

## Notas e Discussões

# Change of entropy in the spontaneous thermalization of two distinct blocks

(*Varição da entropia na termalização espontânea de dois blocos distintos*)

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When two blocks with distinct initial temperatures are placed in thermal contact, heat flows from the hotter block to the cooler until thermal equilibrium is reached. Given the initial temperatures, it is easy to predict the final temperature and, for identical blocks, to find a closed-form expression for the overall change of entropy, showing that it is a positive quantity. The change of entropy in the more general case of distinct blocks, however, is not mentioned in the introductory physics textbooks, which sounds as an indication of very complex mathematics underlying the problem. In this shortnote, I show that this is not the case by deriving a simple expression for the change of entropy and developing an elementary proof, at one-variable calculus level, that this change is a positive quantity.

**Keywords:** heat transfer, thermal equilibrium, entropy change.

Quando dois blocos com temperaturas iniciais distintas são postos em contato térmico, o calor passa do bloco mais quente para o mais frio até que o equilíbrio térmico seja atingido. Dadas as temperaturas iniciais, é fácil prever a temperatura final e, para blocos idênticos, encontrar uma fórmula simples para a variação da entropia do sistema, mostrando que ela é positiva. A variação de entropia no caso mais geral de blocos distintos, entretanto, não é mencionada nos livros-texto de física básica, o que soa como um indicativo de que há aspectos matemáticos muito complexos por trás do problema. Nesta nota, eu mostro que isto não ocorre e o faço determinando uma expressão simples para a variação da entropia e exibindo uma demonstração elementar, ao nível de cálculo diferencial de uma variável, de que tal variação é positiva.

**Palavras-chave:** transferência de calor, equilíbrio térmico, variação da entropia.

When all parts of a material system are at the same temperature, the system is in thermal equilibrium. Otherwise, heat will flow from the hotter parts to the cooler via conduction, convection, and radiation. Although spontaneous thermalization is a common phenomenon in our everyday experience, as well as a topic covered in introductory physics courses, some questions are not addressed in textbooks. For instance, for *distinct* blocks, there would be a simple expression for the change of entropy? Could it be proved that it is a positive quantity? The absence of these discussions in introductory physics textbooks has bothered me since my own undergraduate times because *heat exchange between two blocks is a central paradigm in thermodynamics, underlying the definition of temperature itself when we adopt the macroscopic point of view* [1]. Fortunately, on teaching introductory physics for a class of devoted students, last year, I was asked just about

those questions, which has motivated me to search for an elementary approach, presentable to first year undergraduates with knowledge of calculus of a single variable only, *without mentioning partial derivatives*.

Take into account two distinct blocks with initial (absolute) temperatures  $T_1 > T_2 > 0$ , kept inside an insulated container, initially separated each other by a partition, as seen in Fig. 1(a). At time  $t = 0$ , the partition is removed and the blocks are put in *thermal contact*, see Fig. 1(b). Heat then will flow from the hotter block to the cooler until thermal equilibrium to be reached, at some final temperature  $T_{eq}$ . Given the (in general, distinct) thermal capacities  $C_1$  and  $C_2$  of our blocks, the final temperature  $T_{eq}$  is easily determined as follows. On neglecting any heat exchange other than that between the blocks, all heat that leaves block 1 is absorbed by block 2, *i.e.*  $Q_1 + Q_2 = 0$ . On assuming that both  $C_1$  and  $C_2$  are independent of temperature,

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one has  $C_1 (T_{eq} - T_1) + C_2 (T_{eq} - T_2) = 0$ ,<sup>2</sup> so

$$T_{eq} = \frac{C_1 T_1 + C_2 T_2}{C_1 + C_2}, \quad (1)$$

a well-known result (see, *e.g.*, Sec. 18-8 of Ref. [2]).<sup>3</sup>

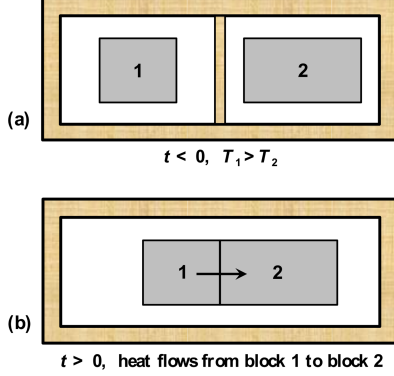


Figure 1 - Our system is an insulated container within which there are two blocks with distinct initial temperatures  $T_1 > T_2$ . Panel (a): initially, the blocks are separated apart by an adiabatic partition. Panel (b): At  $t = 0$  the partition is removed and the blocks are put in thermal contact. Heat will flow from block 1 to block 2, as indicated by the arrow, until thermal equilibrium to be reached, at a final temperature  $T_{eq}$ .

