

Mechanical refraction in action

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We study the analog of Snell's law for particles moving across the interface of two regions with two different potential energies, from two different points of view. First, a simple demonstration involving marbles and a potential step is shown. Then, from a theoretical point of view, this law describing mechanical refraction is derived from the Maupertuis and Jacobi variational principles, in close analogy with the well known derivation of Snell's law for refraction from Fermat's least time principle. Finally, the relativistic version of mechanical refraction is obtained by the Maupertuis principle, by trading the Newtonian dispersion relation with the relativistic one. The pedagogical significance of this treatment is discussed.

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1. Introduction

It is well known that, before the advent of Young's experiments, the hypothesis that light is made up of tiny particles was widely spread, following the influence of Newton. In fact, such a picture seemed capable of explaining light reflection and refraction. We retrace this history in some detail elsewhere [1].

If we consider a beam of light particles impinging on a barrier or an interface between two different transparent media, both the reflection and refraction law follow from conservation of the momentum component parallel to the barrier or interface. In particular, refraction occurs because particles, in going from one medium to the other, change their velocity, thereby changing the total momentum.

For example, consider a particle moving across the interface between two regions, 1 and 2, in which its speed is v_1 and v_2 respectively (cf. Figure 1 and Figure 2 below). Then conservation of the parallel component of $\mathbf{p} = m\mathbf{v}$ gives

$$p_1 \sin \vartheta_1 = p_2 \sin \vartheta_2, \quad (1)$$

which, combined with the definition of \mathbf{p} , translates into

$$v_1 \sin \vartheta_1 = v_2 \sin \vartheta_2. \quad (2)$$

This equation closely resembles Snell's law

$$n_1 \sin \vartheta_1 = n_2 \sin \vartheta_2. \quad (3)$$

However, in order to fully recover the latter it is necessary to assume that light is faster in media with higher refraction indices [1]. Although his assumption could be

perfectly reasonable in the XVII and XVIII centuries, it was subsequently shown to be wrong. In fact, the correct speed dependence for light refraction, which is

$$\frac{1}{v_1} \sin \vartheta_1 = \frac{1}{v_2} \sin \vartheta_2, \quad (4)$$

was first found by Fermat in 1662, from the least time principle. As discussed at length in [1], this result can be also obtained by abandoning Newtonian dynamics, and assuming that light particles obey relativistic dynamics, and are massless.

Equation (2) is the main focus of the present paper. First of all, in Section 2, we propose a simple demonstration in which Eq. (2) can be seen in action. Then, in Section 3, we see how it can be obtained from Maupertuis' least action principle, in parallel with the derivation of Snell's law from the Fermat principle. The derivation is also repeated starting from the variational principle in the Jacobi form. Finally, in Section 4, we show how the same variational principle can give the correct speed dependence if the Newtonian dispersion relation is traded with the relativistic one for massless particles. The necessary background in analytical mechanics is given in the supplementary material, where we also recall for completeness the derivation of Snell's law from Fermat's principle.

2. A simple demonstration

As said above, Eq. (2) can be easily be seen in action with a simple set-up, where a marble moves between two regions of different height, so that there is a rather abrupt difference of potential energy at the interface. Such a set-up can be reproduced by high school students. The demonstration is shown in the auxiliary video, and some

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screenshots of it are superimposed in Figure 1. In the picture, the marble comes from the right, then encounters a step after which its potential energy is higher. In our set-up the step is 0.7 cm high. Thus the marble is faster in the right half of the picture, and slower in the left half. The photographic sequence clearly shows that the slope of the trajectory changes, and that the right half line is closer to the normal to the interface, as expected. Observe that the time sequence can be reversed without affecting the trajectory.

3. Mechanical refraction from variational principles

In this Section we see how Eq. (2) can be derived from variational principles. In particular, we apply both the Maupertuis principle and its Jacobi form, which are outlined in the supplementary material. The derivation closely mimics the usual derivation of Snell's law (3) from Fermat's principle, which also is briefly recalled in the supplementary material.

In the following we refer to Figure 2.

