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Entropic considerations in the two-capacitor problem

(Considerações entrópicas sobre o problema dos dois capacitores)

V.O.M. Lara[1, 2], A.P. Lima[1,3], A. Costa[1]

[1] *Instituto de Física, Universidade Federal Fluminense, Niterói, RJ, Brasil*
[2] *Instituto Federal de Ciência e Tecnologia do Rio de Janeiro, Campus de São Gonçalo, São Gonçalo, RJ, Brasil*
[3] *Centro Federal de Educação Tecnológica Celso Suckow da Fonseca, Itaguaí, RJ, Brasil*
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We study the well-known two-capacitor problem from a new perspective, focusing on the thermodynamic aspects of the discharge process. The free electron gas model is used to describe the electrons' energy levels in both capacitors in the low temperature regime. We assume an isothermal heat exchange between the resistor and the heat reservoir. Even in this limiting case, we obtain a positive entropy variation due to the discharge, which points out the irreversibility of this process.
Keywords: two-capacitor problem, entropy.

Estudamos o problema bem conhecido dos dois capacitor através de uma nova perspectiva, com foco nos aspectos termodinâmicos do processo de descarga. O modelo de gás de elétrons livres é usado para descrever os níveis de energia dos elétrons nos dois condensadores no regime de baixas temperaturas. Consideramos uma troca de calor isotérmica entre o resistor e o reservatório térmico. Mesmo nesse caso limite, obtém-se uma variação positiva da entropia devido à descarga, apontando o caráter irreversível desse processo.
Palavras-chave: problema de dois capacitores, entropia.

1. Introduction

The concept of entropy is intrinsically related to the arrow of time: Ref. [1] at the microscopic level almost all physical processes are time-symmetric, yet at the macroscopic level irreversibility still emerges. Textbooks illustrate this concept by means of a variety of examples, ranging from simple everyday physical processes (such as coffee cooling or the melting of an ice cube) to prototypical examples like the free expansion of an ideal gas or the mixing of two gases. Reference [2], however, thermodynamic descriptions of electrostatic systems are less common. In this work, we consider a simple electrostatic process that is analogous to the free expansion of an ideal gas.

Consider a simple circuit composed of two identical capacitors, each with capacitance C, and one resistor R, all connected in series. Capacitor 1 has initial charge q_0, while capacitor 2 is initially uncharged. The circuit also has a switch that prevents the flow of current, as shown in Fig. 1.

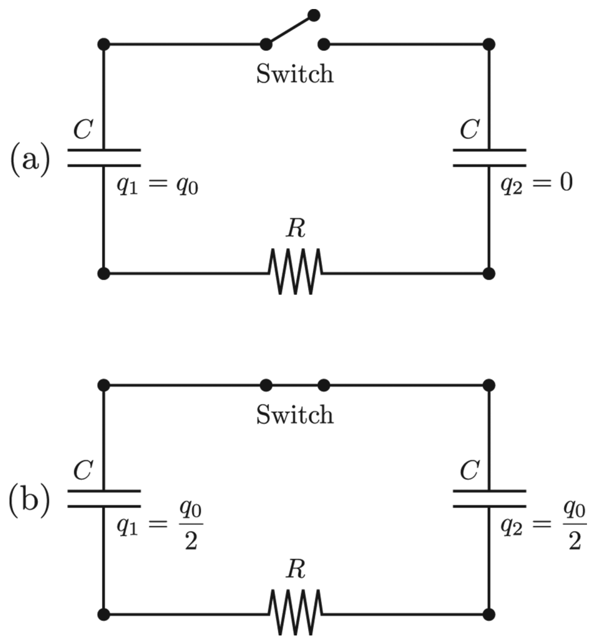

Figura 1 - Illustration of the circuit described in the text. In (a) the switch is open and there is no current flowing. In (b) the switch has been closed and a sufficiently large time has elapsed for the system to reach equilibrium.

[1] E-mail: vitorlara@if.uff.br.

We know intuitively that the discharging process is irreversible: when the switch is closed, the system spontaneously reaches the equilibrium configuration shown in Fig. 1(b) and will not go back to the initial configuration without external interference. There is a simple analogy between this discharging process and the isothermal expansion of an ideal gas (commonly used to obtain the entropy change in a free expansion). In both cases, there is a positive entropy change for the system, due to the redistribution of particles. However, here one must take into account that the relevant particles are fermions, thus obeying a different statistical distribution, which is a fundamental difference between the two processes.

