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Approximate controllability for the semilinear heat equation in $\mathbb{R}^N$ involving gradient terms

SILVANO BEZERRA DE MENEZES*

Departamento de Matemática, Universidade Federal do Pará
66075-110 Belém, PA, Brazil
E-mail: silvano@ufpa.br

Abstract. We prove the approximate controllability of the semilinear heat equation in $\mathbb{R}^N$, when the nonlinear term is globally Lipschitz and depends both on the state u and its spatial gradient ∇u. The approximate controllability is viewed as the limit of a sequence of optimal control problems. In order to avoid the difficulties related to the lack of compactness of the Sobolev embeddings, we work with the similarity variables and use weighted Sobolev spaces.

Mathematical subject classification: 93B05, 73C05, 35B37.

Key words: approximate controllability, optimal control, unbounded domains, weighted Sobolev spaces.

1 Introduction

Let Ω be a domain of $\mathbb{R}^N$ with $N \subseteq 1$. Given $T > 0$ and an open nonempty subset ω of Ω we consider the following semilinear heat equation

$$\left| \begin{array}{l} u_t - \Delta u + f(x, t, u, \Delta u) = h 1_\omega \quad \text{in} \quad Q = \Omega \times (0, T) \\ u = 0 \quad \text{on} \quad \partial\Omega \times (0, T) \\ u(x, 0) = u^0(x) \quad \text{in} \quad \Omega. \end{array} \right. \tag{1.1}$$

In (1.1) 1_ω denotes the characteristic function of ω. Roughly speaking, f is assumed to be mensurable in (x, t) and globally Lipschitz in the variables u and

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∇u. In (1.1) $u = u(x,t)$ is the state and $h = h(x,t)$ is the control function which acts on the system through the subset ω. The approximate controllability problem can be formulated as follows: Given a finite time horizon $T > 0$ and an arbitrary element $u^0 \in L^2(\Omega)$, system (1.1) is said to be approximately controllable at time T in $L^2(\Omega)$ if the reachable set

$$R_{NL}(T) = \{u(x,T)\colon u \text{ is solution of (1.1) with } h \in L^2(Q)\}$$

is dense in $L^2(\Omega)$.

There are numerous works treating the approximate controllability in the linear parabolic framework, see [L2] and its bibliography, when Ω is a bounded set. The first results for nonlinear systems were obtained in [H]. More recently, several situations have been considered by Fabre, Puel and Zuazua [FPZ] for the particular case in which f is a globally Lipschitz function depending only on u., i.e., $f = f(u)$. Their proof is divided in two parts:

a) approximate controllability of the linearized systems;

b) fixed point technique.

This technique cannot be applied when Ω is an unbounded set since the compacteness of Sobolev's embeddings is one of the main ingredients used in b). In L. de Teresa and E. Zuazua [TZ], they proved the approximate controllability of the semilinear heat equation in unbounded domains by an approximation method, for the case $f = f(u)$, f being globally Lipschitz. The method in [TZ] consists in approximating the domain Ω by a sequence of bounded domains $\Omega_R = \Omega \cap B_R$, where B_R denotes the ball centered in zero of radius R. It is then proved that, in the limit as R tends to infinity, the approximate controls in Ω_R provide an approximate control in the unbounded domain Ω.

Later on, L. de Teresa [T] proved the approximate controllability when $\Omega = \mathbb{R}^N$ by an alternative method that consists in writing the heat equation in the similarity variables and using the weighted Sobolev spaces introduced in [EK] to guarantee the compacteness of the Sobolev embeddings. Then, essentially, the methods in [FPZ] apply.

When Ω is a bounded set, L. Fernandez and E. Zuazua [FZ] gave a proof of the approximate controllability for the system (1.1), f being globally Lipschitz, inspired in the work by J.L. Lions [L2] in which, roughly speaking, the approximate controllability is viewed as the limit of a sequence of optimal control problems. More precisely, given $u^0 \in L^2(\Omega)$, $u^1 \in H_0^s(\Omega)$ with $0 < s < 1$ and $k > 0$, let us consider the functional

$$J_k(h) = \frac{1}{2}\int_0^T \int_\omega h^2(x,t)dxdt + \frac{k}{2}\|u_h(T) - u^1\|^2_{H_0^s(\Omega)}$$

where u_h denotes the solution of (1.1) with control h. The functional J_k is well defined for $h \in L^2(\Omega \times (0, T))$. On the other hand, it is shown that, for each $k > 0$, there exists a minimizer h_k of J_k in $L^2(\Omega \times (0, T))$. Let us denote by u_k the solution of (1.1) associated to this minimizer. It is shown in [FZ] that

$$u_k(T) \rightharpoonup u^1 \text{ weakly in } H_0^s(\Omega)$$

as $k \to +\infty$, and therefore $u_k(T) \to u^1$ in $L^2(\Omega)$ as $k \to +\infty$. This fact, combined with the fact that $H_0^s(\Omega)$ is dense in $L^2(\Omega)$, guarantees the approximate controllability of system (1.1).

This paper is devoted to analyze the approximate controllability of system (1.1) in case when $\Omega = \mathbb{R}^N$. We adopt the approach in [FZ] but in the more general case $\Omega = \mathbb{R}^N$. However, in order to avoid the difficulties associated with the lack of compactness of the Sobolev embeddings, we work with similarity variables and the weighted Sobolev spaces as in [T]. We should point out we have essentially combined the techniques from both works [FZ] and [T]. A key point in the proof of our result is aresult of unique continuation by C. Fabre [Fa] in the context of linear heat equations involving gradient terms. The result in [Fa] allow us to conclude that

$$\left.\begin{array}{l} -p_t - \Delta p + \alpha(x,t)p + \operatorname{div}(b(x,t)p) = 0 \\ \quad \text{in} \quad Q = \Omega \times (0,T) \\ p(x,t) = 0 \quad \text{in} \quad q = \omega \times (0,T) \end{array}\right\} \Rightarrow p(x,t) = 0 \quad \text{in} \quad Q$$

provided $a \in L^\infty(Q)$ and $b \in (L^\infty(Q))^N$.

Thus, we consider the domain Ω to be $\mathbb{R}^N$, ω an open nonempty subset of $\mathbb{R}^N$

and analyze the approximate controllability of the system:

$$\left| \begin{array}{l} u_t - \Delta u + f(x, t, u, \nabla u) = h 1_\omega \quad \text{in} \quad \mathbb{R}^N \times (0, T) \\ u(x, 0) = u^0(x) \quad \text{in} \quad \mathbb{R}^N \end{array} \right. \tag{1.2}$$

The main result of this paper is the following:

Theorem 1.1. *Let us assume the following hypotheses:*

$$\left| \begin{array}{l} f(x, t, \theta, \eta) \text{ is a measurable function with respect to} \\ (x, t) \in \mathbb{R}^N \times (0, T) \text{ and a } C^1 \text{ function with respect to} \\ (\theta, \eta) \in \mathbb{R} \times \mathbb{R}^N \end{array} \right. \tag{1.3}$$

$$f(x, t, 0, 0) \in L^2(\mathbb{R}^N \times (0, T)) \tag{1.4}$$

$$\begin{array}{c} \left| \dfrac{\partial f}{\partial \theta}(x, t, \theta, \eta) \right| + \left| \dfrac{\partial f}{\partial \eta}(x, t, \theta, \eta) \right| \leq K_0, \\ \forall\, (x, t, \theta, \eta) \in \mathbb{R}^N \times (0, T) \times \mathbb{R} \times \mathbb{R}^N \end{array} \tag{1.5}$$

where K_0 is a positive constant. Then, for any $u^0 \in L^2(\mathbb{R}^N)$ and $T > 0$, the set of reachable states (at time $T > 0$) given by

$$\begin{aligned} R_{NL}(T) = \{u_{h,u^0}(x, T) \colon u_{h,u^0} \text{ is the solution of (1.2) with} \\ h \in L^2(\mathbb{R}^N \times (0, T))\} \end{aligned}$$

is dense in $L^2(\mathbb{R}^N)$.

