_1), \tag{10}$$

$$E_2 = \mu_2 v_2^2(\Delta s_2 - \Delta z_2). \tag{11}$$

The R and T are given by the ratio

$$R = \frac{E_1}{E_0}, \tag{12}$$

$$T = \frac{E_2}{E_0}. \tag{13}$$

4. Numerical analysis

In this section I explain the numerical evaluation of the wave function. The propagation of the wave is calculated using finite differences in the time domain (FDTD). The string is divided in a large number of cells with length dx. The solution at a time t is evaluated from the solutions of the previous two time intervals $t-dt$ and $t-2dt$ as shown in Fig. 4.

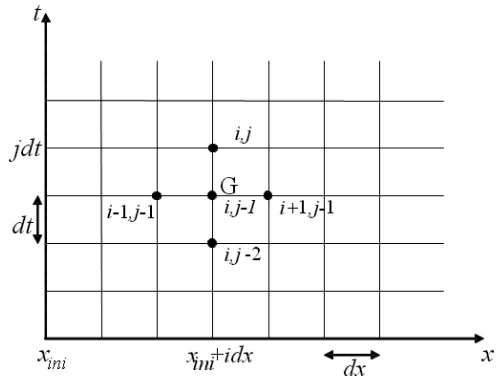

Figure 4 - Schematic procedure of FDTD.

The differential Eq. (1) is written as an equation of differences [12]

$$y(i,j) = (1-r)y(i,j-1) + r\left[y(i+1,j-1) + y(i-1,j-1)\right] - y(i,j-2), \tag{14}$$

where i corresponds to the index of the i^{th} node of the string, j is the j^{th} time interval and

$$r = \left(v\frac{dt}{dx}\right)^2. \tag{15}$$

The condition $0 < r \leq 1$ is necessary for Eq. (14) to converge [12]. The $y(i,j)$ has a direct relation to $y(x,t)$ since each (x,t) has a corresponding cell (i,j) according to $x = x_{ini}+idx$ and $t = jdt$. Initial conditions are

$$y(i,0) = y_0 \exp\left[-\frac{(idx - x_0)^2}{2\Delta x^2}\right], \tag{16}$$

and the boundary conditions are fixed rims

$$y(1,j) = y(i_{tot},j) = 0. \tag{17}$$

The energies (9) to (10) are obtained numerically from the expression

$$E = \sum_{i=i_1}^{i_2} \left\{ \mu(i) v^2 \sqrt{(y(i,j) - y(i-1,j))^2 + dx^2} \right\} - \mu(i) v^2 (i_2 - i_1) dx, \quad (18)$$

where, i_1 and i_2 are the first and last cells of the frame that contains the pulse, $\mu(i)$ is the position depending mass density. Eq. (18) is not applicable if the pulse is interacting with the joint, because the shape of the pulse is changing. Then, Eq. (5) is not valid because there is one more velocity component that depends on the position in a complicated way. To use Eq. (18) the reflected and transmitted pulses must be far from the rims and from the joint, so the numerical propagation must be carried out until this condition has been met.

5. Results

Figure 5 compares two cases in which a pulse passes through a discontinuous joint and a smooth joint segment. It can be seen that the amplitude of the reflected pulse in the smooth joint is smaller and wider, which implies that its energy is lower. However, the transmitted wave is taller, so the total energy is conserved.

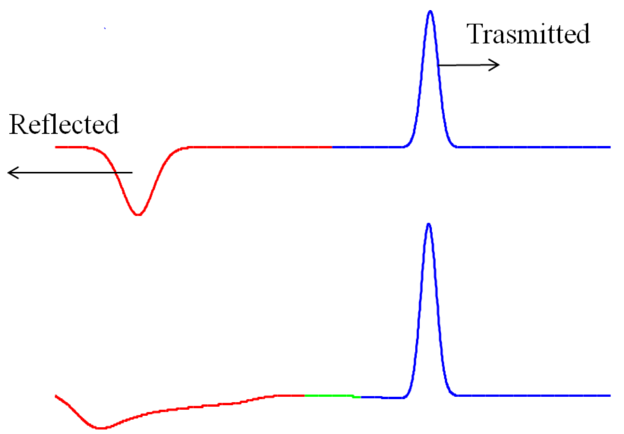

Figure 5 - (color online). Reflection in a discontinuous and in a smooth joint.

Figure 6(a) and (b) show the behavior of R in which a pulse is passing the joint. Figure 6 (a) shows R for $\mu_1 = 1$ kg/m to $\mu_2 = 4$, 9 and 16 kg/m. Figure 6(b) shows the other way around in which the pulse passes from high density $\mu_1 = 4$, 9 and 16 kg/m to low density $\mu_2 = 1$ kg/m. In all cases the reflection decreases as the length of the joint increases. It can be seen that for $\Delta L = 0$ the reflection coefficients in both cases are equal. Figure 7 compares R for two pulse widths. The smaller the width, the smaller is the reflection. In the limit of $\Delta x \ll \Delta L$ then $R \to 0$ and if $\Delta x \gg \Delta L$ then $R \to$ constant independent of Δx or ΔL.