There are many previous works that deal with this simple system using a variety of approaches. In Refs. [3,4], the authors focus on energy considerations, while Refs. [5–7] discuss specifically the electromagnetic radiation produced by the discharge process. Some of those works neglect the circuit's electrical resistance, but include a self-inductance, while others include both resistance and inductance.

In Ref. [8] the author measures experimentally the entropy change of the capacitor-charging process. This process is subdivided into n steps, with the voltage incremented by $\mathcal{V}/n$ during each step. In the limit $n \to \infty$ there is an isothermal heat exchange between the resistor and the heat reservoir, thus implying that the entropy changes of the resistor and the heat reservoir balance: $\Delta S_R + \Delta S_{HR} = 0$. However, the author neglects the entropy change due to the rearrangement of the charges in the capacitor plates.

In this work, we will use the free electron gas model [9] to evaluate the total entropy change during the discharging process. As discussed in Ref. [10], the free electron gas model provides some qualitative and quantitative features of metals at low temperature regime (as compared to the Fermi temperature T_F). We present a quasistatic scheme similar to the one shown in Ref. [8]. As we will see, even considering this scheme we obtain $\Delta S > 0$, in agreement with our intuition about the irreversibility of this process.

2. Entropy calculation

According to the second law of thermodynamics, any process that occurs in an isolated system must have $\Delta S \geq 0$. So, to compute ΔS, we need the entropy change of all components of our system: both capacitors, the heat reservoir, and the resistor. First, we will evaluate ΔS for the heat reservoir and the resistor.

In order to obtain the ΔS of the resistor we need to figure out a quasistatic process which lead the resistor and the heat reservoir from its initial to final states. To do this, we can subdivide the initial electric potential difference q_0/C into n equal intervals, each of them representing an electric potential reservoir (battery) which will be linked with each capacitor and the resistor. This way, we make the capacitor 1 scroll to a sequence of equilibrium states through successive contact with each electric potential reservoirs $\left[\frac{q_0(n-1)}{nc}, \frac{q_0(n-2)}{nc}, ..., \frac{q_0}{2c}\right]$, going down a ladder of potential. The capacitor 2 follows the opposite way, going up the ladder of electric potential reservoirs $\left[\frac{q_0}{nc}, \frac{2q_0}{nc}, ..., \frac{q_0)}{2c}\right]$. In the end both capacitors are at the same potential difference $\frac{q_0}{2C}$. At each ladder step that the capacitors pass through, we link the electric potential reservoir and the capacitor with the very same resistor, that will be the one responsible for the electric energy dissipation to the reservoir. An illustration of this quasistactic process is depicted in Fig. 2

In this scheme, during the discharge process of the capacitor 1 the resistor produces, at each step, an entropy variation,

$$\Delta s_R^{(1)} = \frac{C\mathcal{V}_0}{2nT} = \frac{q_0^2}{2Cn^2T}. \tag{1}$$

Analogously, the charge process of the capacitor 2 has an entropy variation per step of,

$$\Delta s_R^{(2)} = \frac{C\mathcal{V}_0}{2nT} = \frac{q_0^2}{2Cn^2T}. \tag{2}$$

Summing the entropy variation contribution of each step and both capacitors, we obtain the total entropy variation due to the resistor, which depends of how many steps the difference potential is subdivided,

$$\Delta S_R = \frac{q_0^2}{2nCT}. \tag{3}$$

So, if we take $n \to \infty$ the entropy produced by the Joule heating process vanishes and, even in this limiting case, the discharge is irreversible due to the rearrangement of the charges in the capacitors plates.

The free electron gas model was the first attempt at a microscopic description of the properties of metals. It was introduced by Drude [10] and later modified by Sommerfeld [9,10] to include the quantum nature of the electrons. The model assumes that conduction electrons are confined to a box with impenetrable walls. Inside the box the electrons are subject to a uniform background potential. Although this model is very simple, it describes qualitatively some important features of metals, such as the Wiedemann-Franz Law and the fact that the electronic contribution to the specific heat is proportional to T at temperatures much less than the Fermi temperature [9, 10].