Remark 1.1.

a) As we mentioned above, the method of proof applies in the case where Ω is a cone of $\mathbb{R}^N$ (i.e. $\lambda\Omega = \Omega$, $\forall\, \lambda > 0$). Extending Theorem 1.1 to the case of general open unbounded sets Ω is an open problem. At this respect, the approximation method developed in [TZ] may be the most suitable tool.

b) By an approximation argument, the assumption that f is C^1 in the variables $(u, \nabla u)$ can be easily relaxed. It is sufficient f to be Lipschitz in those variables.

c) The assumption that f is globally Lipschitz in $(u, \nabla u)$ might seem to be artificial. But it is not. As it is shown in E. Fernández-Cara [F], there are nonlinearities growing at infinity in a slightly superlinear way and for which the approximate controllability fails.

The rest of the paper is organized as follows: in section 2, we introduce the similarity variables and use weighted Sobolev spaces. We prove basic results on existence, uniqueness and regularity of solutions. In section 3 we state some preliminary results concerning the solutions by transposition and we prove the approximate controllability of the linear equation. Section 4 is devoted to prove the main result. Finally, in section 5, we briefly comment the case of general conical domains.

2 Similarity variables and weighted Sobolev spaces

In this section we recall some basic facts about the similarity variables and weighted Sobolev spaces for the heat equation. We refer to [EK] and [K] for further developments and details. As we said above, to prove Theorem 1.1 we follow the approach in [FZ]. Thus, first we consider the problem of approximate controllability for the linearized system with potentials:

$$\begin{cases} u_t - \Delta u + a(x,t)u + b(x,t) \cdot \nabla u = h 1_\omega \quad \text{in} \quad \mathbb{R}^N \times (0,T) \\ u(x,0) = u^0 \quad \text{in} \quad \mathbb{R}^N \end{cases} \tag{2.1}$$

with $a(x,t) \in L^\infty(\mathbb{R}^N \times (0,T))$ and $b(x,t) \in (L^\infty(\mathbb{R}^N \times (0,T)))^N$. The approximate controllability of (2.1) is then obtained as a singular limit of a sequence of optimal control problems. But, to do that, the lack of compactness of the Sobolev embedding in $\mathbb{R}^N$ is an obstacle. To avoid it we work on the similarity variables of the heat equation and the weighted Sobolev spaces introduced in [EK] that we recall now.

We introduce the new space-time variables

$$y = \frac{x}{\sqrt{t+1}}; \quad s = \log(t+1). \tag{2.2}$$

Then, given $u = u(x,t)$ solution of (2.1) we introduce $v(y,s) = e^{sN/2}u(e^{s/2}y, e^s - 1)$. It follows that u solves (2.1) if and only if v satisfies

$$\left| \begin{array}{l} v_s - \Delta v - \dfrac{y \cdot \nabla v}{2} + A(y,s)v + B(y,s) \cdot \nabla v - \dfrac{N}{2}v = \varphi(y,s)1_{\omega'}(s) \\ \quad \text{in} \quad \mathbb{R}^N \times (0,S) \\ v(y,0) = v^0(y) = u^0(y) \quad \text{in} \quad \mathbb{R}^N \end{array} \right. \tag{2.3}$$

where

$$\left| \begin{array}{l} A(y,s) = e^s a(e^{s/2}y, e^s - 1) \\ B(y,s) = e^{s/2} b(e^{s/2}y, e^s - 1) \\ \varphi(y,s) = e^{s(N+2)/2} h(e^{s/2}y, e^2 - 1) \\ S = \log(T+1) \\ w'(s) = e^{-s/2}\omega \end{array} \right. \tag{2.4}$$

The elliptic operator appearing in (2.3) may also be written as

$$Lv = -\Delta v - \frac{y \cdot \nabla v}{2} = -\frac{1}{K(y)} \operatorname{div}\, (K(y) \nabla v) \tag{2.5}$$

where $K = K(y)$ is the Gaussian weight $K(y) = \exp(|y|^2/4)$. We introduce the weighted L^p-spaces:

$$L^p(K) = \left\{ f : |f|_{L^p(K)} = \left[\int_{\mathbb{R}^N} |f(y)|^p K(y) dy \right]^{1/p} < \infty \right\}.$$

For $p = 2$, $L^2(K)$ is a Hilbert space and the norm $|\cdot|_{L^2(K)}$ is induced by the inner product $(f,g) = \int_{\mathbb{R}^N} f(y)g(y)K(y)dy$. We then define the unbounded operator L on $L^2(K)$ by setting $Lf = -\Delta f - \frac{y \cdot \nabla f}{2}$ as above, and $D(L) = \{f \in L^2(K) \colon Lf \in L^2(K)\}$. Integrating by parts it is easy to see that

$$\int_{\mathbb{R}^N} (Lf) f\, K\, dy = \int_{\mathbb{R}^N} |\nabla f|^2 K\, dy. \tag{2.6}$$

Therefore it is natural to introduce the weighted H^1-space:

$$H^1(K) = \left\{ f \in L^2(K) \colon \frac{\partial f}{\partial y_i} \in L^2(K), i = 1, \ldots, n \right\}$$

endowed with the norm $\| f \|_{H_1(K)} = \left[\int_{\mathbb{R}^N} (|f|^2 + | \bigtriangledown f|^2) K\, dy \right]^{1/2}$. In a similar way, for any $s \in \mathbb{N}$ and multi-index α we may introduce the space

$$H^s(K) \left\{ f \in L^2(K) \colon D^\alpha f \in L^2(K),\ \forall \alpha, |\alpha| \le s \right\}.$$

The following properties were proved in [EK] and [K]:

$$\int_{\mathbb{R}^N} K(y)|f(y)|^2|y|^2 dy \le c \int_{\mathbb{R}^N} K(y)| \bigtriangledown f|^2 dy \tag{2.7}$$

$$\text{The embedding } H^2(K) \hookrightarrow L^2(K) \text{ is compact.} \tag{2.8}$$

$$L \colon H^1(K) \to (H^1(K))^* \tag{2.9}$$

is an isomorphism where $(H^1(K))^*$ denotes the dual space of $H^1(K)$.

$$D(L) = H^2(K) \tag{2.10}$$

$$L^{-1} \colon L^2(K) \to L^2(K) \text{ is self-adjoint and compact.} \tag{2.11}$$

Since the operator L defined above has compact inverse in $L^2(K)$, the equation

$$\left| \begin{array}{l} v_s + Lv + A(y,s)v + B(y,s) \cdot \bigtriangledown v - \dfrac{N}{2} v = \varphi(y,s) 1_{\omega'(s)} \\ \quad \text{in} \quad \mathbb{R}^N \times (0, S) \\ v(y,0) = v^0(y) \quad \text{in} \quad \mathbb{R}^N \end{array} \right. \tag{2.12}$$

can be studied in the same manner as the heat equation in a bounded region Ω of $\mathbb{R}^N$. Let us recall some important and useful facts about the spaces appearing this paper. First, we introduce some notation. In fact, given two separable Hilbert spaces V and H such that $V \subset H$ with continuous embedding, V being dense in H, let us consider the Hilbert space $W(0,T,V,H) = \{u \in L^2(0,T,V) \colon u_t \in L^2(0,T,H)\}$, equipped with the norm

$$\|u\|_{W(0,T,V,H)} = \left(|u|^2_{L^2(0,T,V)} + |u_t|^2_{L^2(0,T,H)} \right)^{1/2} \text{ see, for instance [DL].}$$

We have

$$\begin{aligned}
&W(0, T, H^1(K), (H^1(K))^*) \subset L^2(0, T, L^2(K)) \\
&\quad \text{with compact embedding} && (2.13)\\
&W(0, T, H^1(K), (H^1(K))^*) \subset C([0, T], L^2(K)) \\
&\quad \text{with continuous embedding} && (2.14)\\
&W(0, T, H^2(K), L^2(K)) \subset L^2(0, T, H^1(K)) \\
&\quad \text{with compact embedding} && (2.15)\\
&W(0, T, H^2(K), L^2(K)) \subset C([0, T], H^1(K)) \\
&\quad \text{with continuous embedding} && (2.16)
\end{aligned}$$

We represent by $(\, , \,)$, $| \cdot |$, $((\, , \,))$ and $\| \cdot \|$ the inner product and norm, respectively, of $L^2(K)$ and $H^1(K)$. We observe that the operator L is defined by the triplet

$$\left\{H^1(K), L^2(K), ((\, , \,))\right\}, \text{ with } ((u, v)) = \int_{\mathbb{R}^N} \nabla u \cdot \nabla v K(y) dy.$$

Therefore we have the following result about existence and uniqueness of the linear system (2.12). (See, for instance, [L1]).

