Revista Brasileira de Ensino de Física, vol. 45, e20230235 (2023)
www.scielo.br/rbef
DOI: https://doi.org/10.1590/1806-9126-RBEF-2023-0235

Some applications of Laplace transforms in models with impulses or discontinuous forcing functions

Diego Miranda Gonçalves[1], Lucas Antonio Caritá[*2]

[1]Universidade Estadual Paulista "Júlio de Mesquita Filho", Rio Claro, SP, Brasil.
[2]Instituto Federal de Educação, Ciência e Tecnologia de São Paulo, São José dos Campos, SP, Brasil.

Received on August 20, 2023. Revised on November 9, 2023. Accepted on November 14, 2023.

Commonly, in Ordinary Differential Equations courses, equations with impulses or discontinuous forcing functions are studied. In this context, the Laplace Transform of the Dirac delta function and unit step function is taught, which are used as forcing functions in theoretical equations. However, application in real situations is also an important part of the learning process. In this sense, most books are flawed regarding the practical applications of this type of equation. Therefore, the purpose of this work is to study the solution of differential equations under the action of discontinuous forcing functions or impulses, contextualized in Physics or Engineering problems, using Laplace transforms. For this, this article analyzes some physical systems that are not so explored in the literature, such as the galvanometer, a circuit used inside an ammeter or voltmeter to measure current or voltage. In addition, we also studied an R-L-C circuit (Resistor, Inductor and Capacitor), using the Laplace Transform to find the capacitor voltage, demonstrating an extremely useful way for solving electrical circuits in series or in parallel.
Keywords: Differential equations, Laplace transforms, Mathematical modeling.

1. Introduction

Differential equations can describe the way certain quantities vary with time, for example, mass-spring systems, electrical circuits in series, oscillations of vibrating membranes, or heat flows through isolated conductors [1]. In general, these equations are linked to initial conditions that present the initial state of such systems. According to [2], differential equations of the type

$$m\frac{d^2x}{dt^2}+\beta\frac{dx}{dt}+kx=f(t) \tag{1}$$

or

$$L\frac{d^2q}{dt^2}+R\frac{dq}{dt}+\frac{1}{C}q=E(t) \tag{2}$$

are examples of models of systems that have "forcing functions" that can represent either an external force $f(t)$ or an impressed voltage $E(t)$. It is very common for these forcing functions to be discontinuous and, thus, the more traditional methods for solving this type of equation can become very laborious. For example, the voltage applied to a circuit can be piecewise continuous and periodic, and behave like the "step" or "sawtooth" functions, as exemplified in Figure 1.

It is also common for differential equations to be under the influence of forcing functions with some kind of "impulse", as exemplified in Figure 2.

In cases where the equations are under the action of discontinuous or impulse forcing functions (as in Equations (1) and (2)), solving the differential equation can be laborious.

The Laplace transform provides important help in solving problems of this type. The name is due to the mathematician Pierre-Simon Laplace (1749–1827) who studied celestial mechanics and probability theory.

The first to study integral transforms as tools for solving differential equations was Euler (1707–1783) [5]. The results obtained by Euler were incorporated by Laplace in an article called *Théorie Analytique des Probabilités* at the beginning of the 19th-century [4]. According to [1], it was Spitzer (1737–1880) who associated Laplace's name with the transform

$$Y(s)=\int_a^b e^{st}y(t)dt \tag{3}$$

which was elaborated by Euler, but used extensively by Laplace.

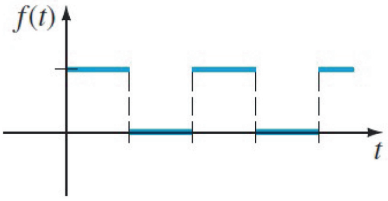
Also, according to [1], in 1910, Bateman (1882–1946) applied the following transform

$$Y(s)=\int_0^{+\infty} e^{-st}y(t)dt \tag{4}$$

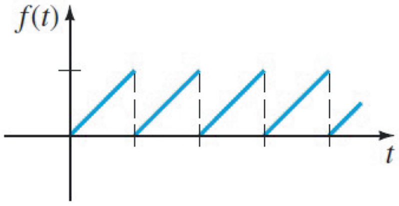
later named Laplace transform by Berntein (1880–1968), in Rutherford's radioactive decay equation (1871–1937)

$$\frac{dy}{dt}=-\lambda y \tag{5}$$

*Correspondence email address: lucasacarita@gmail.com



(a) Step function.



(b) Sawtooth function.

Figure 1: Examples of discontinuous forcing functions. Source: Zill [3].

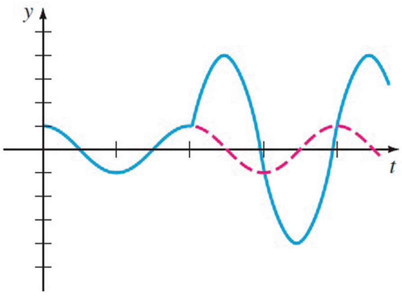


Figure 2: Example of forcing function with impulse. The dashed line shows how the function would continue if the impulse had not been applied. Source: Zill [3].

Such a transform began to be discussed in order to precisely justify some operational rules, which were used by Heaviside (1850–1925) at the end of the 19th century, with the aim of solving the equations of Maxwell's Electromagnetic Theory (1831–1879) [4]. With that, he contributed strongly to Maxwell's Electromagnetic Theory by reducing the 38 equations into only 4 fundamental equations of such theory [4].

At the beginning of the 20th century, after many attempts, this method of solving equations was successful after much work and dedication by mathematicians such as Bromwich (1875–1929), Carson (1886–1940) and Van der Pol (1889–1959) [6].

The Laplace transform is an integral transform with kernel $K(s,t) = e^{-st}$ and constitutes a fundamental tool to solve initial value problems in differential equations. For this reason, it is widely used as a tool in studies of electrical circuits, systems and signals, control and automation, probability and statistics, bioengineering, mechanical engineering, and other areas [5].

Often, in Calculus or Differential Equations courses at universities, the tools are studied but without an appropriate context. Functions such as the unit step or the Dirac delta appear in initial value problems, students learn how to compute their transforms, but there is not a glimmer of application. Thus, the purpose of this work is to study the solution of differential equations under the action of discontinuous forcing functions or impulses, contextualized in typical problems of Physics or Engineering, using Laplace transforms.

2. Mathematical Background

2.1. Laplace transform

Let $f : [0, +\infty[\subset \mathbb{R} \longrightarrow \mathbb{C}$ be a function and s be a real or complex parameter. We define the *Laplace transform* of f as

$$F(s) = \mathcal{L}(f(t)) = \int_0^{+\infty} e^{-st} f(t)\, dt \tag{6}$$

when the improper integral converges.

In [7], you can find a demonstration that if f has exponential order[1] γ piecewise continuous on $[0, x]$ for all $x > 0$. Then the integral of Equation (6) is convergent, if $Re(s) > \gamma$.