Let us consider a point particle which moves on the plane, and suppose that the plane is divided into two parts in which the potential energy of the particle assumes two different values, U_1 and U_2 , so that its speed changes from v_1 to v_2 when crossing the interface between the two regions (the detailed relation between the potentials and the speeds does not concern us, and can be found e.g. in [2]). We choose our axes in such a way that the interface lies on the x -axis. Suppose for definiteness that the particle starts moving from point A in the region with potential energy U_1 , and arrives at point B in the region with potential energy U_2 . Notice that we do not specify

which speed is higher, or equivalently, which potential energy is lower, despite in Figure 2, for definiteness, we consider the $U_1 > U_2$ case. We remark that, in the case in which $U_2 > U_1$, we must make the additional assumption that the particle starts moving from A with a kinetic energy which is larger than the potential energy difference $U_2 - U_1$. Both points A and B are fixed. Our aim is to find out which trajectory is followed. We are not interested in the equations of motion, but only in the shape of the trajectory; moreover, the time of arrival in the point B is not fixed and energy is conserved. We are thus in the situation in which the Maupertuis principle can be applied (cf. supplementary material). Recall that this principle amounts to the statement:

$$\delta S_0 = \delta \int_{t_A}^{t_B} \sum_i p_i \dot{q}_i dt = 0. \quad (5)$$

In this case the action splits into two parts, one referring to the motion from A to the point at the interface P (where the particle is expected to change its velocity), and the other to the motion from P to B . In each of the two regions we can write

$$\sum_i p_i \dot{q}_i = \mathbf{p} \cdot \mathbf{v} = pv = mv^2, \quad (6)$$

where in the second equality we used the fact that $\mathbf{p} = m\mathbf{v}$ and \mathbf{v} are parallel. Therefore we can write

$$\begin{aligned} 0 &= \delta \int_{t_A}^{t_P} mv_1^2 dt + \delta \int_{t_P}^{t_B} mv_2^2 dt \\ &= mv_1^2 \delta(t_P - t_A) + mv_2^2 \delta(t_B - t_P), \end{aligned} \quad (7)$$

where in the second equality we brought the constant factors mv^2 outside the variations. At this point we write

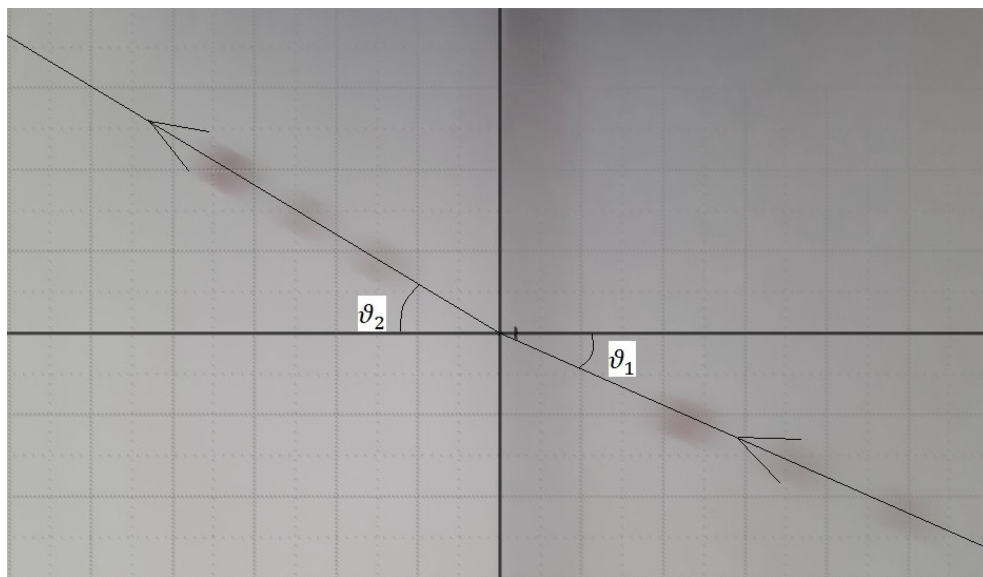


Figure 1: A mechanical experiment to see refraction of particles. In this case we are considering a marble which goes from a region where potential energy is lower (on the right) to a region where potential energy is higher (on the left). The thin line indicates the trajectory.