Proposition 2.1. *Given $\varphi \in L^2(0, S, L^2(K))$ and $v^0 \in L^2(K)$, there exists a unique solution v in the space $W(0, S, H^1(K), L^2(K))$ of problem* (2.12). *Moreover, if $v^0 \in H^1(K)$, then $v \in W(0, S, H^2(K), L^2(K))$.* □

3 Solutions by transposition and the linear case

In this section we give a precise definition of (2.12) in the sense of transposition and prove an existence and uniqueness result. The main question we are concerned here consists in finding a solution p of the parabolic problem:

$$\left| \begin{array}{l} -p_s + Lp + Ap - \operatorname{div}(Bp) - \dfrac{N}{2} p - \dfrac{1}{2} y \cdot Bp = 0 \\ \quad \text{in} \quad \mathbb{R}^N \times (0, S) \\ p(y, S) = f \quad \text{in} \quad \mathbb{R}^N, \end{array} \right. \tag{3.1}$$

when $f \in (H^1(K))^*$. Here $(H^1(K))^*$ denotes the dual of $H^1(K)$. A function $p \in L^2(0, S, L^2(K))$ is called an solution of (3.1) obtained by transposition if

$$\int_0^S \int_{\mathbb{R}^N} p(y, s)\varphi(y, s)K(y)dy = \langle f, v(S)\rangle$$

for any solution v of problem (2.12) with $\varphi \in L^2(0, S, L^2(K))$ and $v^0(y) = 0$ in $\mathbb{R}^N$. We represent by $\langle\ ,\ \rangle$ the duality pairing between $(H^1(K))^*$ and $H^1(K)$. We have

Proposition 3.1. *If* $f \in (H^1(K))^*$ *then there exists an unique solution* $p \in L^2(0, S, L^2(K)) \cap C([0, S], (H^1(K))^*)$ *of problem* (3.1).

Proof. Let us consider the linear form $F\colon L^2(0, S, L^2(K)) \to \mathbb{R}$ defined by

$$F(\varphi) = \langle f, v(S)\rangle \text{ for all } \varphi \in L^2(0, S, L^2(K)) \tag{3.2}$$

where v is the solution of (2.12), with $v^0 = 0$, corresponding to φ.

Since $v^0 = 0$, it is easy see that

$$\|v(s)\|_{H^1(K)} \le C|\varphi|_{L^2(0,S,L^2(K))}. \tag{3.3}$$

Thus, F is a continuous linear form in $L^2(0, S, L^2(K))$. By Riesz's representation theorem, there exists a unique $p \in L^2(0, S, L^2(K))$ such that

$$F(\varphi) = \int_0^S \int_{\mathbb{R}^N} p\varphi K(y)\, dyds, \text{ for all } \varphi \in L^2(0, S, L^2(K)). \tag{3.4}$$

The uniqueness is consequence of the Du Bois Raymond Lemma. Moreover,

$$|F|_{L^2(0,S,L^2(K))} = |p|_{L^2(0,S,L^2(K))}. \tag{3.5}$$

Thus, by (3.2), (3.3) and (3.5), we have

$$|p|_{L^2(0,S,L^2(K))} \le C|f|_{(H^1(K))^*}. \tag{3.6}$$

Since $f \in (H^1(K))^*$, there exists $(f_m)_{m\in\mathbb{N}}$ with $f_m \in L^2(K)$ for all m, such that

$$f_m \to f \text{ strongly in } (H^1(K))^*. \tag{3.7}$$

Let $(p_m)_{m\in\mathbb{N}}$ be the sequence of solutions corresponding to f_m for each m. Obviously, the function $p_n - p_m$ is the solution by transposition corresponding to $f_n - f_m$.

From (3.6) and (3.7) we have that $(p_m)_{m\in\mathbb{N}}$ is a Cauchy sequence in $L^2(0, S, L^2(K))$ and

$$p_m \to p \text{ strongly in } L^2(0, S, L^2(K)). \tag{3.8}$$

On the other hand,

$$\begin{aligned} p'_m - p'_n &= L(p_m - p_n) + A(p_m - p_n) - \operatorname{div}(B(p_m - p_n) \\ &\quad - \frac{N}{2}(p_m - p_n) - \frac{1}{2} y \cdot B(p_m - p_n) \\ &\quad \text{in} \quad L^2(0, S, (H^2(K))^*). \end{aligned} \tag{3.9}$$

Then, for each $\psi \in L^2(0, S, H^2(K))$ we have

$$\begin{aligned} |\langle p'_m - p'_n, \psi\rangle| &\le |\langle p_m - p_n, L\psi\rangle| + |\langle A(p_m - p_n), \psi\rangle| \\ &+ |\langle p_m - p_n, B \cdot \nabla\psi\rangle| + \frac{N}{2}|\langle p_m - p_n, \psi\rangle| \\ &\le c|p_m - p_n|_{L^2(0,S,L^2(K))}|\psi|_{L^2(0,S,H^2(K))} \end{aligned}$$

and this implies that $(p'_m)_{m\in\mathbb{N}}$ is a Cauchy sequence in $L^2(0, S, (H^2(K))^*)$.

Therefore, we have

$$p_m \to p \text{ strongly in } W(0, S, L^2(K), (H^2(K))^*)$$

and this space is continuously embedded in $C([0, S], (H^1(K))^*)$. We obtain $p_m \to p$ strongly in $C([0, S], (H^1(K))^*)$. In particular, $p \in C([0, S], (H^1(K))^*)$. $\square$

Through this work we set the notation $q' = \{(y, s), s \in (0, S), y \in w'(s)\}$, where $\omega'(s) = e^{-s/2}$ and ω is an open nonempty subset of $\mathbb{R}^N$.

To prove the approximate controllability of system (2.12) we employ the theorem of unique continuation due to C. Fabre [Fa], Theorem 1.4 of [Fa]. We have

Proposition 3.2. *Let w be an open and nonempty set of $\mathbb{R}^N$, $\omega'(s) = e^{-s/2}\omega$ and q' defined above. Assume that $A(y,s) \in L^\infty(\mathbb{R}^N \times (0,S))$, $B(y,s) \in (L^\infty(\mathbb{R}^N \times (0,S)))^N$. Let $p \in L^2(0,S,H^1(K))$ be such that*

$$\left| \begin{array}{l} -p_s + Lp + Ap - \ div\ (Bp) - \dfrac{N}{2}p - \dfrac{1}{2}y \cdot Bp = 0 \\ \quad in \quad \mathbb{R}^N \times (0,S) \\ p = 0 \quad in \quad q' \end{array} \right. \tag{3.10}$$

Then $p \equiv 0$.

Proof. L, A and B do satisfy the conditions of Theorem 1.4 due Fabre [Fa] and the assumption of p implies that $p \in L^2_{loc}(0,S,H^1_{loc}(\mathbb{R}^N))$. Consequently $p \equiv 0$.