With the result for  $T_{eq}$  in hands, let us determine the overall change of entropy in our system – *i.e.*, its variation from the initial *nonequilibrium* state (at  $t = 0$ ) to the final equilibrium state, in which the blocks attain the same temperature  $T_{eq}$ .<sup>4</sup> We begin with the simpler case of *identical blocks*, for which  $C_1 = C_2 \equiv C$ . From Eq. (1), it promptly follows that  $T_{eq} = (T_1 + T_2)/2$ . As this spontaneous heat transfer involves an internal thermal irreversibility, with an equilibrium state only in the final state, we cannot directly replace the actual irreversible process with a reversible one because *the initial state itself is not an equilibrium state*. For block 1, imagine that it is put in contact successively with a series of reservoirs ranging in temperature from  $T_1$  to  $T_{eq}$ . We then have a very great number  $N$  of almost reversible processes and then the change of entropy can be determined by adding  $\delta Q/T$  for all these processes and taking the limits  $N \rightarrow \infty$  and  $\delta Q \rightarrow 0$ . The result is

$$\begin{aligned} \Delta S_1 &= \int_{R_1} \frac{\delta Q}{T} = \int_{T_1}^{T_{eq}} \frac{C dT}{T} = C \ln \left( \frac{T_{eq}}{T_1} \right) \\ &= -C \ln \left( \frac{T_1}{T_{eq}} \right), \end{aligned} \quad (2)$$

where  $R_1$  is the sequence of reversible processes described (see Sec. 9.8 of Ref. [1]).

<sup>3</sup>Since  $T_{eq}$  is a weighted mean of  $T_1$  and  $T_2$  (with statistical weights  $C_1$  and  $C_2$ , respectively), then  $T_2 < T_{eq} < T_1$ .

<sup>4</sup>Note that at any time  $t < 0$  the system is assumed to be composed by two insulated subsystems with distinct temperatures which are kept, each, in thermodynamic equilibrium. Of course, this demands the partition itself to be adiabatic.

<sup>5</sup>This theorem establishes that, given a real function  $f(x)$  that is continuous on an interval  $[a, b]$  and differentiable on the open  $(a, b)$ , if  $f(a) = f(b)$  then there is at least one number  $c$  in  $(a, b)$  such that  $f'(c) = 0$ . See, *e.g.*, Sec. 3.2 of Ref. [3].

cribed above. Analogously, for block 2 one has

$$\Delta S_2 = \int_{R_2} \frac{\delta Q}{T} = \int_{T_2}^{T_{eq}} \frac{C dT}{T} = C \ln \left( \frac{T_{eq}}{T_2} \right). \quad (3)$$

From Eqs. (2) and (3), one has

$$\begin{aligned} \Delta S_{system} &= \Delta S_1 + \Delta S_2 = C [2 \ln T_{eq} - \ln(T_1 T_2)] \\ &= 2C \left( \ln T_{eq} - \ln \sqrt{T_1 T_2} \right). \end{aligned} \quad (4)$$

For identical blocks,  $T_{eq} = (T_1 + T_2)/2$ , which is the arithmetic mean of  $T_1$  and  $T_2$ . It is easy to see that, for any positive numbers  $T_1 \neq T_2$ , the arithmetic mean is greater than the geometric mean  $\sqrt{T_1 T_2}$  (by squaring both means), then  $\ln T_{eq} > \ln \sqrt{T_1 T_2}$ , so  $\Delta S_{system} > 0$ . As our system is isolated from its neighborhood, the change of entropy of the universe  $\Delta S_{univ}$  equals  $\Delta S_{system}$  and then  $\Delta S_{univ} > 0$ , in agreement to the entropic version of the 2nd law of thermodynamics. This proof of increase of entropy can be found in some introductory physics textbooks (*e.g.*, Sec. 20-3 of Ref. [2]), though not in the general form given here, for their authors prefer to attribute numerical values for the initial temperatures and then compute  $\Delta S_{system}$ .