We define $\mathbb{L}$ as the set of all functions of type $f : [0, +\infty[\subset \mathbb{R} \longrightarrow \mathbb{C}$ such that the Laplace transform exists for some value of s. A function $f : [0, +\infty[\subset \mathbb{R} \longrightarrow \mathbb{C}$ is said to be admissible if it is piecewise continuous at $[0, x]$ for all $x > 0$ and have exponential order γ. Admissible functions obviously belong to $\mathbb{L}$. However, there are functions in $\mathbb{L}$ that do not satisfy one or both of the conditions for being admissible. For example, $f(t) = 2t\, e^{t^2} \cos(e^{t^2})$, has a Laplace transform, that is, it belongs to $\mathbb{L}$, however, is not of exponential order γ for any $\gamma > 0$.

A relevant observation is that the Laplace transform is linear, that is, given $f_1 \in \mathbb{L}$ for $Re(s) > \alpha$ and $f_2 \in \mathbb{L}$ for $Re(s) > \beta$, then $c_1 f_1 + c_2 f_2 \in \mathbb{L}$ to $Re(s) > max\,\{\alpha, \beta\}$, and

$$\mathcal{L}(c_1 f_1 + c_2 f_2) = c_1 \mathcal{L}(f_1) + c_2 \mathcal{L}(f_2) \tag{7}$$

for arbitrary constants c_1 and c_2.

In addition, the following theorems hold, whose proofs can, for example, be found in [1]:

Theorem 1 *Let f be a differentiable function of exponential order γ, with f' piecewise continuous on $[0, x]$ for all $x > 0$. Then,*

$$\mathcal{L}(f'(t)) = s\mathcal{L}(f(t)) - f(0) \tag{8}$$

for $Re(s) > \gamma$.

[1] It is understood that a function f has exponential order γ if there are constants $M > 0$ and $\gamma > 0$ such that for some $t_0 \geq 0$, $|f(t)| \leq Me^{\gamma t}$, $\forall\, t \geq t_0$.

 DOI: https://doi.org/10.1590/1806-9126-RBEF-2023-0235

Theorem 2 *Let f and f' be differentiable functions of exponential order γ, with f'' piecewise continuous on $[0, x]$, for all $x > 0$. Then,*

$$\mathcal{L}(f''(t)) = s^2\mathcal{L}(f(t)) - sf(0) - f'(0) \tag{9}$$

for $Re(s) > \gamma$.

More general:

Theorem 3 *Suppose that $f, f', \dots, f^{(n-1)}$ are differentiable in $[0, +\infty[$ and of exponential order γ with $f^{(n)}$ continuous piecewise on $[0, x]$ for all $x > 0$. Then,*

$$\mathcal{L}(f^{(n)}(t)) \\ = s^n\mathcal{L}(f(t)) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0). \tag{10}$$

We will think about the inverse process of the transform, knowing a function $F(s)$ we will find $f(t)$, if possible. If $F(s) = \mathcal{L}(f(t))$, then we define $f(t) = \mathcal{L}^{-1}(F(s))$. Such a definition makes sense if we can guarantee uniqueness, since different functions can have the same Laplace transform. Lerch's theorem guarantees the conditions we need.

Theorem 4 (Lerch's Theorem) *Let $f, g : [0, +\infty[\subset \mathbb{R} \longrightarrow \mathbb{C}$ be continuous functions with $\mathcal{L}(f(t)) = F(s)$ and $\mathcal{L}(g(t)) = G(s)$ such that $F(s) = G(s)$, $Re(s) > \alpha$. Then $f(t) = g(t)$.*

A very detailed proof of this theorem can be seen in [8]. The inverse transform is also linear.

One of the practical features of the Laplace transform is that it can be applied to discontinuous functions f. In these instances, it must be borne in mind that when the inverse transform is invoked, there are other functions with the same $\mathcal{L}^{-1}(F(s))$ [1]. However, this does not occur if the points where the original functions differ are finite, even if they are discontinuous. In other words, functions which differ by a finite number of points on $[0, +\infty[$ have the same Laplace transform.

The Laplace transform is a very important tool to solve Initial Value Problems (IVP) in a very simplified way. The method basically works by transforming a complicated differential equation in the variable t into an algebraic equation, simpler to deal with, in the complex variable s. For example, considering the notations $F(s) = \mathcal{L}(f(t))$ and $Y(s) = \mathcal{L}(y(t))$ for the Laplace transforms of the functions f and y on the t variable, the Initial Value Problem

$$y'' + 4y' + 5y = f(t) \tag{11}$$

with $y(0) = 0$ and $y'(0) = 0$, applying the transform in Equation (11) and using linearity, it is taken into

$$Y(s) = \frac{F(s)}{s^2 + 4s + 5}. \tag{12}$$

Obtaining the IVP solution $y(t)$ explicitly is possible by applying the inverse transform concept and using linearity once more, since $y(t) = \mathcal{L}^{-1}(Y(s))$.

2.2. Solving IVPs with Laplace transforms

We have seen that $\mathcal{L}\left(y^{(n)}(t)\right)$, when it exists, depends on $y(t)$ and all of its derivatives of order less than n at $t = 0$. In this way, the Laplace Transform is suitable for initial value problems with constant coefficients, being able to transform $y(t)$ into a simple algebraic function $Y(s)$. So let us consider an initial value problem

$$a_n\frac{d^ny(t)}{dt^n} + a_{n-1}\frac{d^{n-1}y(t)}{dt^{n-1}} + \cdots \\ + a_1\frac{dy(t)}{dt} + a_0y(t) = g(t) \tag{13}$$
$$y(0) = y_0 \quad y'(0) = y_0', \dots, y^{(n-1)}(0) = y_0^{(n-1)}$$

where $a_1, \dots, a_n$ and $y_0, y_0', \dots, y_0^{(n-1)}$ are constants, as well as $g, y, y', \dots, y^{(n)} \in \mathbb{L}$. The function g is called a forcing function and the solution $y = y(t)$ is called the answer. By the linearity of the Laplace transform, we get

$$a_n\mathcal{L}\left(\frac{d^ny(t)}{dt^n}\right) + a_{n-1}\mathcal{L}\left(\frac{d^{n-1}y(t)}{dt^{n-1}}\right) + \cdots \\ + a_0\mathcal{L}\left(y(t)\right) = \mathcal{L}\left(g(t)\right). \tag{14}$$

By Theorem 3 we find

$$a_n\left[s^nY(s) - s^{n-1}y(0) - \cdots - y^{(n-1)}(0)\right] \\ + a_{n-1}\left[s^{n-1}Y(s)s^{n-2}y(0) - \cdots - y^{(n-2)}(0)\right] \\ + \cdots + a_0Y(s) = G(s), \tag{15}$$

where $G(s) = \mathcal{L}(g(t))$ and $Y(s) = \mathcal{L}(y(t))$.

That is,

$$\left[a_ns^n + a_{n-1}s^{n-1} + \cdots + a_0\right]Y(s) \\ = a_n\left[s^{n-1}y_0 + \cdots + y_0^{(n-1)}\right] \\ + a_{n-1}\left[s^{n-2}y_0 + \cdots + y_0^{(n-2)}\right] + \cdots + G(s). \tag{16}$$

Thus, isolating $Y(s)$, we obtain $y(t)$ as follows

$$y(t) = \mathcal{L}^{-1}\left(Y(s)\right). \tag{17}$$

The scheme presented in Figure 3 shows us the step by step to solve a IVP as explained.