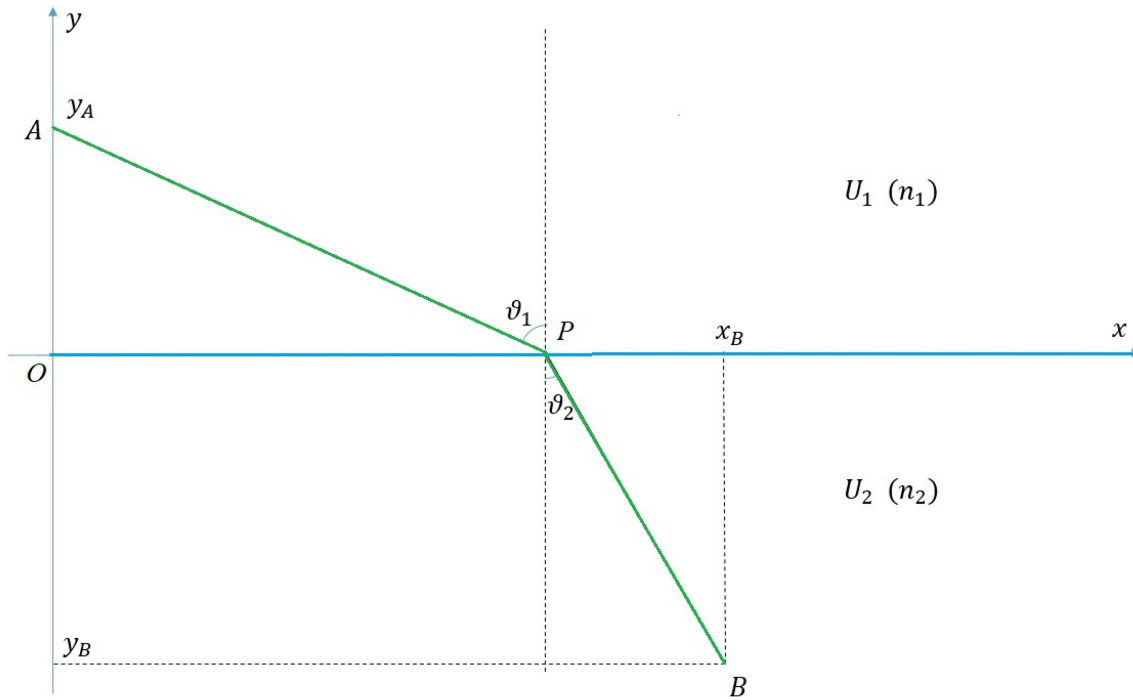


Figure 2: The path followed by a particle moving from point A in the region where its velocity is v_1 to point B in the region where its velocity is v_2 . The interface between the two regions coincides with the x -axis.

the time intervals as

$$t_P - t_A = \frac{\sqrt{x_P^2 + y_A^2}}{v_1};$$

$$t_B - t_P = \frac{\sqrt{(x_B - x_P)^2 + y_B^2}}{v_2}, \quad (8)$$

from which we get:

$$\delta(t_P - t_A) = \frac{x_P}{v_1 \sqrt{x_P^2 + y_A^2}} \delta x_P = \frac{\sin \vartheta_1}{v_1} \delta x_P; \quad (9)$$

$$\delta(t_B - t_P) = -\frac{x_B - x_P}{v_2 \sqrt{(x_B - x_P)^2 + y_B^2}} \delta x_P$$

$$= -\frac{\sin \vartheta_2}{v_2} \delta x_P, \quad (10)$$

hence

$$0 = \delta S_0 = (mv_1 \sin \vartheta_1 - mv_2 \sin \vartheta_2) \delta x_P, \quad (11)$$

i.e., since the variation δx_P is arbitrary

$$v_1 \sin \vartheta_1 = v_2 \sin \vartheta_2, \quad (12)$$

which is Eq. (2).