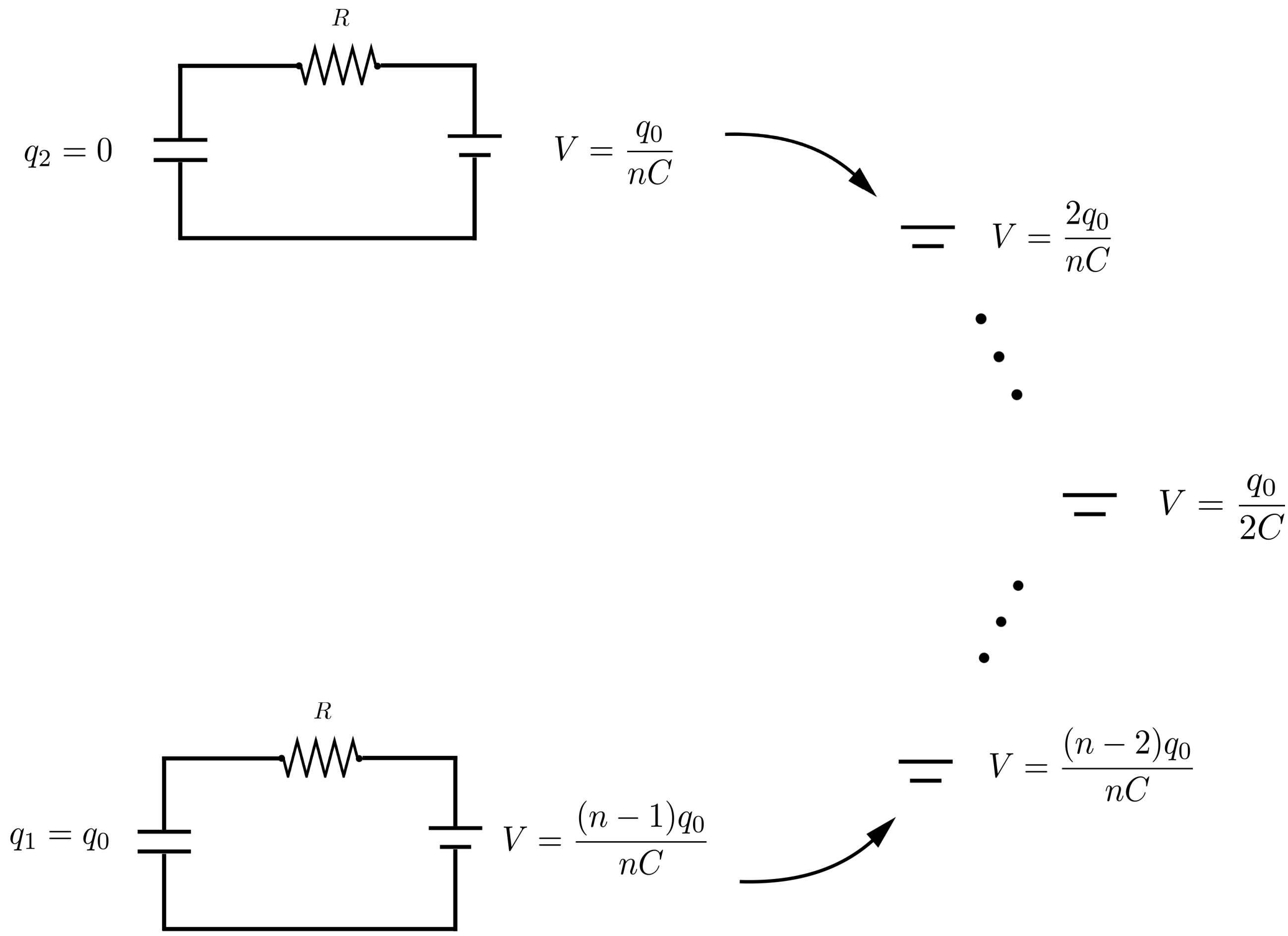

Figura 2 - Illustration of the quasistactic electric discharge process.