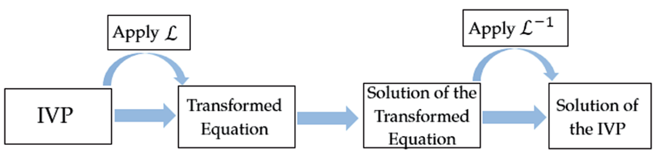


Figure 3: Procedure for solving IVPs with Laplace transform. Source: The authors.

2.3. Discontinuous forcing functions and impulses

In modeling some Physics situations, such as electrical circuits or mechanical vibrations, it is common to use non-homogeneous ODEs with constant coefficients. The non-homogeneous term is called forcing (or external force) and is commonly a piecewise continuous function with discontinuous jumps (therefore discontinuous). We can use the unit step function to represent discontinuous jumps without the need to specify the function piece by piece. Also very common are forcing functions with impulses, and for that we can use the unit impulse function.

We will present the step and unit impulse functions as forcing and the calculation of their transforms.

2.3.1. Unit step function

In this section we will develop a concept that is widely used, mainly in electrical physics, called the unit step function.

Consider $a \in \mathbb{R}$. We define u_a the unit step function of index a, as

$$u_a(t) = \begin{cases} 0, & 0 \le t < a \\ 1, & t \ge a \end{cases}. \tag{18}$$

Another notation, also quite useful, is $u_a(t) = u(t-a)$. If $a = 0$, we can denote $u_0(t) = u(t)$.

After defining the unit step function we can calculate its transform

$$\begin{aligned} \mathcal{L}\left(u_a(t)\right) &= \int_0^{+\infty} e^{-st} u_a(t)\, dt \\ &= \int_0^a e^{-st} 0\, dt + \int_a^{+\infty} e^{-st}\, dt \\ &= \int_a^{+\infty} e^{-st}\, dt \\ &= -\frac{e^{-st}}{s}\Big|_a^{+\infty} \\ &= \frac{e^{-as}}{s} \end{aligned} \tag{19}$$

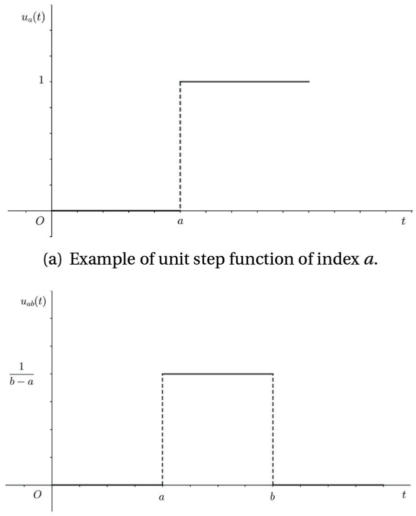
since $Re(s) > 0$.

In this way, we can also write

$$\mathcal{L}^{-1}\left(\frac{e^{-sa}}{s}\right) = u_a(t). \tag{20}$$

Consider $a, b \in \mathbb{R}$ with $0 \le a < b$. We define u_{ab}, the Step Function of index ab, as

$$u_{ab}(t) = \frac{1}{b-a}(u_a(t) - u_b(t)) = \begin{cases} 0, & t < a \\ \dfrac{1}{b-a}, & a \le t < b \\ 0, & t \ge b \end{cases}. \tag{21}$$



(a) Example of unit step function of index a.

(b) Example of unit step function of index ab.

Figure 4: Unit step function examples. Source: The authors.

Then

$$\begin{aligned} \mathcal{L}(u_{ab}(t)) &= \int_0^{+\infty} e^{-st} u_{ab}(t)\, dt \\ &= \int_0^a e^{-st} 0\, dt + \int_a^b \frac{e^{-st}}{b-a}\, dt + \int_b^{+\infty} e^{-st} 0\, dt \\ &= \int_a^b \frac{e^{-st}}{b-a}\, dt \\ &= -\frac{e^{-st}}{s(b-a)}\Big|_a^b \\ &= \frac{e^{-as} - e^{-bs}}{s(b-a)}. \end{aligned} \tag{22}$$

In this way, we can also write

$$\mathcal{L}^{-1}\left(\frac{e^{-as} - e^{-bs}}{s(b-a)}\right) = u_{ab}(t). \tag{23}$$

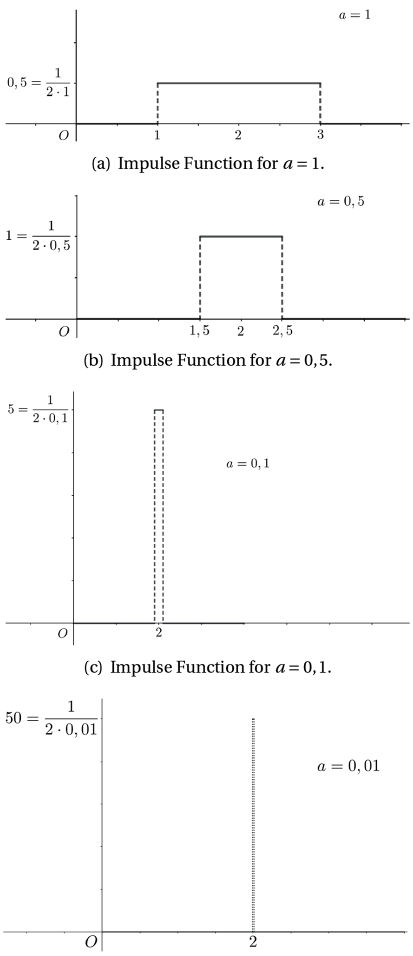
The step functions, graphed in Figure 4, are used to denote in a more simplified way some functions that have discontinuous jumps, as we will see later. Such functions can also represent an "on/off" duality; u_a, for example, serves as a tool to describe a function turned on from a value $a \in \mathbb{R}$ and turned off before that (or vise versa).

2.3.2. Unit impulse function

The function

$$\delta_a(t - t_0) = \begin{cases} 0, & 0 \le t < t_0 - a \\ \dfrac{1}{2a}, & t_0 - a \le t < t_0 + a \\ 0, & t \ge t_0 + a \end{cases} \tag{24}$$

is called a unit impulse, with $a > 0$ and $t_0 > 0$. Generally used to mathematically model external forces of large



(a) Impulse Function for $a = 1$.

(b) Impulse Function for $a = 0,5$.

(c) Impulse Function for $a = 0,1$.

(d) Impulse Function for $a = 0,01$.

Figure 5: Impulse Functions examples. Source: The authors.

amplitudes and in short periods, mainly in mechanical and electrical systems subject to external force actions. For small values of a, $\delta_a(t-t_0)$ is a practically constant function of great intensity acting for a small time interval close to the instant t_0. Figure 5 illustrates the behavior of $\delta_a(t-2)$ as $a \to 0$.

Such a function can be used to apply an impulse to a system. For example, a vibrating airplane wing could be struck by lightning, a mass on a spring could be given a sharp blow by a ball peen hammer, and a ball (baseball, golf ball, tennis ball) could be sent soaring when struck violently by some kind of club (baseball bat, golf club, tennis racket) [2].