Now, we have all the ingredients to prove our result:

Theorem 3.1 (Approximate controllability in $H^1(K)$). *Given bounded, measurable potentials A and B with $S > 0$, for every $v^0 \in H^1(K)$, the set of reachable states at time $S > 0$, $\hat{R}_L(S) = \{v_{\varphi,v^0}(S) \colon v_{\varphi,v^0}$ is the solution of (2.12) with $\varphi \in L^2(0,S,L^2(K))\}$ is dense in $H^1(K)$.*

Proof. The proof follows the arguments in [FZ].

Because of the linearity of (2.12), we can assume without loss of generality that $v^0 = 0$ and we denote the solution $v_{\varphi,0}$ by v_φ. Given a fixed element $v_d \in H^1(K)$ and $k \in \mathbb{N}$, we introduce the following optimal control problem

$$P(k) \quad \left\{ \begin{array}{l} \text{minimize } J_k(\varphi) = \dfrac{1}{2}\displaystyle\int_q^{\prime} \varphi^2 K(y)\,dyds + \dfrac{k}{2}\|v_\varphi(S) - v_d\|^2_{H^1(K)} \\ \text{over } \varphi \in L^2(0,S,L^2(K)) \end{array} \right.$$

For all $k \in \mathbb{N}$ the functional J_k is lower semicontinuous and strictly convex. Observe that the functional J_k is coercive in the Hilbert space of $L^2(0,S,L^2(K))$ of functions with support in q'. Then, there exists a solution φ_k of (P_k) for each $k \in \mathbb{N}$. The derivative of $J_k(\varphi)$ is given by

$$J'_k(\varphi)\xi = \int_{q'} \varphi\xi K(y)\,dyds + k((v_\varphi(S) - v_d, v_\xi(S)))_{H^1(K)} \tag{3.11}$$

for all $\xi \in L^2(0, S, L^2(K))$. If we evaluate $J_k'(\varphi) \cdot \xi$ in the solution φ_k of (P_k) we must have

$$\int_{q'} \varphi_k \xi K(y)\, dyds + k((v_k(S) - v_d, v_\xi(S))) = 0 \tag{3.12}$$

for all $\xi \in L^2(0, S, L^2(K))$, with $v_k(S) = v_{\varphi_k}(S)$.

Taking into account that $J_k(\varphi_k) \leq J_k(0) = \dfrac{k}{2}\|v_d\|^2_{H^1(K)}$ for every $k \in \mathbb{N}$, we deduce that $\{v_k(S) - v_d\}_{k\in\mathbb{N}}$ is a bounded sequence in $H^1(K)$ and $\left\{\dfrac{1}{\sqrt{k}}\varphi_k 1_{\omega'}\right\}_{k\in\mathbb{N}}$ is bounded in $L^2(0, S, L^2(K))$. Then we can extract a subsequence $\{v_k(S) - v_d\}_{k\in\mathbb{N}}$ such that

$$v_k(S) - v_d \rightharpoonup \psi \text{ weakly in } H^1(K). \tag{3.13}$$

From (3.12) we obtain:

$$((\psi, v_\xi(S))) = 0 \text{ for all } \xi \in L^2(0, S, L^2(K)). \tag{3.14}$$

By transposition we define the adjoint state p as the unique solution (cf. section 3) of the problem:

$$\left|\begin{array}{l} -p_s + Lp + Ap - \text{ div } (Bp) - \dfrac{N}{2}p - \dfrac{1}{2}y \cdot Bp = 0 \\ \quad \text{in} \quad \mathbb{R}^N \times (0, S) \\ p(y, S) = L\psi \quad \text{in} \quad \mathbb{R}^N \end{array}\right. \tag{3.15}$$

We obtain

$$\begin{aligned} \int_0^S \int_{\mathbb{R}^N} p(y, s)\left[z_s + Lz + Az + B \cdot \nabla z - \frac{N}{2}z\right] K(y)\, dyds \\ = \langle L\psi, z(S)\rangle \end{aligned} \tag{3.16}$$

for all $z \in W(0, S, H^2(K), L^2(K))$, such that $z(0) = 0$.

The solution p of (3.15) defined by (3.16) has the regularity:

$$p \in L^2(0, S, L^2(K)) \cap \mathcal{C}([0, S], (H^1(K))^*) \text{ since } L\psi \in (H^1(K))^*),$$

cf. Proposition 3.1.

Let us consider z the solution of (2.12) when $\varphi = \xi$; this solution we denote by v_ξ. From relation (3.16) it follows that

$$\int_0^S \int_{\mathbb{R}^N} p(y)\,\xi K(y) 1_{\omega'}\, dyds = ((\psi, v_\xi(S))) = 0,\ \forall\, \xi.$$

It follows that $p(x,t) = 0$, a.e. $(y,s) \in w' \times (0,S)$ and by Proposition 3.2 we have $p(x,t) = 0$, a.e. $(y,s) \in \mathbb{R}^N \times (0,S)$. Since $p \in \mathcal{C}([0,S], (H^1(K))^*)$ and $p(S) = L\psi$ with $\psi \in H^1(K)$ we have $\psi = 0$. From (3.13) we have

$$v_k(S) \rightharpoonup v_d \text{ weakly in } H^1(K).$$

The strong convergence follows from (3.12) with $\xi = \varphi_k$. Indeed, letting $k \to \infty$ in

$$\frac{1}{k}\int_{q'} \varphi_k^2 K(y)\, dyds + \|v_k(S) - v_d\|^2_{H^1(K)} = -((v_k(S) - v_d, v_d))_{H^1(K)}$$

we obtain that $v_k(S) \to v_d$ strongly in $H^1(K)$. □

Corollary 3.1. *For every $v^0 \in H^1(K)$ and $m \in [0,1)$, the set $\hat{R}_L(S)$ is dense in $H^m(K)$.*

Proof. This is a direct consequence of Theorem 3.1 and the density of $H^1(K)$ in $H^m(K)$. □

Corollary 3.2. *For every $v^0 \in L^2(K)$, the set $\hat{R}_L(S)$ is dense in $L^2(K)$.*

Proof. Let us consider $v^1 \in L^2(K)$ and $\varepsilon > 0$. Since $H^1(K) \subset L^2(K)$ with dense inclusion, there exists a sequence $\{v_n^0\} \subset H^1(K)$ such that $v_n^0 \to v^0$ strongly in $L^2(K)$.

From Corollary 3.1, we know the existence of controls φ_n such that the solution $v^n_{\varphi_n, v_0^n}$ of (2.12) with $\varphi = \varphi_n$ satisfies

$$\left| v^n_{\varphi_n, v_n^0}(S) - v_1 \right|_{L^2(K)} < \frac{\varepsilon}{2}\,.$$

Let $\widetilde{N} > 0$ be such that $|v^0_{\widetilde{N}} - v_0|_{L^2(K)} \leq e^{-c\frac{S}{2}}\frac{\varepsilon}{2}$ where $c > 0$ is a constant that will appear bellow. Consider v_{φ,v^0} the solution of (2.12) corresponding to v^0 and $\varphi = \varphi_{\widetilde{N}}$. Let $z = v_{\varphi,v^0} - v_{\varphi,v^0_{\widetilde{N}}}$. Then z satisfies

$$\left| \begin{array}{l} z_s + Lz + Az + B\cdot\nabla z - \dfrac{N}{2}z = 0 \quad \text{in} \quad \mathbb{R}^N \times (0,S) \\ z(y,0) = z_0(y) = v^0 - v^0_{\widetilde{N}} \quad \text{in} \quad \mathbb{R}^N \end{array} \right. \tag{3.17}$$

We multiply (3.17) by $zK(y)$ and integer over $\mathbb{R}^N$. Then, there exists a constant $c > 0$ such that $|z(S)|_{L^2(K)} \leq e^{c\frac{S}{2}}|z_0|_{L^2(K)}$ and, therefore,

$$\left|v_\varphi, v^0(S) - v^1\right|_{L^2(K)} \leq |z(S)|_{L^2(K)} + \left|v_{\varphi,v^0_{\widetilde{N}}}(S) - v^1\right|_{L^2(K)} \leq \varepsilon\,. \qquad \square$$

We conclude with the

Theorem 3.2. *Given $u^0 \in L^2(\mathbb{R}^N)$ the set of reachable states at time $T > 0$, $R_L(T) = \{u_{h,u^0}(T)\colon uh, u^0$ is the solution of (2.1) with $h \in L^2(\mathbb{R}^N \times (0,T))\}$ is dense in $L^2(\mathbb{R}^N)$.*

Proof. Let us consider $u^1 \in L^2(\mathbb{R}^N)$ and $\varepsilon > 0$. We divide the proof into three steps.