Since the model assumes that electrons do not interact with each other, its solution consists of finding the eigenstates of a quantum particle confined to a box and filling up these states according to Fermi-Dirac statistics. In order to obtain the expression for the electronic contribution to the entropy, we will use a well-known result for the heat capacity in this model [10, 11]

$$C_V = \frac{N_e k_B \pi^2}{2} \frac{k_B T}{\epsilon_F}, \tag{4}$$

where k_B is the Boltzmann constant and ϵ_F is the Fermi energy which is given by

$$\epsilon_F = \frac{\hbar^2 k_F^2}{2m} = \frac{\hbar^2}{2m}\left(\frac{3\pi^2}{V}\right)^{2/3} N_e^{2/3}, \tag{5}$$

where N_e is the total number of free electrons, $\hbar$ is the Planck constant divided by 2π, m is the electron mass, and V is the volume of the macroscopic sample. Expression (4) results from the Sommerfeld expansion for the heat capacity, which is accurate as long as the temperature is small compared to the Fermi temperature $T_F = \epsilon_F / k_B$. This condition is easily satisfied at room temperature, at which $T/T_F \sim 10^{-2}$.

Using the thermodynamic identity

$$C_V = T\left(\frac{\partial S}{\partial T}\right)_V, \tag{6}$$

in the free electron model, it is clear that the entropy S is equal to the heat capacity [2]

$$S = \frac{N_e k_B \pi^2}{2} \frac{k_B T}{\epsilon_F}. \tag{7}$$

Since we assume that the discharging process occurs at constant temperature, we can rewrite Eq. (7) as

$$S(T, V, N) = A(T, V) N_e^{1/3}, \tag{8}$$

where

$$A(T, V) \equiv \frac{k_B^2 \pi^2 m}{\hbar^2}\left(\frac{V}{3\pi^2}\right)^{2/3} T. \tag{9}$$

Note that the function $A(T, V)$ is always positive.

Now that we have the entropy as a function of all the relevant parameters of the problem, we are able to evaluate the entropy change of the system due to the

[2] If you prefer a derivation that goes from the beggining, see Ref. [13].

discharge process. We denote the entropy of each capacitor in the initial (I) and final (F) configuration by $S_I^{(i)}$ and $S_F^{(i)}$, respectively, where the superscript (i) indicates which capacitor (1 or 2). The system's total entropy change is the sum of the entropy changes of each capacitor,

$$\Delta S = \Delta S_1 + \Delta S_2 = (S_F^{(1)} - S_I^{(1)}) + (S_F^{(2)} - S_I^{(2)}). \quad (10)$$

Before we go on with the calculations, let's make an observation about the free electron gas model. We assume that the piece of metal that we are modelling possess N atoms, and that each one of these atoms contributes with a electrons to our sample (typically, a is 1 or 2). When an atom donates a electrons, it will be positively charged, with ae, where e is the electron's fundamental charge. So, when we consider all the sample, we conclude that our piece of metal is uncharged because the charge of the electrons cancels out the charge of the ions, although there exist free electrons.

Furthermore, we are considering metal plates that could be positively or negatively charged. We know that the excess charge q that a macroscopic metal can support is very small, when compared to the number of valence electrons in the sample. To see a good discussion on this subject, see Ref. [12], when Feynman introduces electric forces.

With these observations in mind, let us continue with our computations. Consider first the two plates of capacitor 1. Initially, one plate has an excess charge q_0, and the other, $-q_0$. We know that the electric charge is quantized, so we have $q_0 = N_0 e$, where N_0 is the excess number of electrons in the plate. Remembering that the total number of electrons in a plate is Na plus the excess charge, we have

$$\Delta S_1 = A(T,V)\left[\left(Na + \frac{q_0}{2e}\right)^{1/3} + \left(Na - \frac{q_0}{2e}\right)^{1/3} - \right. \\ \left. - \left(Na + \frac{q_0}{e}\right)^{1/3}\left(Na - \frac{q_0}{e}\right)^{1/3}\right], \quad (11)$$

because in the equilibrium situation the excess charge in each plate is $\pm q_0/2$, as discussed before.