The name unit impulse comes from the following property

$$
\begin{aligned}
&\int_0^{+\infty} \delta_a(t-t_0)\,dt \\
&\quad= \int_0^{t_0-a} 0\,dt + \int_{t_0-a}^{t_0+a} \frac{1}{2a}\,dt + \int_{t_0+a}^{+\infty} 0\,dt \\
&\quad= \frac{1}{2a} t\Big|_{t_0-a}^{t_0+a} \\
&\quad= \frac{1}{2a}(t_0+a-t_0+a) \\
&\quad= \frac{2a}{2a} = 1. \qquad (25)
\end{aligned}
$$

Let us now introduce the Dirac delta function. Suppose that f is a continuous function on $t = t_0$. The unit impulse $\delta(t-t_0)$ such as

$$\int_{-\infty}^{+\infty} \delta(t-t_0) f(t)\,dt = f(t_0) \qquad (26)$$

is called the Dirac delta function[2].

In order to achieve Equation (26), we define $F(t) = \int_a^t f(x)\,dx$ and compute

$$
\begin{aligned}
\int_{-\infty}^{+\infty} \delta(t-t_0) f(t)\,dt &= \lim_{a\to 0} \int_{-\infty}^{+\infty} \delta_a(t-t_0) f(t)\,dt \\
&= \frac{1}{2a}\int_{-a}^{a} f(t)\,dt \\
&= \frac{F(a)-F(-a)}{2a} \\
&= F'(0) \\
&= f(t_0). \qquad (27)
\end{aligned}
$$

Equation (26) is known as filter property. Using this property, although $\delta(t-t_0)$ is not a proper function, we are still able to obtain its Laplace transform, resulting in

$$\mathcal{L}(\delta(t-t_0)) = e^{-st_0}, \qquad (28)$$

for $t_0 > 0$.

Also, if $t_0 = 0$, we consider

$$\mathcal{L}(\delta(t)) = \lim_{t_0\to 0^+} \mathcal{L}(\delta(t-t_0)) = 1. \qquad (29)$$

3. Applications

After the previous theoretical foundation, this section presents some applications using the Laplace transform as a mathematical tool for solving ordinary differential equations that govern physical phenomena such as mass-spring systems and electric circuits.

3.1. Discussing Newton's 2nd law

Newton's Second Law is a foundational principle of classical dynamics, providing a framework for understanding various phenomena in classical mechanics. It is often associated with Newton's First Law, especially in cases where the net force acting on a particle is zero. In such instances, a particle remains either in uniform motion along a straight line or at rest, as there is no acceleration ($\mathbf{a} = \mathbf{0}$).

[2] Actually $\delta(t-t_0)$ is not a function in the usual sense of the word.

In some contexts, the Second Law is oversimplified by being reduced to a mere definition of force. However, this reduction undermines its physical depth because forces acting on a particle are determined by its interactions with other particles in a given inertial reference frame. Consequently, the force $\mathbf{F}$, which represents the resultant of these interactions, dictates the state of the particle.

In classical mechanics, the Second Law enables the definition of inertial mass m. In this realm of physics, mass is typically considered a constant, maintaining its fixed nature as an inherent property of the particle. Special relativity, on the other hand, introduces mass variation with velocity, thereby extending the applicability of the Second Law beyond classical mechanics and into the domain of relativistic mechanics, which encompasses particles moving at significant fractions of the speed of light in a vacuum.

The equation

$$\mathbf{F} = m\mathbf{a} \tag{30}$$

does not characterize the original formulation of Newton himself in relation to the 2nd Law. Newton began his study by defining it from the momentum or linear momentum that says:

"The momentum is measured directly by velocity and its mass". That is, the linear momentum of a given particle is the product of its mass and velocity

$$\mathbf{p} = m\mathbf{v}. \tag{31}$$

By differentiating the Equation (31) with respect to t and assuming that the mass m does not vary with time, we obtain

$$\frac{d\mathbf{p}}{dt} = m\frac{d\mathbf{v}}{dt} = m\mathbf{a}. \tag{32}$$

Using Equation (30) we arrive at the following expression

$$\frac{d\mathbf{p}}{dt} = \mathbf{F} \tag{33}$$

which evidences the formulation given by Newton about the 2nd Law:

"The change in momentum is proportional to the force impressed, and has the direction of the force."

That is, force is the time rate of variation of momentum.

Based on this definition, an application using Newton's 2nd Law and the Laplace Transform can be discussed for its mathematical resolution.

3.1.1. Impulsive force

According to [9] (Chapter 15 – page 739), an impulsive force acting on a particle of mass m, by Newton's 2nd Law, is

$$m\frac{d^2x}{dt^2} = F(t) \tag{34}$$

where $x(t)$ is the position of the particle, $F = P\delta(t)$ and P is a constant. Applying the Laplace transform on both sides of the Equation (34), we get

$$m\mathcal{L}(x'') = P\mathcal{L}(\delta(t)) \tag{35}$$

$$ms^2X(s) - msx(0) - mx'(0) = P. \tag{36}$$

Given the initial conditions for position $x(0) = 0$ and velocity $x'(0) = 0$, we have

$$ms^2X(s) = P. \tag{37}$$

Then,

$$X(s) = \frac{P}{ms^2}. \tag{38}$$

Applying the inverse Laplace transform,

$$\begin{aligned} \mathcal{L}^{-1}(X(s)) &= \frac{P}{m}\mathcal{L}^{-1}\left(\frac{1}{s^2}\right) \\ x(t) &= \frac{P}{m}t. \end{aligned} \tag{39}$$

Differentiating the Equation (39) with respect to t, we get

$$\frac{dx(t)}{dt} = \frac{P}{m}. \tag{40}$$

3.1.2. Ballistic galvanometer

According to [10] (Chapter 5 – page 287), a ballistic galvanometer is employed for the measurement of electric current, primarily in ammeters and voltmeters. It consists of a coil connected to its terminals. Consequently, any change in flux within the coil induces an electromotive force (e.m.f.) denoted as ε. As a result, the electric current i is directly proportional to the e.m.f. registered by the galvanometer.

This particular type of galvanometer is often referred to as a D'Arsonval galvanometer[3]. The current passing through the ballistic galvanometer is transient, characterized by impulsive behavior, and exists only in response to variations in the magnetic flux associated with the coil. Consequently, the galvanometer records transient current pulses as they pass through it. The measurement process yields non-constant deflection, and the galvanometer's scale is calibrated such that it allows for the measurement of both the current and the corresponding flux changes generated by these pulses.

The angular deflection of the galvanometer (mobile system) is directly proportional to the electric current but only when the magnetic flux produced by the current reaches its maximum value. This condition necessitates a mobile system with a high moment of inertia I, enabling

[3] For more details the reader can consult the book [10], page 76, section 2.7.

the system to oscillate for extended periods, typically between 10 and 15 seconds. The moment of inertia is increased by adding weights to the moving system, resulting in prolonged, damped oscillations with a low damping ratio. This low damping ensures that the initial deflection is of substantial magnitude, easily discernible on a numerical scale.