For *distinct blocks*, our rigorous analysis of the change of entropy demands the following result.

**Theorem 1 (A positive function)** *Given a real constant  $\alpha > 0$ , let*

$$f(x) \equiv (1 + \alpha) \ln \left( \frac{x + \alpha}{1 + \alpha} \right) - \ln x$$

*be a real function defined for all  $x \geq 1$ . Then  $f(x) > 0$  for all  $x > 1$ .*

*Proof.* The derivative of the given function  $f(x)$  is

$$f'(x) = \frac{1 + \alpha}{x + \alpha} - \frac{1}{x} = \frac{\alpha(x - 1)}{x(x + \alpha)},$$

so  $f(x)$  is a differentiable (hence continuous) function for all  $x \geq 1$ . Being  $\alpha > 0$ , then  $f'(x) = 0$  for  $x = 1$  only, and  $f'(x) > 0$  for all  $x > 1$ . This implies that  $f(x)$  is an increasing function for all  $x \geq 1$ , therefore  $f(x) \geq f(1) = 0$  for all  $x \geq 1$ . Now, assume (towards a contradiction) that there is some  $x > 1$  for which  $f(x) = 0$ . From Rolle's theorem,<sup>5</sup> this would imply the existence of at least one point  $c$ ,  $1 < c < x$ , such that  $f'(c) = 0$ , which contradicts the fact that  $f'(x) > 0$  for all  $x > 1$ . Then,  $f(x) > 0$  for all  $x > 1$ . □

By following the same strategy employed for identical blocks – *i.e.*, to divide the actual irreversible thermalization process into an infinity of reversible stages –, one has

$$\begin{aligned}\Delta S_1 + \Delta S_2 &= \int_{T_1}^{T_{eq}} \frac{C_1 dT}{T} + \int_{T_2}^{T_{eq}} \frac{C_2 dT}{T} \\ &= C_2 \ln\left(\frac{T_{eq}}{T_2}\right) - C_1 \ln\left(\frac{T_1}{T_{eq}}\right).\end{aligned}\quad (5)$$

From the expression for  $T_{eq}$  in Eq. (1), one finds

$$\begin{aligned}\Delta S_{system} &= C_2 (\ln T_{eq} - \ln T_2) - C_1 (\ln T_1 - \ln T_{eq}) \\ &= C_2 \ln\left(\frac{C_1 x + C_2}{C_1 + C_2}\right) + C_1 \ln\left(\frac{C_1 + C_2/x}{C_1 + C_2}\right) \\ &= C_2 \ln\left(\frac{C_1 x + C_2}{C_1 + C_2}\right) + C_1 \ln\left(\frac{C_1 x + C_2}{(C_1 + C_2)x}\right) \\ &= (C_1 + C_2) \ln\left(\frac{C_1 x + C_2}{C_1 + C_2}\right) - C_1 \ln x,\end{aligned}\quad (6)$$

where  $x \equiv T_1/T_2$ . On dividing both sides by  $C_1$ , one has

$$\frac{\Delta S_{system}}{C_1} = \left(1 + \frac{C_2}{C_1}\right) \ln\left(\frac{x + C_2/C_1}{1 + C_2/C_1}\right) - \ln x. \quad (7)$$

Finally, on putting  $\alpha = C_2/C_1$ , which is of course a positive constant, Eq. (7) reduces to

$$\frac{\Delta S_{system}}{C_1} = (1 + \alpha) \ln\left(\frac{x + \alpha}{1 + \alpha}\right) - \ln x, \quad (8)$$

From Theorem 1, we know that the function at the right-hand side is *positive* for any  $x > 1$ , so  $\Delta S_{system}/C_1 > 0$  whenever  $T_1 > T_2$ , which implies that  $\Delta S_{system} > 0$ . I could not find a such proof in literature.

For those interested in approaching this topic in a deeper mathematical level, *e.g.* by analyzing other irreversible processes (or paths) with the same initial and final states, in a more advanced course (*e.g.*, *Classical Thermodynamics* for physicists or *Heat and Mass Transfer* for engineers), a recent paper by Stilck and Brum (see Ref. [4]) would be more appropriate.

## References

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