Step 1. The case $u^0 = 0,\ u^1 \in L^2(K)$.

Let us consider $v^1(y) = (T+1)^{N/2}u^1((T+1)^{\frac{1}{2}}y) \in L^2(K)$. From Corollary 3.2 we know the existence of $\varphi \in L^2(0,S,L^2(K))$ such that $\left|v(S) - v^1\right|_{L^2(K)} \leq \varepsilon$, with $S = \log(T+1)$. Then

$$u(x,t) = (1+t)^{-\frac{N}{2}}v\left(\frac{x}{\sqrt{1+t}}, \log(1+t)\right)$$

is the solution of (2.1) with

$$h(x,t) = (1+t)^{-\frac{N}{2}-1}\varphi\left(\frac{x}{\sqrt{1+t}}, \log(1+t)\right)$$

and

$$
\begin{aligned}
\left|u(T)-u^1\right|^2_{L^2(\mathbb{R}^N)} &\leq \int_{\mathbb{R}^N}(1+t)^{-N}e^{|x|^2}4(1+T) \\
&\quad \left|v\left(\frac{x}{\sqrt{1+t}},S\right)-v^1\left(\frac{x}{\sqrt{1+t}}\right)\right|^2 dx \\
&\leq \int_{\mathbb{R}^N} K(1+T)^{-\frac{N}{2}}\left|v(y,S)-v^1(y)\right|^2 dy \\
&\leq (1+T)^{-\frac{N}{2}}\left|v(S)-v^1\right|^2_{L^2(K)} \leq \varepsilon^2.
\end{aligned}
$$

Step 2. The case $u^0=0,\ u^1\in L^2(\mathbb{R}^N)$.

Since $L^2(K)\subset L^2(\mathbb{R}^N)$ with dense inclusion, there exists a sequence $\{u^1_n\}$ and $\widetilde{N}>0$ such that $\left|u^1_n-u^1\right|_{L^2(\mathbb{R}^N)}\leq\dfrac{\varepsilon}{2}$ for every $n>\widetilde{N}$.

From the first step, we know the existence of controls h_n such that u_n the solution of (2.1) with $h=h_n$ satisfies

$$\left|u_n(T)-u^1_n\right|_{L^2(\mathbb{R}^N)}\leq\frac{\varepsilon}{2}\,.$$

Then, the solution u of (2.1) with $h=h_{\widetilde{N}}$ satisfies

$$\left|u(T)-u^1\right|_{L^2(\mathbb{R}^N)}\leq\varepsilon\,. \qquad \square$$

Step 3. The general case, i.e. $u^0,u^1\in L^2(\mathbb{R}^N)$ arbitrary.

We write $u=z+Y$ where z is the solution of

$$\left|\begin{array}{l} z_t-\Delta z+a(x,t)z+b(x,t)\cdot\nabla z=0 \quad \text{in} \quad \mathbb{R}^N\times(0,T)\\ z(x,0)=u^0(x) \quad \text{in} \quad \mathbb{R}^N. \end{array}\right. \tag{3.18}$$

Then $z(T)\in L^2(\mathbb{R}^N)$. We construct $h=h(u^1-z(T))$ such that the solution of

$$\left|\begin{array}{l} Y_t-\Delta Y+a(x,t)Y+b(x,t)\cdot\nabla Y=h1_\omega \quad \text{in} \quad \mathbb{R}^N\times(0,T)\\ Y(x,0)=0 \quad \text{in} \quad \mathbb{R}^N \end{array}\right. \tag{3.19}$$

satisfies

$$\left|Y(T)-(u^1-z(T))\right|_{L^2(\mathbb{R}^N)}\leq\varepsilon. \tag{3.20}$$

Therefore in (2.1) it is enough to chose $h=h(u^1-z(T))$ since the unique solution is $u=z+Y$ and in view of (3.20) we conclude the proof. $\square$

4 The non linear case

We will now study the same problem for the semilinear heat equation (1.2). As in the linear case, a change of variables $v(y,s) = e^{s\frac{N}{2}} u(e^{s/2}y, e^s - 1)$ transforms the system (1.2) in

$$\left|\begin{array}{l} v_s + Lv + g(y,s,v,\nabla v) - \dfrac{N}{2}v = \varphi(y,s)1_{\omega'(s)} \quad \text{in} \quad \mathbb{R}^N \times (0,S) \\ v(y,0) = v^0(y) \quad \text{in} \quad \mathbb{R}^N \end{array}\right. \tag{4.1}$$

with $g(y,s,v,\nabla v) = e^{s(N+2)/2} f(e^{s/2}y, e^s - 1, e^{-sN/2}v, e^{-s(N+2)/2} \nabla v)$ and $\varphi(y,s) = e^{s(N+2)/2} h(e^{s/2}y, e^2 - 1)$.

We must remark the function $g = g(y,s,\theta,\eta)$ possesses the same properties as f. In particular, g is globally Lipschitz in the variables (θ,η).

Observe that by (1.4) and because $L^2(K)$ is dense in $L^2(\mathbb{R}^N)$ we can consider $f(x,t,0,0)$ in $L^2(0,T,L^2(K))$. This guarantees that $g(y,s,0,0) \in L^2(0,S,L^2(K))$. Thus, we suppose that f also satisfies

$$f(x,t,0,0) \in L^2(0,T,L^2(K)), \tag{4.2}$$

besides (1.3) and (1.5).

From the embedding (I1)-(I4) and g globally Lipschitz, it follows the following result on existence and uniueness for system (4.1).

Proposition 4.1. *Suppose f satisfying the conditions* (1.3), (1.5) *and* (4.2) *above. Then, given $\varphi \in L^2(0,S,L^2(K))$ and $v^0 \in L^2(K)$, there exists a unique solution v in the space $W(0,S,H^1(K),(H^1(K))^*)$ of problem* (4.1). *Moreover, if $v^0 \in H^1(K)$, hence $v \in W(0,S,H^2(K),L^2(K))$.*

Proof. The proof is the same given for Theorem 1 in [FZ]. □

We observe that, in view of Proposition 4.1 and change of variables, if $u^0 = v^0 \in L^2(K)$ and $f \in L^2(0,T,L^2(K))$ then there exists a unique solution $u \in W(0,T,H^1(K),(H^1(K))^*)$ of problem (1.2).

As usual, it is possible to derive the continuous dependence of the solution with respect to the initial data:

Proposition 4.2 (Continuity with respect to the data). *Suppose g satisfying the conditions above.*

a) Let us suppose $v_m^0 \to v^0$ *in* $L^2(K)$ *and* $\varphi_m \to \varphi$ *in* $L^2(0, S, L^2(K))$. *Then,* $v_{\varphi_m, v_m^0} \to v_{\varphi, v^0}$ *in* $W(0, S, H^1(K), (H^1(K))^*)$.

b) Let us suppose $v_m^0 \to v^0$ *in* $L^2(K)$ *and* $\varphi_m \to \varphi$ *in* $H^1(0, S, L^2(K))$. *Then,* $v_{\varphi_m, v_m^0} \to v_{\varphi, v^0}$ *in* $W(0, S, H^2(K), (L^2(K))^*)$. □

To show the approximate controllability property of the semilinear heat equation, we are going to introduce a family of optimal control problems. A previous step for establishing the optimality conditions corresponding to their solutions in the study of the differentiability of the functional involved.