Finally, this effect can be characterized mathematically, where the resulting torque τ found in this electrical device is ki, where i is a current pulse and k is a constant. Knowing that i is a current of fast duration, we have

$$ki = kq\delta(t) \tag{41}$$

where q is the total charge and $\delta(t)$ is the Dirac delta function. By the fundamental principle of dynamics for rotations [10], we obtain

$$\tau = I\frac{d^2\theta}{dt^2}$$
$$I\frac{d^2\theta}{dt^2} = kq\delta(t) \tag{42}$$

where $\theta(t)$ is the angular displacement of the galvanometer needle and I is the moment of inertia.

From the Equation (42), the Laplace transform can be applied

$$I\mathcal{L}(\theta'') = kq\mathcal{L}(\delta(t)) \tag{43}$$

$$I(s^2\Theta(s) - s\theta(0) - \theta'(0)) = kq \tag{44}$$

where $\Theta(s) = \mathcal{L}(\theta(t))$. As $\theta(0) = 0$ and $\theta'(0) = 0$ indicate the angular displacement and angular velocity, respectively, at the initial instants, we have

$$Is^2\Theta(s) = kq \tag{45}$$

$$\Theta(s) = \frac{kq}{I}\frac{1}{s^2}. \tag{46}$$

Applying the inverse Laplace transform,

$$\mathcal{L}^{-1}(\Theta(s)) = \frac{kq}{I}\mathcal{L}^{-1}\left(\frac{1}{s^2}\right)$$
$$\theta(t) = \frac{kq}{I}t. \tag{47}$$

Differentiating the Equation (47) with respect to t, we have

$$\frac{d\theta(t)}{dt} = \frac{kq}{I}$$
$$I\frac{d\theta(t)}{dt} = kq \tag{48}$$
$$I\omega = kq$$

where $I\omega$ is called angular momentum (denoted by L).

Therefore, from Equation (48) we certify that the function of the current pulse i is to transfer kq units of angular momentum to the galvanometer.

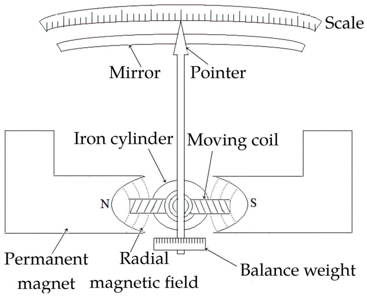


Figure 6: Example of a galvanometer. Source: The authors.

Figure 6 shows the configuration of the ballistic galvanometer. The moving coil (protected by an iron cylinder) is surrounded by a magnetic field produced by a permanent magnet. With a current, it is possible to impose another magnetic field, which from the magnetic field produced by the magnet has a resultant field. This field is responsible for generating a magnetic force and, consequently, a torque capable of moving the pointer, which is connected to an axis also connected to the mobile coil. The balance weight helps to balance pointer torque and the Mirror assists in measuring electrical current.

3.2. Coupled systems

The Laplace transform can be applied to determine the solution of a system excited by a forcing function, which includes harmonic and periodic systems, transforming differential equations into simpler algebraic equations. Another very important advantage is the fact that it handles discontinuous equations very easily, in addition to taking into account the initial conditions.

We will discuss a system of coupled differential equations, where $x_1(t)$ and $x_2(t)$ are mixed, as modeled in [11] (Chapter 5 – page 192). However, by employing the Laplace transform method it is possible to decouple them into two simple algebraic equations in the variable s. Applying the inverse Laplace transform, the two solutions $x_1(t)$ and $x_2(t)$ are found.

3.2.1. Mass-spring system with two boxcars

Consider two boxcars with masses $m_1 = M$ and $m_2 = m$ that are connected by a spring with spring constant K, as shown in Figure 7. The boxcar with mass $m_1 = M$ is subject to an impulsive force $F = F_0\delta(t)$. Let's determine the solutions $x_1(t)$ and $x_2(t)$ using the Laplace Transform method, considering that the displacements and velocities are zero at the initial instant.

By the fundamental principle of dynamics, the equations of motion of the two boxcars can be explained as

$$Mx_1'' + Kx_1 - Kx_2 = F_0\delta(t) \tag{49}$$

$$-Kx_1 + mx_2'' + Kx_2 = 0. \tag{50}$$

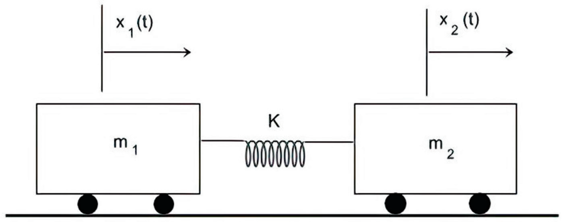


Figure 7: Two boxcars attached to a spring of spring constant K. Source: The authors.

In Equation (49) applying the Laplace transform, we get

$$M\mathcal{L}(x_1'') + K\mathcal{L}(x_1) - K\mathcal{L}(x_2) = F_0\mathcal{L}(\delta(t)). \tag{51}$$

Thus,

$$M(s^2X_1(s) - sx_1(0) - x_1'(0)) + KX_1(s) - KX_2(s) = F_0. \tag{52}$$

Once $x_1(0) = 0$ and $x_1'(0) = 0$,

$$Ms^2X_1(s) + KX_1(s) - KX_2(s) = F_0. \tag{53}$$

Similarly in Equation (50)

$$-K\mathcal{L}(x_1) + m\mathcal{L}(x_2'') + K\mathcal{L}(x_2) = 0 \tag{54}$$

$$-KX_1(s) + m(s^2X_2(s) - sx_2(0) - x_2'(0)) + KX_2(s) = 0. \tag{55}$$

Once $x_2(0) = 0$ and $x_2'(0) = 0$,

$$-KX_1(s) + ms^2X_2(s) + KX_2(s) = 0. \tag{56}$$

After some manipulations

$$X_1(s) = \frac{F_0(ms^2 + K)}{s^2(Mms^2 + K(M+m))} \tag{57}$$

$$X_2(s) = \frac{F_0K}{s^2(Mms^2 + K(M+m))}. \tag{58}$$

After partial fractions in Equations (57) and (58), we get

$$X_1(s) = \frac{F_0}{M+m}\left(\frac{1}{s^2} + \frac{m}{M}\left(\frac{1}{s^2 + K\left(\frac{1}{M} + \frac{1}{m}\right)}\right)\right) \tag{59}$$

$$X_2(s) = \frac{F_0}{M+m}\left(\frac{1}{s^2} - \frac{1}{s^2 + K\left(\frac{1}{M} + \frac{1}{m}\right)}\right). \tag{60}$$

Taking $\omega^2 = K\left(\frac{1}{M} + \frac{1}{m}\right)$ as the angular frequency of the mass-spring system, we have

$$X_1(s) = \frac{F_0}{M+m}\left(\frac{1}{s^2} + \frac{m}{\omega M}\left(\frac{\omega}{s^2+\omega^2}\right)\right) \tag{61}$$

$$X_2(s) = \frac{F_0}{M+m}\left(\frac{1}{s^2} - \frac{1}{\omega}\left(\frac{\omega}{s^2+\omega^2}\right)\right). \tag{62}$$

Applying the inverse Laplace transform to Equations (61) and (62), we have

$$x_1(t) = \frac{F_0}{M+m}\left(t + \frac{m}{M\omega}\sin(\omega t)\right) \tag{63}$$

$$x_2(t) = \frac{F_0}{M+m}\left(t - \frac{1}{\omega}\sin(\omega t)\right). \tag{64}$$

We obtained the displacement solutions of the mass-spring system with two blocks attached to a spring with spring constant K.