The same result can be obtained by starting directly from the variational principle in Jacobi's form. In that case in place of (5) we have (cf. the supplementary material)

$$\delta S_0 = \delta \int_A^B \sqrt{2mT} dl = 0. \quad (13)$$

Again the integral can be split into two parts, so we get

$$\delta S_0 = mv_1 \delta \int_A^P dl + mv_2 \delta \int_P^B dl$$

$$= m(v_1 \delta \overline{AP} + v_2 \delta \overline{PB}), \quad (14)$$

where $\overline{AP} = \sqrt{x_P^2 + y_A^2}$ and $\overline{PB} = \sqrt{(x_B - x_P)^2 + y_B^2}$. Therefore we have:

$$\delta \overline{AP} = \delta \sqrt{x_P^2 + y_A^2} = \frac{x_P}{\sqrt{x_P^2 + y_A^2}} \delta x_P = \sin \vartheta_1 \delta x_P, \quad (15)$$

and analogously

$$\delta \overline{PB} = \delta \sqrt{(x_B - x_P)^2 + y_B^2}$$

$$= -\frac{x_B - x_P}{\sqrt{(x_B - x_P)^2 + y_B^2}} \delta x_P$$

$$= -\sin \vartheta_2 \delta x_P. \quad (16)$$

Therefore

$$\delta S_0 = m(v_1 \sin \vartheta_1 - v_2 \sin \vartheta_2) \delta x_P = 0, \quad (17)$$

which gives

$$v_1 \sin \vartheta_1 = v_2 \sin \vartheta_2. \quad (18)$$

Thus we have shown that Eq. (2), which follows from Newtonian mechanics, can be put in a variational context just as Snell's law. As we know, the former does not describe refraction of light correctly, since it has the wrong dependence on speeds [1].

4. Snell's law from the Maupertuis principle and relativity

In this section we see how the use of the relativistic dispersion relation for light particles allows to derive the correct refraction law.

Let us start from the Maupertuis principle written in the form:

$$\delta S_0 = \delta \int_A^B \sum_i p_i dq_i = 0, \quad (19)$$

which does not use time explicitly and hence is more suited to a relativistic context. In our case

$$\sum_i p_i dq_i = \mathbf{p} \cdot d\mathbf{q} = p dl = \frac{E}{v} dl. \quad (20)$$

We recall (see [1]) that for a relativistic massless particle $E = pv$. In our case energy is conserved in both regions, therefore $E = p_1 v_1 = p_2 v_2$, and we can write:

$$0 = \delta \int_A^P \frac{E}{v_1} dl + \delta \int_P^B \frac{E}{v_2} dl = \frac{E}{v_1} \delta \overline{AP} + \frac{E}{v_2} \delta \overline{PB}. \quad (21)$$

Using (15) and (16), we obtain

$$\delta S_0 = E \left(\frac{1}{v_1} \sin \vartheta_1 - \frac{1}{v_2} \sin \vartheta_2 \right) \delta x_P, \quad (22)$$

from which

$$\frac{1}{v_1} \sin \vartheta_1 = \frac{1}{v_2} \sin \vartheta_2, \quad (23)$$

which, upon multiplication of both sides by c and defining $n_i = \frac{c}{v_i}$, is just Snell's law (3).

5. Conclusions

In this paper we have studied mechanical refraction, i.e. how the trajectory of a particle gets bent when crossing the interface between two regions in which it moves with different speeds, for example two regions in which it has different potential energies. We described a simple set-up in which the motion can be visualized, and then studied the problem from a variational point of view. The addition of derivations of (2) from variational principles gives a useful complement to this picture for undergraduate students. In fact, these topics are usually mentioned only in passing in university courses (a less detailed derivation of the mechanical refraction law from Maupertuis' principle can be found in [3]). Despite the inability of Eq. (2) to correctly describe light refraction, such a presentation allows students to see the Maupertuis variational principle and its Jacobi form at work within a simple yet nontrivial situation, and also the use of the former in a relativistic context. Moreover, the simple demonstration we described, due to its simplicity, can be proposed to high school students, and to undergraduates as well.

Supplementary material

The following online material is available for this article:

The Maupertuis' and Jacobi variational principles. Snell's law from Fermat's principle.

References

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