Proposition 4.3 (Differentiability with respect to the data). *Suppose that g satisfies the conditions above. Given* $v^0 \in H^1(K)$, *the functional* $F\colon L^2(0, S, L^2(K)) \to H^1(K)$ *defined by* $F(\varphi) = v_{\varphi, v^0}(S)$ *is differentiable. Moreover,* $DF(\varphi)\psi = z_\psi(S)$, *where* z_ψ *is the unique solution in* $W(0, S, H^2(K), L^2(K))$ *of the following linearized problem*

$$\left| \begin{array}{l} z_s + Lz + \dfrac{\partial g}{\partial \theta}(y, s, v_{\varphi, v^0}, \nabla v_{\varphi, v^0}) z + \dfrac{\partial g}{\partial \eta}(y, s, v_{\varphi, v^0}, \nabla v_{\varphi, v^0}) \\ \cdot \nabla z - \dfrac{N}{2} z = \psi \quad in \quad \mathbb{R}^N \times (0, S) \\ z(y, 0) = 0 \quad in \quad \mathbb{R}^N \end{array} \right. \tag{4.3}$$

Proof. The functional F is well defined by Proposition 4.1 and the embedding (2.16). Given $\varphi, \psi \in L^2(0, S, L^2(K))$ and $\lambda \in (0, 1)$, let us denote $v_\lambda = v_{\varphi+\lambda\psi, v^0}$, $v = v_{\varphi, v^0}$ and $z_\lambda = \frac{1}{\lambda}(v_\lambda - v)$. To show that F is Gâteaux differentiable we have to prove that

$$z_\lambda(S) \to z_\psi(S) \quad \text{in} \quad H^1(K) \quad \text{as} \quad \lambda \to 0. \tag{4.4}$$

It is clear that for each λ we have

$$\left| \begin{array}{l} z_{\lambda,s} + Lz_\lambda + \dfrac{1}{\lambda}\big[g(y,s,v_\lambda,\nabla v_\lambda) - g(y,s,v,\nabla v)\big] - \dfrac{N}{2} z_\lambda = \psi \\ \quad \text{in} \quad \mathbb{R}^N \times (0,S) \\ z_\lambda(y,0) = 0 \quad \text{in} \quad \mathbb{R}^N \end{array} \right. \tag{4.5}$$

where $z_{\lambda,s}$ denotes $\dfrac{dz_\lambda}{ds}$.

By the Mean Value Theorem, we have

$$\begin{aligned} g_\lambda(y,s) &= \frac{1}{\lambda}(g(y,s,v_\lambda(y,s),\nabla v_\lambda(y,s)) - g(y,s,v(y,s),\nabla v(y,s)) \\ &= \frac{\partial g}{\partial \theta}(y,s,w_\lambda(y,s))z_\lambda + \frac{\partial g}{\partial \eta}(y,s,w_\lambda(y,s),\nabla w_\lambda(y,s)) \cdot \nabla z_\lambda. \end{aligned} \tag{4.6}$$

where $w_\lambda = v + \xi(v_\lambda - v)$, $0 \le \xi \le 1$ depends on y, s and λ. Applying (4.5) to $z_\lambda K(y)$, integrating by parts, using the fact that g is globally Lipschitz in the variable $(v, \nabla v)$ and Young's inequality, we deduce the existence of a constant $c_1 > 0$ such that

$$\begin{aligned} |z_\lambda(\tilde{s})|^2 \; + \; \int_0^{\tilde{s}} \int_{\mathbb{R}^N} |\nabla z_\lambda|^2 \, K \, dyds &\le K_1 \int_0^{\tilde{s}} |z_\lambda(s)|^2_{L^2(K)} \, ds \\ + \int_0^{\tilde{s}} |\psi(s)|^2_{L^2(K)} \, ds, \; \forall \tilde{s} \in [0,S] \quad &\text{and} \quad \forall \lambda \in (0,1). \end{aligned} \tag{4.7}$$

Using Gronwall inequality, we have

$$|z_\lambda(s)|_{L^2(K)} \le e^{c_1 s} \, |\psi|_{L^2(0,S,L^2(K))} \, , \; \forall s \in [0,S], \; \forall \lambda \in (0,1).$$

Thus, the sequence $\{z_\lambda\}$ is bounded in $\mathcal{C}([0,S], L^2(K))$ and by (4.7), $\{z_\lambda\}$ is bounded in $L^2(0,S,H^1(K))$. In fact, $|z_\lambda|_{L^2(0,S,H^1(K))} \le e^{c_1 S} \, |\psi|_{L^2(0,S,L^2(K))}$, $\forall \lambda \in (0,1)$. Taking into account this estimate together with the equation (4.5), (4.6) and hypothesis on g, we conclude that there exists a positive constant c_2 (independent on λ and ψ) such that

$$|z_\lambda|_{W(0,S,H^1(K),(H^1(K))^*)} \le c_2 \, |\psi|_{L^2(0,S,L^2(K))} \, , \; \forall \lambda \in (0,1).$$

In view of the expression of g_λ and using classical estimates, we deduce the existence of a positive constant c_3 (still independent of λ and ψ) such that

$$|z_\lambda|_{W(0,S,H^2(K),(L^2(K))} \le c_3 \, |\psi|_{L^2(0,S,L^2(K))} \, , \; \forall \lambda \in (0,1). \tag{4.8}$$

Thus, by extracting subsequences,

$$z_\lambda \rightharpoonup \hat{z} \quad \text{weakly in} \quad W(0, S, H^2(K), L^2(K)), \quad \text{as} \quad \lambda \to 0. \tag{4.9}$$

Combining this convergence with the compact imbedding (2.15) and Proposition (4.2b), we deduce that

$$g_\lambda \to \frac{\partial g}{\partial \theta}(y, s, v, \nabla v)\hat{z} + \frac{\partial g}{\partial \eta}(y, s, v, \nabla v) \cdot \nabla \hat{z} \quad \text{in} \quad L^2(0, S, L^2(K)).$$

It is then easy to see, by the well-posedness properties of (4.5), that $\hat{z} = z_\psi$ and that the convergence in (4.9) holds in the strong topology. Using the embedding (2.16), we obtain (4.4). □

We have the following result on approximate controllability for the system (4.1):

Theorem 4.4. *For each $v^0 \in H^1(K)$ the set of admissible states at $S > 0$,*

$$\hat{R}_{NL}(S) = \left\{ v_{\varphi,v^0}(S) : v_{\varphi,v^0} \text{ is solution of (4.1) with } \varphi \in L^2(0, S, L^2(K)) \right\}$$

is dense in $H^m(K)$, for each $0 \leq m < 1$.

Proof. We know that $H^1(K)$ is dense in $H^m(K)$ for $0 \leq m < 1$. Then, it is sufficient to prove that $\hat{R}_{NL}(S)$ is dense in $H^1(K)$ with the norm of $H^m(K)$ for $0 \leq m < 1$.

In fact, let us take $v_d \in H^1(K)$, $0 \leq m < 1$ and consider the functional defined in $L^2(0, S, L^2(K))$ by:

$$J_k(\varphi) = \frac{1}{2} \int_{q'} K(y)\varphi^2 \, dyds + \frac{k}{2} \left| v_{\varphi,v^0}(S) - v_d \right|^2_{H^m(K)}.$$

This functional is lower semicontinuous and coercive in the Hilbert space of $L^2(0, S, L^2(K))$ of functions with support in q'. It follows that the minimization problem (P_k) given by:

$$\text{Min } J_k(\varphi) \ \text{ for all } \ \varphi \in L^2(0, S, L^2(K)) \tag{P_k}$$

has at least one solution φ_k.