Analogously, for the capacitor 2 we have

$$\Delta S_2 = A(T,V)\left[\left(Na + \frac{q_0}{2e}\right)^{1/3} + \left(Na - \frac{q_0}{2e}\right)^{1/3} - \right. \\ \left. 2\,(Na)^{1/3}\right]. \quad (12)$$

Now, to obtain ΔS, we substitute Eqs. (11) and (12) into Eq. (10)

$$\Delta S = A(T,V)(Na)^{1/3}\left[2\,(1+x)^{1/3} + 2\,(1-x)^{1/3} - \right. \\ \left. (1+2x)^{1/3} - (1-2x)^{1/3} - 2\right], \quad (13)$$

where we have defined $x \equiv q_0/2Nae$, the ratio between the excess charge and the number of valence electrons in the neutral sample. The expression obtained for ΔS in Eq. (13) is always positive, as shown in Fig. 3.

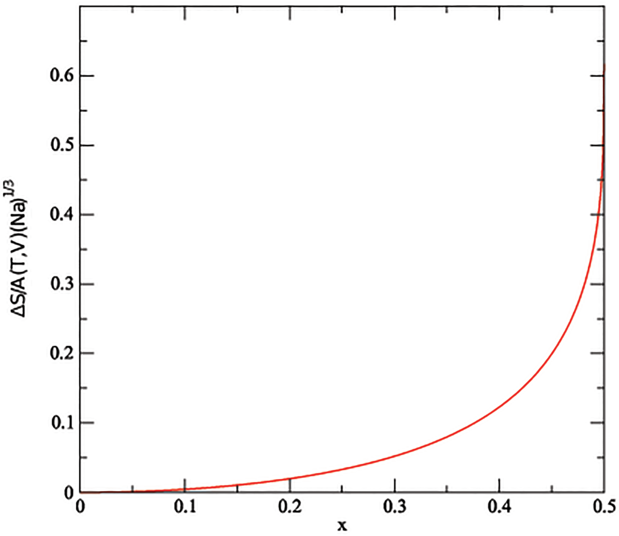

Figura 3 - Entropy change as a function of the ratio between the excess charge and the total number of valence electrons.

Although we obtained an expression for ΔS and have shown that $\Delta S > 0$, we can go a little further. As we discussed earlier, typically we have $x \ll 1$. Therefore, we can expand the Eq. (13), using the binomial expansion for all terms. Doing this, and collecting only second order terms, we find

$$\Delta S = \frac{4A(T,V)x^2(Na)^{1/3}}{9} = \frac{A(T,V)q_0^2(Na)^{-5/3}}{9e^2}. \quad (14)$$

In this case, we needed to consider second order terms in the expansion for ΔS, because the first order term vanishes. This can be noticed visually in Fig. 3, as the slope of the curve vanishes as $x \to 0$.

It is worthwhile to compare the entropy variation in Eq. (14) with the entropy change due to the heat exchange between the resistor and the heat reservoir. This determines whether the entropy variation calculated here is relevant to the irreversibility of the whole process. This ratio is given by

$$\frac{\Delta S_R}{\Delta S} = \frac{e^2\hbar^2(Na)}{nk_B^2T^2Cm_e}\rho^{2/3} \sim \frac{10^7}{n}, \quad (15)$$

where $\rho = \frac{Na}{V}$ is the valence electron density.

In Eq. (15) we used the values of the constants and considered $Na \sim 10^{20}$ and $C \sim 10^{-3}$ (the best value of C for typical values for the capacitance). Although the value of the ratio shows that ΔS is small when compared to ΔS_R, this difference could be reduced as one could arbitrarily grows the number of steps n in the quasistatic discharge process.

3. Energy calculation

Up to now we have discussed the entropy change in the discharging process. We can also discuss what happens to the energy. First, we note a somewhat counterintuitive fact. The electrostatic energy change ΔE_{El} of the system, between the initial and the final configurations does not depend on the resistance R. As long as $R > 0$,[3] we have

$$\Delta E_{El} = -\frac{q_0^2}{4C}. \tag{16}$$

However note that Eq. (16) only takes into account the electrostatic interaction. In order to obtain the total energy variation we should consider the energy change due to the rearrangement of the energy levels of the free electron gas. At room temperature the total energy of the free electron gas is well approximated by its ground state energy given by

$$E = \frac{3}{5} N \epsilon_F. \tag{17}$$

Although in the entropic calculation we used the first order Sommerfeld expansion, in the energy calculation the use of the ground state energy is justified by the fact that in the latter case $\Delta E \neq 0$ even for $T = 0$ K. For the entropy, however, $\Delta S = 0$ for $T = 0$ K and going beyond zero order is essential.