3.3. Circuits

Circuits are primordial applications in the field of Electromagnetism, such as circuits that operate with power on large scales or microcomputer circuits that have low voltage [12]. In this section, in the theoretical development, it is essential to present the main components of a circuit.

3.3.1. Components of a circuit

Resistor

Figure 8 illustrates a resistor represented by the letter R. It can be called ohmic since it obeys Ohm's Law, having a potential drop in the direction of the electric current i through its extremes, where $V = V_1 - V_2$. So, the potential is given by

$$V = Ri. \tag{65}$$

By Joule effect, the resistor transforms electrical energy into thermal energy and its dissipated power is

$$P = i^2R. \tag{66}$$

Capacitor

Figure 9 shows a capacitor, another important element defined by the letter C its capacitance. One of its plates or reinforcements has a charge q and the other has a charge $-q$, and these charges can vary over time, as long

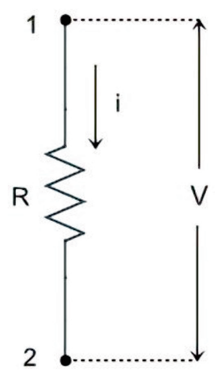


Figure 8: Resistor. Source: The authors.

 DOI: https://doi.org/10.1590/1806-9126-RBEF-2023-0235

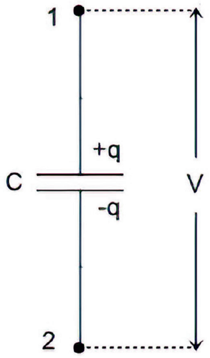


Figure 9: Capacitor. Source: The authors.

as they are in a quasi-steady state. Its potential or also potential drop $V = V_1 - V_2$ between the plates is given

$$V = \frac{q}{C}. \tag{67}$$

The function of a capacitor is to store electrical energy. Then its energy is

$$U = \frac{1}{2}CV^2 = \frac{q^2}{2C}. \tag{68}$$

Inductor

Figure 10 indicates an inductor represented by the letter L. Taking the closed circuit 1234, and knowing that 3 and 4 are considerably close to 1 and 2, we have

$$\begin{aligned} \varepsilon &= \oint_{1234} \mathbf{E} \cdot \mathbf{dl} = -L\frac{di}{dt} = \int_3^4 \mathbf{E} \cdot \mathbf{dl} \\ &= -(V_4 - V_3) = -(V_1 - V_2) = -V \end{aligned} \tag{69}$$

that is,

$$V = L\frac{di}{dt} \tag{70}$$

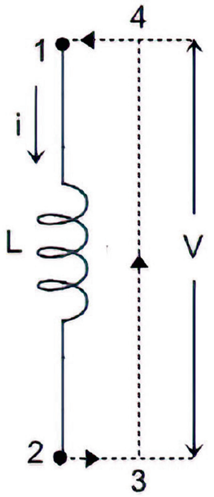


Figure 10: Inductor. Source: The authors.

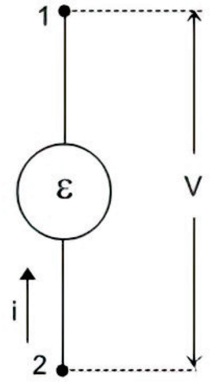


Figure 11: Generator. Source: The authors.

the potential drop at the ends of the inductor in the direction of current i.

In an inductor there is storage of magnetic energy given by

$$U = \frac{1}{2}Li^2. \tag{71}$$

Generator

Figure 11 exemplifies a generator, a source of electromotive force represented by the letter ε. It is responsible for giving energy to the system while the other elements receive energy. The generator is crossed by an electric current i in the opposite direction in relation to the potential drop, so that

$$V_1 - V_2 = V = -\varepsilon \tag{72}$$

which is the potential drop for this case. The generator supplies power at a rate of εi.

3.3.2. The Kirchhoff's laws

Kirchhoff's 1st Law

Consider the circuit represented by Figure 12 with only one mesh, in which each rectangle illustrates a passive electrical component R, L or C. It is known that

$$\int_1^2 \mathbf{E} \cdot \mathbf{dl} = -\int_1^2 dV = V_1 - V_2 = -\varepsilon \tag{73}$$

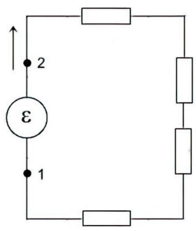


Figure 12: Circuit with one mesh. Source: The authors.

is the voltage drop between the extremes 1 and 2.

$$\varepsilon_1+\varepsilon_2+\varepsilon_3+\cdots+\varepsilon_n=\sum_{k=1}^{n}\varepsilon_k=0. \tag{74}$$

From Equation (74) we can state that the sum of all voltage drops along a loop of a circuit is zero.

This is an algebraic sum, where the voltage drop is positive if we take a current path and negative if we take a counter current path.

Kirchhoff's 2nd law

Now consider a circuit as shown in Figure 13 which has two meshes. Points A and B are called nodes and represent the junction of two or more elements of a circuit. Thus, we can state the following law:

The algebraic sum of all currents that arrive and depart from a node is zero. That is,

$$i_1+i_2+i_3+\cdots+i_n=\sum_{k=1}^{n}i_k=0. \tag{75}$$

Applying the Equation (75) in the circuit of Figure 13 from node B (it could be from node A, since we would get the same result), we have

$$i_3=i_1-i_2. \tag{76}$$

Thus, the currents i_1 and i_2 are the independent variables, that is, currents circulating in the loops as shown in Figure 14.

3.3.3. R-C circuit in series

Consider a circuit formed by a capacitor and a resistor, as shown in Figure 15. Consider that at instant $t=0$ the capacitor is discharged and connected to a battery with

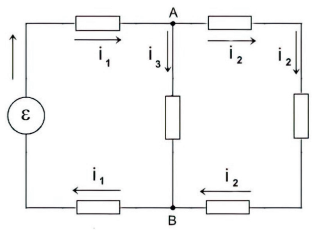


Figure 13: Circuit with two mashes. Source: The authors.

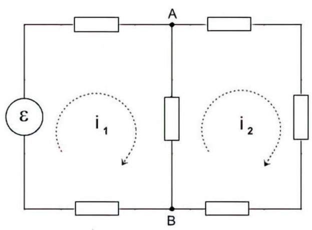


Figure 14: Current in circuit with two loops. Source: The authors.

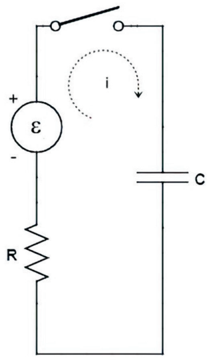


Figure 15: R-C Circuit. Source: The authors.

emf ε. Turning the switch on, we can use Kirchhoff's 1st law to obtain the differential equation that governs such a circuit

$$Ri(t)-\varepsilon+\frac{q(t)}{C}=0 \tag{77}$$

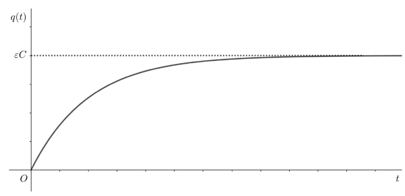
where $i(t)$ corresponds to the electric current at the instant t and $q(t)$ the charge of the capacitor stored at the same instant.