Thanks to Proposition 4.3, the first order optimality condition associated with the minimization problem (P_k) at this minimum gives

$$J_k'(\varphi_k)\psi = \int_{q'} K(y)\varphi_k\psi\, dyds + k(v_k(S) - v_d, z_\psi^k(S))_{H^m(K)} = 0 \qquad (4.10)$$

for all $\psi \in L^2(0, S, L^2(K))$, where $v_k = v_{\varphi_k, v^0}$ and z_ψ^k is the unique function in $W(0, S, H^2(K), L^2(K))$ solution of the problem

$$\left| \begin{array}{l} z_s + Lz + \dfrac{\partial g}{\partial \theta}(y, s, v_k, \nabla v_k)z + \dfrac{\partial g}{\partial \eta}(y, s, v_k, \nabla v_k) \cdot \nabla z \\ \quad - \dfrac{N}{2}z = \psi 1_{\omega'} \quad \text{in} \quad \mathbb{R}^N \times (0, S) \\ z(y, 0) = 0 \quad \text{in} \mathbb{R}^N \end{array} \right. \qquad (4.11)$$

By the minimum condition we have $J_k(\varphi_k) \leq J_k(0)$ for every $k \in \mathbb{N}$. It follows that $\{v_k(S) - v_d\}_{k\in\mathbb{N}}$ is a bounded sequence in $H^m(K)$ and $\left\{ \dfrac{1}{\sqrt{k}} \varphi_k 1_{\omega'} \right\}_{k\in\mathbb{N}}$ is a bounded sequence in $L^2(0, S, L^2(K))$. Thus, by extracting subsequences,

$$v_k(S) - v_d \rightharpoonup \theta \text{ weakly in } H^m(K). \qquad (4.12)$$

Moreover, there exist $c \in L^\infty(\mathbb{R}^N \times (0, S))$ and $d \in (L^\infty(\mathbb{R}^N \times (0, S)))^N$ such that

$$\left| \begin{array}{l} \dfrac{\partial g}{\partial \theta}(y, s, v_k, \nabla v_k) \rightharpoonup c \text{ weak* } L^\infty(\mathbb{R}^N \times (0, S)) \\ \dfrac{\partial g}{\partial \eta}(y, s, v_k, \nabla v_k) \rightharpoonup d \text{ weak* } (L^\infty(\mathbb{R}^N \times (0, S)))^N \end{array} \right. \qquad (4.13)$$

Let z_ψ be the unique solution of the problem

$$\left| \begin{array}{l} z_s + Lz + c(y, s)z + d(y, s) \cdot \nabla z - \dfrac{N}{2}z = \psi 1_{\omega'} \;\; \text{in} \;\; \mathbb{R}^N \times (0, S) \\ z(y, 0) = 0 \quad \text{in} \quad \mathbb{R}^N. \end{array} \right. \qquad (4.14)$$

Suppose that

$$z_\psi^k(S) \rightharpoonup z_\psi(S) \text{ strongly in } H^m(K). \qquad (4.15)$$

By (4.10), (4.12) and (4.15) we obtain:

$$\frac{1}{\sqrt{k}} \int_{q'} \frac{\varphi k}{\sqrt{k}} \psi K\, dyds + (v_k(S) - v_d, z_\psi^k(S))_{H^m(K)} = 0$$

and when $k \to \infty$, we have

$$(\theta, z_\psi(S))_{H^m(K)} = 0, \text{ for each } \psi \in L^2(0, S, L^2(K)). \tag{4.16}$$

By Theorem 3.1 the set $\{z_\psi(S)\}$ is dense in $H^1(K)$ and, therefore, in $H^m(K)$, when ψ varies in $L^2(0, S; L^2(K))$. Consequently, (4.16) implies that $\theta \equiv 0$. Therefore, according to (4.12), $v_k(S) \rightharpoonup v_d$ weakly in $H^m(K)$. Since the embedding $H^m(K) \subset H^{m'}(K)$ is compact for $m' < m$, we conclude the strong convergence in $H^{m'}(K)$, for any $m' < m$. Taking into account that this holds for all $m < 1$, we conclude the density in $H^m(K)$, as stated in Theorem 4.4.

To complete the proof we need to proof the convergence (4.15). In fact, multiplying (4.11) by $K(y)z_\psi^k$, integrating in $\mathbb{R}^N$ and observing that

$$\left|\frac{\partial g}{\partial \theta}(y, s, v_k, \nabla v_k)\right| + \left|\frac{\partial g}{\partial \eta}(y, s, v_k, \nabla v_k)\right| < K_0 \tag{4.17}$$

we have

$$\left|z_\psi^k(s)\right|^2 + \int_0^S \|z_\psi^k(s)\|^2 ds < c_1. \tag{4.18}$$

Multiplying (4.11) by $K(y)(z_\psi^k)'$ and integrating in $\mathbb{R}^N$, we obtain:

$$\int_0^S \left|(z_\psi^k)'\right|^2 ds + \|z_\psi^k(s)\|^2 < c^2. \tag{4.19}$$

From (4.18) and (4.19) it follows that $\{z_\psi^k\}_{k\in\mathbb{N}}$ is bounded in $W(0, S, H^2(K), L^2(K))$. Since $W(0, S, H^2(K), L^2(K))$ is compactly embedded in $L^2(0, S, H^1(K))$, there exists subsequences such that

$$\left|\begin{array}{l} z_\psi^k \to \chi \text{ strongly in } L^2(0, S, H^1(K)) \\ z_\psi^k \to \chi \text{ weakly in } W(0, S, H^2(K), L^2(K)) \end{array}\right. \tag{4.20}$$

From (4.15) and (4.20) we can pass to the limit in (4.11) obtaining that χ is the solution of (4.14) and by uniqueness $\chi = z_\psi$. On the other hand, from (4.20) and the continuous embedding I4) we have:

$$z_\psi^k \rightharpoonup z_\psi \text{ weakly in } H^1(K)$$

and k tends to $+\infty$. The embedding of $H^1(K)$ into $H^m(K)$ into $H^m(K)$ being compact, for $m < 1$. This proves (4.15). □

Corollary 4.1. *Given $v^0 \in L^2(K)$, the set $\hat{R}_{NL}(S)$ is dense in $L^2(K)$.*

Proof. As in the proof of Corollary 3.2, let us consider $v^1 \in L^2(K)$ and $\varepsilon > 0$. Then, there exists a sequence $\{v_n^0\}_{n\in\mathbb{N}} \subset H^1(K)$ such that $v_n^0 \to v^0$ strongly in $L^2(K)$. Moreover, as we saw previously, there exists a sequence of controls φ_n such that v_{φ_n, v_n^0}, the solution of (4.1) corresponding to v_n^0 and φ_n, satisfies

$$\left|v_{\varphi_n, v_n^0}(S) - v^1\right|_{L^2(K)} \leq \frac{\varepsilon}{2}.$$

Let $\widetilde{N} > 0$ be such that

$$\left|v_{\widetilde{N}}^0 - v^0\right|_{L^2(K)} \leq e^{-\frac{cS}{2}} \frac{\varepsilon}{2},$$

where $c > 0$ is a constant that will be timely introduced.

Consider v_{φ, v^0} the solution of (4.1) corresponding to v^0 and $\varphi = \varphi_{\widetilde{N}}$. Let $z = v_{\varphi, v^0} - v_{\varphi, v_{\widetilde{N}}^0}$. Then, z satisfies

$$\left|\begin{array}{l} z_s + Lz + g(y, s, v_{\varphi,v^0}, \nabla v_{\varphi,v^0}) - g(y, s, v_{\varphi,v_{\widetilde{N}}^0}, \nabla v_{\varphi,v_{\widetilde{N}}^0}) \\ \quad - \dfrac{N}{2} z = 0 \quad \text{in} \quad \mathbb{R}^N \times (0, S) \\ z(y, 0) = z^0(y) = v^0 - v_{\widetilde{N}}^0 \quad \text{in} \quad \mathbb{R}^N. \end{array}\right. \tag{4.21}$$

We multiply (4.21) by $zK(y)$ and integer over $\mathbb{R}^N$. Since g satisfies (4.17), we obtain

$$|z(S)|_{L^2(K)} \leq e^{\frac{cS}{2}} \left|z^0\right|_{L^2(K)}, \text{ where } c = K_0 + \frac{K_0}{2} + \frac{N}{2}, \text{ and therefore}$$

$$\left|v_{\varphi,v^0}(S) - v^1\right|_{L^2(K)} \leq |z(S)|_{L^2(K)} + \left|v_{\varphi,v_{\widetilde{N}}^0}(S) - v^1\right|_{L^2(K)} \leq \varepsilon.$$

□

We conclude with the proof of Theorem 1.1.