Using Eq. (17) to calculate the energy variation of the four capacitor plates, we obtain (see Fig. 4), analogously to the entropy calculation,

$$\Delta E_{FE} = B(V) N^{5/3} \Big[2(1+x)^{5/3} + 2(1-x)^{5/3} - (1+2x)^{5/3} - (1-2x)^{5/3} - 2 \Big], \tag{18}$$

where

$$B(V) = \frac{3}{5} \left(\frac{\hbar^2}{2m} \right) \left(\frac{3\pi^2}{V} \right)^{2/3}. \tag{19}$$

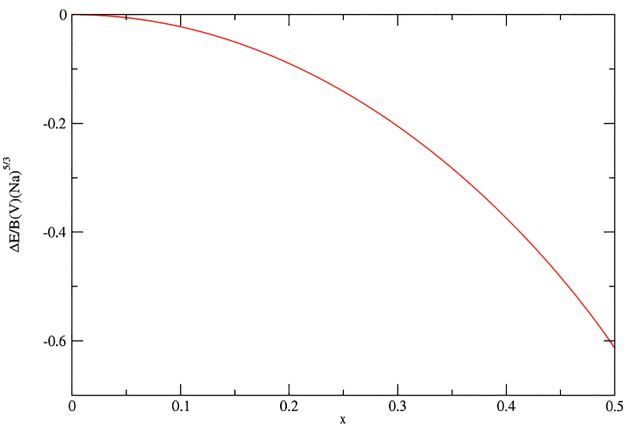

Figura 4 - Energy change as a function of the ratio between the excess charge and the total number of valence electrons.

Expanding Eq. (18) for $x \ll 1$ we obtain

$$\Delta E_{FE} = -\frac{20}{9} B(V) N^{5/3} x^2. \tag{20}$$

Thus, the total energy variation is given by

$$\Delta E = -\frac{q_0^2}{4C} - \frac{20}{9} B(V) N^{5/3} x^2. \tag{21}$$

4. Conclusions

In this work we introduced a new approach for the two-capacitor problem. Instead of looking at it from an electromagnetic point of view, we focus on its thermodynamic properties, mainly the entropy change due to the discharge process. Our goal was to show that this process is irreversible, which means that the entropy variation must be always positive ($\Delta S > 0$). We assumed that the resistor is in thermal contact with a heat reservoir at the same temperature, which leads to $\Delta S_R + \Delta S_{HR} = 0$. Thus, all the entropy variation is due to the change in the electrons' distributions in the plates of the capacitors. We assumed that the valence electrons in the metal plates can be modeled by the Free Electron Gas Model. Under these considerations, we obtained $\Delta S > 0$ and $\Delta E < 0$ for the whole process.

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References

[1] J.J. Halliwell, J. Pérez-Mercader and H.W. Zurek (eds), *Physical Origins of Time Assymmetry* (Cambridge University Press, New York, 1996).

[2] D.V. Schroeder, *An Introduction to Thermal Physics* (Addison-Wesley, San Francisco, 2000).

[3] C. Cuvaj, Am. J. Phys. **36**, 909 (1968).

[4] R.A. Powell, Am. J. Phys. **47**, 460 (1979).

[5] S.M. Al-Jaber and S.K. Salih, Eur. J. Phys. **21**, 341 (2000).

[6] T.B. Boykin, D.Hite and N. Singh, Am. J. Phys. **70**, 415 (2002).

[7] T.C. Choy, Am. J. Phys. **72**, 663 (2004).

[8] F. Heinrich, Am. J. Phys. **54**, 742 (1986).

[9] A. Sommerfeld, Rev. Mod. Phys. **3**, 1 (1931).

[10] N.W. Ashcroft and N.D. Mermin, *Solid State Physics* (Saunders, Philadelphia, 1976).

[3] The case $R = 0$ should be analyzed more carefully. See, for instance Refs. [5, 6].

[11] C. Kittel, *Introduction to Solid State Physics* (Wiley & Sons, New York, 2004), 8^{a} ed.

[12] R.P. Feynman, R.B. Leighton and M. Sands, *The Feynman Lectures on Physics* (Addison-Wesley Longman, Boston, 2005).

[13] A.L. Fetter and J.D. Walecka, *Quantum Theory of Many-Particle Systems* (Dover, Mineola, 2003).