In Figure 16 (a) we observe that the charging current of the capacitor drops exponentially with the passage of time. If the capacitor is initially charged and we happen to remove the battery, consequently causing the capacitor to be connected exclusively to R, being discharged through it, the capacitor discharges with the same exponential law.

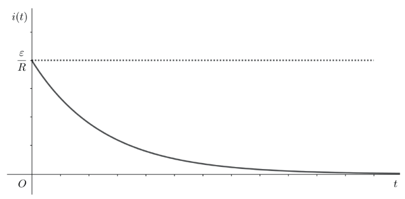
Considering $q(0)=0$ and $i(0)=\dfrac{\varepsilon}{R}$ the initial conditions of Equation (77) and knowing that $i=\dfrac{dq}{dt}$, we get

$$Rq'+\frac{1}{C}q-\varepsilon=0 \tag{78}$$

$$Rq'+\frac{1}{C}q=\varepsilon. \tag{79}$$



(a) $q(t)$.



(b) $i(t)$.

Figure 16: Graphs of $q(t)$ and $i(t)$ for R-C circuit. Source: The authors.

 DOI: https://doi.org/10.1590/1806-9126-RBEF-2023-0235

Applying the Laplace transform in the Equation (79), the initial condition $q(0) = 0$ and using the Theorem 1, we have

$$R\mathcal{L}(q') + \frac{1}{C}\mathcal{L}(q) = \varepsilon\mathcal{L}(1) \tag{80}$$

$$RsQ(s) - Rq(0) + \frac{1}{C}Q(s) = \varepsilon\frac{1}{s}. \tag{81}$$

After some manipulations,

$$Q(s) = \frac{\varepsilon C}{s(sRC+1)}. \tag{82}$$

Using Partial Fractions in Equation (82)

$$Q(s) = \varepsilon C\left(\frac{1}{s} - \left(\frac{1}{s+\frac{1}{RC}}\right)\right). \tag{83}$$

Therefore, using the inverse Laplace transform on the Equation (83), we have

$$q(t) = \varepsilon C(1 - e^{-\frac{t}{RC}}). \tag{84}$$

We got the load function of the R-C Circuit. It can be seen that when $t \to +\infty$ the following approximation occurs $q(t) = \varepsilon C$, according to Figure 16(a) and Equation (84).

By making a time derivative in Equation (84) we find the function of electric current. So,

$$i(t) = \frac{\varepsilon}{R}e^{-\frac{t}{RC}}. \tag{85}$$

Finally, the electric current decays exponentially with time and goes to zero when $t \to +\infty$. Such a conclusion we also note in Figure 16(b).

3.3.4. R-L circuit in series

Now consider a circuit formed by a resistance R and an inductor L, as shown in Figure 17. By turning on the switch and applying Kirchhoff's 1st law, we obtain the differential equation that governs this system

$$L\frac{di(t)}{dt} + Ri(t) - \varepsilon = 0 \tag{86}$$

where $i(t)$ corresponds to the electric current at instant t.

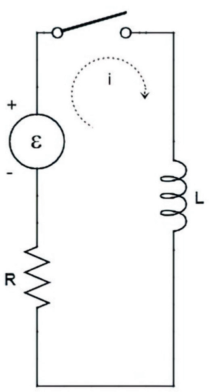


Figure 17: R-L Circuit. Source: The authors.

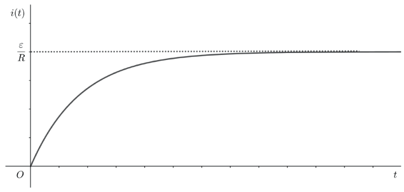


Figure 18: Graph of $i(t)$ to R-L. Source: The authors.

Considering $i(0) = 0$ as the initial condition of Equation (86), we obtain

$$Li' + Ri - \varepsilon = 0 \tag{87}$$

$$Li' + Ri = \varepsilon. \tag{88}$$

Applying the Laplace transform to the Equation (88) and the Theorem 1

$$L\mathcal{L}(i') + R(i) = \varepsilon\mathcal{L}(1) \tag{89}$$

$$L(sI(s) - i(0)) + RI(s) = \varepsilon\frac{1}{s}. \tag{90}$$

After some manipulations,

$$I(s) = \frac{\varepsilon}{s(sL+R)}. \tag{91}$$

Using Partial Fractions in Equation (91)

$$I(s) = \varepsilon\left(\frac{1}{Rs} - \frac{L}{R(sL+R)}\right). \tag{92}$$

By using the inverse Laplace transform in the Equation (92) we find the function that governs the current, so

$$i(t) = \frac{\varepsilon}{R}\left(1 - e^{-\frac{R}{L}t}\right). \tag{93}$$

Figure 18 and Equation (93) indicate that the current exponentially approaches the asymptotic value given by Ohm's law, since when $t \to +\infty$ the current $i = \dfrac{\varepsilon}{R}$.

3.3.5. R-L-C circuit

According to [13–15] and [16], R-L-C circuits, that is, circuits that have a resistor, inductor and capacitor are usually called linear, and can use the transform of Laplace to transform the functions $i(t)$ (electric current) and $v(t)$ (voltage) in the domain $t \in \mathbb{R}$ into functions in the domain $s \in \mathbb{C}$ (this is also called the frequency domain). A circuit can be analyzed in this variable both qualitatively and quantitatively. Finally, the voltages that power the circuit are functions in the form of the unit step $u(t)$.

Models for the elements in the domain $s \in \mathbb{C}$

Resistor

In Figure 19 sets the voltage of a resistor in the domain of s.

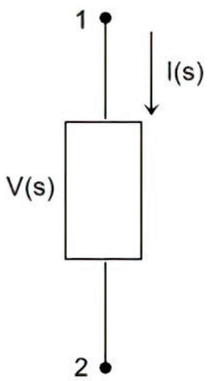


Figure 19: Resistor voltage in the domain of s. Source: The authors.

The voltage of the resistor in the domain t is

$$v(t) = Ri(t) \tag{94}$$

and in the domain s is

$$V(s) = RI(s). \tag{95}$$

The voltage across the resistor obeys Ohm's Law.

Inductor

Figure 20 represents the configuration of the inductor voltage in the domain of s.

The inductor voltage $v(t)$ in domain t is

$$v(t) = L\frac{di}{dt}(t). \tag{96}$$

After applying the Laplace transform and using Theorem 1 we have the inductor voltage in the domain s

$$V(s) = LsI(s) - Li(0). \tag{97}$$

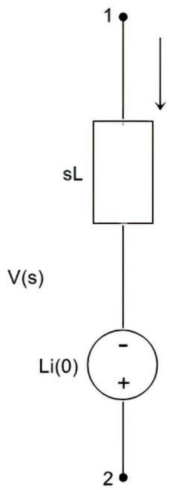


Figure 20: Inductor voltage in the domain of s. Source: The authors.