Proof of Theorem 1.1. Let us consider $\alpha > 0$ and $u^1 \in L^2(\mathbb{R}^N)$. As in the proof of Theorem 3.2 we divide into several steps.

First step. The case u^0, u^1 in $L^2(K)$.

We make the change of variables $v^1(y) = (1+T)^{N/2}u^1((1+T)^{1/2}y)$ and $S = \log(1+T)$. Then $v^1 \in L^2(K)$ and by Corollary 4.1, there exists $\varphi \in L^2(0, S, L^2(K))$ such that the solution v of (4.1) corresponding to v^0 and φ, satisfies

$$\left|v(S) - v^1\right|_{L^2(K)} \leq \alpha.$$

We define

$$u(x,t) = (1+t)^{-N/2} v\left(\frac{x}{\sqrt{1+t}}, \log(1+t)\right).$$

Then u is solution of (1.2) with

$$h(x,t) = (1+t)^{-\frac{N}{2}-1}\varphi\left(\frac{x}{\sqrt{1+t}}, \log(1+t)\right)$$

and $\left|u(T) - u^1\right|^2_{L^2(\mathbb{R}^N)} \leq \alpha^2$ (cf. Theorem 3.2, Step 1).

Second step. The case $u \in L^2(K)$, $u^1 \in L^2(\mathbb{R}^N)$.

Since $L^2(K) \subset L^2(\mathbb{R}^N)$ with dense inclusion, there exists a sequence $\{u^1_n\}_{n\in\mathbb{N}} \subset L^2(K)$ such that $u^1_n \to u^1$ strongly in $L^2(\mathbb{R}^N)$. From the First Step, we know that existence of controls h_n such that u_n the solution of (1.2) satisfies $\left|u_n(T) - u^1_n\right|_{L^2(\mathbb{R}^N)} \leq \dfrac{\alpha}{2}$. Let $\widetilde{N} > 0$ such that for every $n > \widetilde{N}$, $\left|u^1 - u^1_n\right|_{L^2(\mathbb{R}^N)} \leq \dfrac{\alpha}{2}$.

Then, u the solution of (1.2) with $h = h_{\widetilde{N}}$ satisfies $\left|u(T) - u^1\right|_{L^2(\mathbb{R}^N)} \leq \alpha$.

Third step. The case $u^0 \in L^2(\mathbb{R}^N)$.

Then there exists a sequence $\{u^0_n\}$ with $u^0_n \in L^2(K)$ such that $u^0_n \to u^0$ strongly in $L^2(\mathbb{R}^N)$. Given $u^1 \in L^2(\mathbb{R}^N)$ and $\alpha > 0$, as we saw previously, there exists a sequence of controls h_n such that u_n, the solution of (1.2) corresponding to u^0_n and h_n, satisfies

$$\left|u_n(T) - u^1\right|_{L^2(\mathbb{R}^N)} \leq \frac{\alpha}{2}.$$

Let $\widetilde{N} > 0$ be such that

$$\left|u^0_{\widetilde{N}} - u^0\right|_{L^2(\mathbb{R}^N)} \leq e^{-(K_0 + \frac{K_0^2}{2})\frac{T}{2}}\frac{\alpha}{2},$$

K_0 the constant given in (1.5).

Consider u the solution of (1.2) corresponding to u^0 and $h = h_{\widetilde{N}}$. Let $z = u - u_{\widetilde{N}}$. Then, z satisfies

$$\left| \begin{array}{l} z_t - \Delta z + f(x,t,u,\nabla u) - f(x,t,u_{\widetilde{N}},\nabla u_{\widetilde{N}}) = 0 \\ \quad \text{in} \quad R^N \times (0,T) \\ z(x,0) = z^0(x) = u^0 - u^0_{\widetilde{N}} \quad \text{in} \quad \mathbb{R}^N. \end{array} \right. \tag{4.22}$$

We multiply (4.22) by z and integer over $\mathbb{R}^N$. Since f satisfies (1.5), we obtain

$$\frac{1}{2}\frac{d}{dt}\,|z|^2_{L^2(\mathbb{R}^N)} + \|z\|^2_{H^1(\mathbb{R}^N)} \le K_0\,|z|^2_{L^2(\mathbb{R}^N)} + \frac{K_0^2}{2}\,|z|^2_{L^2(\mathbb{R}^N)} + \frac{1}{2}\,\|z\|^2_{H^1(\mathbb{R}^N)}.$$

Then, $|z(T)|_{L^2(\mathbb{R}^N)} \le e^{(K_0 + \frac{K_0^2}{2})\frac{T}{2}}\,\left|z^0\right|_{L^2(\mathbb{R}^N)}$, and therefore

$$\left|u(T) - u^1\right|_{L^2(\mathbb{R}^N)} \le |z(T)|_{L^2(\mathbb{R}^N)} + \left|u_{\widetilde{N}}(T) - u^1\right|_{L^2(\mathbb{R}^N)} \le \alpha. \qquad \square$$

5 Conical domains

Consider a cone-like domain Ω satisfying:

$$0 \in \bar{\Omega}, \quad \forall \lambda > 0, \quad \forall x \in \Omega, \quad \lambda x \in \Omega. \tag{5.1}$$

All the results of these paper hold for the semilinear heat equation (1.1) when Ω is a cone-like domain. In fact, consider a domain Ω satisfying (5.1). Then if one consider the evolution equation

$$\left| \begin{array}{l} u_t - \Delta u + f(x,t,u,\nabla u) = h 1_\omega \quad \text{in} \quad Q = \Omega \times (0,T) \\ u = 0 \quad \text{on} \quad \partial\Omega \times (0,T) \\ u(x,0) = u^0(x) \quad \text{in} \quad \Omega \end{array} \right. \tag{5.2}$$

we can study the controllability of the equation (5.2) in the same way as the case of the whole space $\mathbb{R}^N$. Observe that defining v, g and φ like in this section we obtain that v satisfies

$$\left| \begin{array}{l} v_s + Lv + g(y,s,v,\nabla v) - \dfrac{N}{2} v = \varphi 1_{\omega'(s)} \quad \text{in} \quad \Omega \times (0,S) \\ v = 0 \quad \text{on} \quad \partial\Omega \times (0,S) \\ v(y,0) = v^0(y) \quad \text{in} \quad \Omega \end{array} \right.$$

The operator $Lf = -\frac{1}{K}\mathrm{div}(K \bigtriangledown f)$, $K(y) = \exp(|y|^2/4)$, is self-adjoint in $L^2(K, \Omega)$ with compact inverse and

$$D(L) = \{f \in H_0^1(K, \Omega) \colon Lf \in L^2(K, \Omega)\},$$

where

$$L^2(K, \Omega) = \left\{ v \colon \int_\Omega |v|^2 K(y) dy < \infty \right\},$$

$$H_0^1(K, \Omega) = \left\{ v \colon \int_\Omega \left(|v|^2 + |\bigtriangledown v|^2 \right) K(y) dy < \infty, \quad v\|_{\partial\Omega} = 0 \right\} \quad \text{and}$$

$$H^m(K, \Omega) = \left\{ v \in L^2(K, \Omega) \colon \|v\|_{H^m(K,\Omega)} = \left[\sum_{|\alpha| \le m} \|D^\alpha v\|^2_{L^2(K,\Omega)} \right]^{1/2} < \infty \right\}.$$

Indeed, the methods of the previous sections apply. First of all, the approximate controllability of the linearized equation can be obtained as a consequence of the unique continuation result in [Fa]. Then, the approximate controllability for the semilinear system can be proved by viewing this property as the limit of optimality conditions of penalized optimal control problems.

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