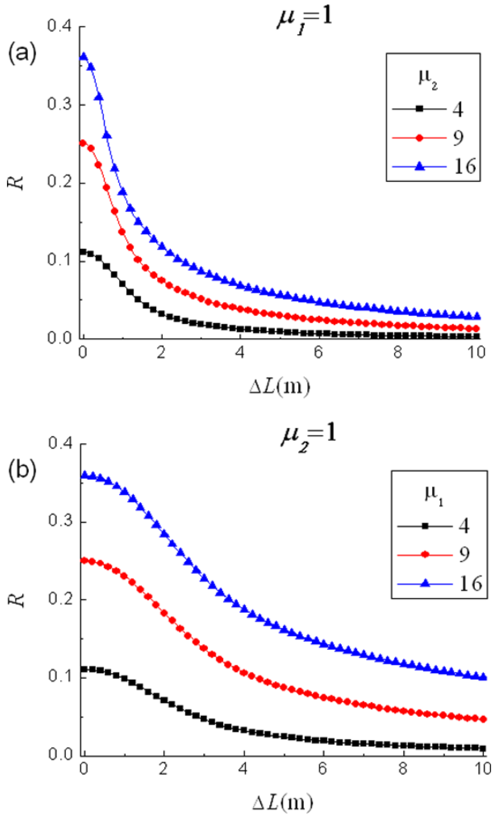

Figure 6 - (color online). Reflection coefficient as a function of ΔL as the pulse passes (a) from low to high or (b) from high to low densities.

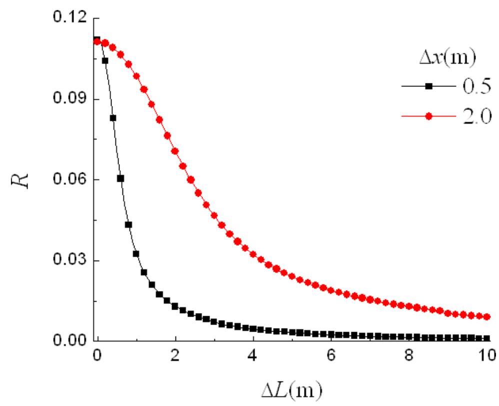

Figure 7 - (color online). Short pulses have smaller reflection coefficients at the joint segment.

The transmission coefficient is simply T=1-R, so the curves for T are not shown here.

6. Analogies with other physical systems

As any analogy this analysis only provides insights to some extent to other physical systems, after which it fails due to their particularities. Nevertheless, this analysis is worthwhile, since it helps comprehending qualitatively the transmission of light in anti-reflection coatings and tsunamis. I'll describe these examples briefly.

6.1. Anti-reflection coatings

The refraction index variation between two media always causes some light reflection at the interface. It is tempting to make an analogy between the index of refraction and wavelength in optics and the mass density and the pulse width respectively in non-uniform strings. Some anti-reflection coatings deal with a smooth gradient of the refractive index and these coatings or transitions show indeed very good anti-reflection behavior. An interesting example is the moth eye, in which anti-reflection pyramidal nanostructures have grown at the interface. These structures are smaller than the wavelength of visible light, so the light passes as if the pyramids are a continuous layer of varying index [13].

6.2. Tsunamis

The Earth crust accommodation can cause the seabed to lift, lower or displace to the sides. In any case, a large water column is displaced. Also in this case it is tempting to make a comparison between the depth of the water with the mass density and the water displacement with the pulse on a string. Then the decreasing depth of the seabed is analogous to a string joint with decreasing mass density. When the pulse is transmitted to a low density string its amplitude increases. The same is expected in water, which explains why a small amplitude at the deep see becomes a tsunami at the beach. In this case however, the analogy is weak because it fails in two other noticeable characteristics: the pulse in the string is faster and wider in the lighter string whereas tsunami waves become slower and narrower in shallow waters. Water waves have different boundary conditions and involve mass conservation; so, for the amplitude to increase, more water must be taken from the front, rear and beneath the wave and Eq. (1) does not account for this. It is interesting to see the simulation of a tsunami in Ref. [14] and to compare this with the pulse in a string in the simulation in Ref. [15].

7. Conclusion

The simulation of pulses in a non-uniform string leads to the following conclusions:

The reflection coefficient is reduced when the transition between two strings is smooth in terms of mass density. An equivalent phenomenon is observed at the interface of two transparent media where the optical reflection also decreases largely upon a smooth transition of the refractive index.

The reflection coefficient depends on the ratio of the pulse width and the length of the joint segment. The analogy in anti-reflective coatings is that these are designed to be efficient in a certain wavelength band, whereas for large wavelengths the coating is too short to diminish the reflection appreciably.

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