Capacitor

Figure 21 shows the capacitor voltage in the domain of s.

The capacitor current $i(t)$ in domain t is

$$i(t) = C\frac{dv}{dt}(t). \tag{98}$$

After applying the Laplace transform and Theorem 1, we get

$$I(s) = CsV(s) - Cv(0). \tag{99}$$

Step by step analysis of a linear circuit with Laplace transform

- Transform circuit from the t domain to the s domain using the presented elements;
- Circuit analysis in the s domain with the same laws and methods as in the t domain;
- Transform the obtained solution (voltage or current) back to the domain t.

Solution and Analysis of an R-L-C circuit using Laplace transform

Consider the R-L-C circuit represented in Figure 22. Let's determine the voltage on the capacitor $v(t)$ for $t > 0$, considering the initial conditions $i(0) = -1A$ and $v(0) = 5V$ and that $v_s(t) = 10u(t)V$, where $v_s(t)$ is the source voltage and $u(t)$ is a unit step function.

Figure 23 exemplifies the transformed circuit for the variable s using the Laplace transform.

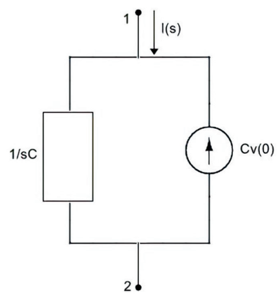


Figure 21: Capacitor voltage in the domain of s. Source: The authors.

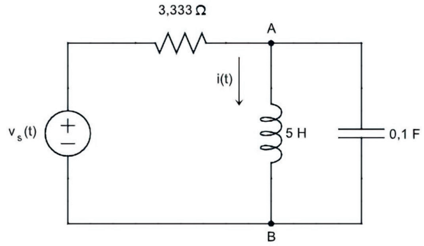


Figure 22: Circuit R-L-C in domain t. Source: The authors.

 DOI: https://doi.org/10.1590/1806-9126-RBEF-2023-0235

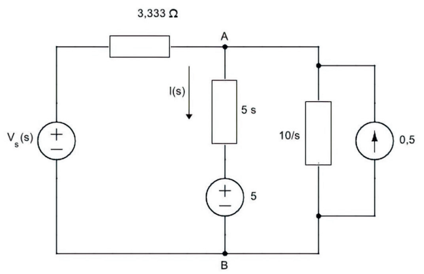


Figure 23: Circuit R-L-C in domain s. Source: The authors.

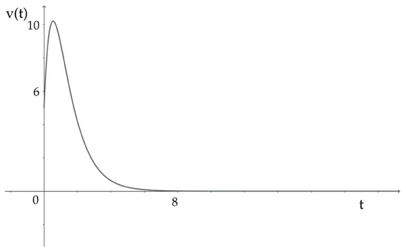


Figure 24: Graph of $v(t)$. Source: The authors.

By Kirchhoff's 2nd law, we get

$$\frac{V(s)-V_s(s)}{\left(\frac{10}{3}\right)}+\frac{V(s)-5}{5s}+\frac{V(s)}{\left(\frac{10}{s}\right)}-\frac{1}{2}=0 \qquad (100)$$

$$\frac{3}{10}\left(V(s)-\frac{10}{s}\right)+\frac{1}{5s}(V(s)-5)+\frac{s}{10}V(s)-\frac{1}{2}=0 \qquad (101)$$

$$3s\left(V(s)-\frac{10}{s}\right)+2(V(s)-5)+s^2V(s)-5s=0 \qquad (102)$$

$$(s^2+3s+2)V(s)-30-10-5s=0 \qquad (103)$$

$$(s^2+3s+2)V(s)-40-5s=0 \qquad (104)$$

$$V(s)=\frac{5s+40}{s^2+3s+2}. \qquad (105)$$

After decomposition by partial fractions, we have

$$V(s)=\frac{35}{s+1}-\frac{30}{s+2}. \qquad (106)$$

Applying the inverse Laplace transform

$$v(t)=35e^{-t}u(t)-30e^{-2t}u(t). \qquad (107)$$

For $t>0$,

$$v(t)=35e^{-t}-30e^{-2t}. \qquad (108)$$

Figure 24 shows the voltage on the capacitor.

4. Final Remarks

In this work, we show, in a didactic way, examples of applications of discontinuous forcing functions or impulses in IVPs and the solution via Laplace transform.

Functions such as unit step and unit impulse are commonly studied in Ordinary Differential Equations courses, but their application in real situations involving such forcing functions is scarce in general books. This text was written to fill this gap.

In summary, we seek to understand some physical phenomena using PVIs with ordinary differential equations with impulse or step functions, solve them with Laplace transform, and transform functions and their derivatives in the variable t into simple algebraic functions in the variable s. After this transformation, we apply the inverse Laplace transform to obtain the functions that govern such phenomena.

References

[1] J.L. Schiff, *The Laplace Transform: Theory and Applications* (Springer Science & Business Media, New York, 1999).

[2] D.G. Zill and M.R. Cullen, *Equações Diferenciais* (Pearson Education, São Paulo, 2001), v. 1, 3 ed.

[3] D.G. Zill, *A First Course in Differential Equations With Modeling Applications* (Cengage Learning, New York, 2012).

[4] D.A.V. Tonidandel and A.E.A. de Araújo, Rev. Bras. Ensino Fís. **34**, 2601 (2012).

[5] M.D. Ortigueira and J.T. Machado, Mathematics **8**, 8 (2020).

[6] M.R. Spiegel, *Laplace Transforms* (McGraw-Hill, New York, 1965).

[7] D.M. Gonçalves and L.A. Caritá, Revista Eletrônica Paulista de Matemática **23**, 53 (2023).

[8] D.M. Gonçalves and L.A. Caritá, *Pormenores do Teorema de Lerch. Anais do XXII Encontro Latino Americano de Pós-Graduação* (Paraíba, 2022). Available in: https://www.inicepg.univap.br/cd/INIC_2022/anais/\trabalhos_ciencias_exatas_e_da_terra.html

[9] G. Arfken and H. Weber, *Física Matemática: Métodos Matemáticos para Engenharia e Física* (Elsevier, Rio de Janeiro, 2007).

[10] U.A. Bakshi and A.V. Bakshi, *Electrical Measurements and Measuring Instruments* (Technical Publications Pune, 2008).

[11] S.S. Rao, *Física Vibrações Mecânicas* (Pearson Education do Brasil, São Paulo, 2009), 4 ed.

[12] H.M. Nussenzveig, *Curso de Física Básica: Eletromagnetismo* (Editora Blucher, São Paulo, 2015), v. 3.

[13] S. Haykin and B.V. Veen, *Signals and Systems* (John Wiley & Sons, Nova Jersey, 1998).

[14] D.J. Irwin and R.M. Nelms, *Basic Engineering Circuit Analysis* (Wiley, Nova Jersey, 2015).

[15] B.P. Lathi, *Signals, Systems and Communication* (Wiley, Nova Jersey, 1966).

[16] A.V. Oppenheim, S. Hamid and A.S. Willsky, *Signals and Systems* (Prentice Hall, Londres, 1996).