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Size exclusion during particle suspension transport in porous media: stochastic and averaged equations

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Abstract. A population balance model is formulated for transport of stable particulate suspensions in porous media. Equations for particle and pore size distributions were derived from the stochastic *Master equation*. The model accounts for particle flow reduction due to restriction for large particles to move through small pores. An analytical solution for low particle concentration is obtained for general particle and pore size distributions. The averaged equations differ significantly from the traditional deep bed filtration model.

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Key words: deep bed filtration, particle size distribution, pore size distribution, size exclusion mechanism, stochastic model, averaging.

Introduction

Injection of particulate suspensions in porous media has long been recognised to cause formation damage, which occurs due to particle capture by the porous media. This phenomenon is named deep bed filtration and its understanding is essential for industrial and environmental technologies such as water flooding of oil reservoirs, water filtration, wastewater treatments, etc.

Different physical mechanisms (electrical forces, size exclusion, diffusion, bridging, etc.) are responsible for the capture of suspended particles during

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fluid injection. In the current work, the size exclusion mechanism (where large particles are captured by thin pores) is discussed.

The core-scale deep bed filtration model operates with averaged concentrations for suspended and deposited particles [8, 10]. The model is widely used in industrial filtration [8, 10, 14], in poor quality water injection into oil reservoirs [17, 24] and in environmental studies [7, 12]. Several analytical solutions for direct prediction problems [8, 14, 24] and for inverse data treatment problems [3, 4, 24] were obtained. The model exhibits a good agreement with laboratory data and can be used for prediction purposes, such as forecast of well injectivity decline based on laboratory coreflood data.

Nevertheless, the model does not distinguish between different mechanisms of formation damage. Also, the model operates with phenomenological filtration coefficient and does not include real parameters such as pore and particle sizes, ionic strength and electrical charges. Therefore, it cannot be used for diagnostic predictive purposes based on real physical parameters, such as forecast of well injectivity decline based on pore and particle sizes data.

For the size exclusion mechanism, the larger the particles are and the smaller the pores are, the more intensive the capture is, resulting in major formation damage [1, 18, 20]. Simultaneous zoom of the porous media and flowing particle ensemble would not affect the capture rate. Therefore, the filtration coefficient in the size exclusion mechanism should be a monotonically increasing function of the particle-pore size ratio. Nevertheless, several attempts to correlate the formation damage with sizes of particles and pores were unsuccessful [4, 16]. This failure means that either size exclusion mechanisms did not dominate in the laboratory experiments studied, or that the phenomenological model for average concentrations does not describe adequately the size exclusion filtration. One way around the problem is micro-scale modelling of each capture mechanism.

The capture rate due to the size exclusion mechanism is determined by the concentration of particles that are larger than the pores. Therefore, it is essential to consider the particle and pore size distributions in modelling the size exclusion filtration. The stochastic approach using size distributions has been used to model the deep bed filtration in pore networks [12, 19, 21].

A phenomenological deep bed filtration model accounting for pore and particle size distributions is also present in the literature. The population balance equations are derived in [20-22] from continuity equations for each particle size. The closed system of equations is presented. The model captures change of pore and particle size distributions due to different retention mechanisms. Several analytical solutions have been obtained and interpreted physically. Nevertheless, the model assumes that particles of any size move with carrier water velocity, while in reality particles move only through large pores resulting in reduction of particle flow.

In the current paper, a population balance model for filtration with size exclusion mechanism is derived from the stochastic *Master equation* [11]. The particle flow reduction due to restriction for large particle flow through small pores is considered. The derived non-linear system allows for linearization for low suspended particle concentration. An analytical solution for injection of low particle concentration suspension is presented. Averaging of the developed model allows proposing corrections to the widely used classical deep bed filtration theory.

In section 1, the classical deep bed filtration model is presented. Its stochastic generalization accounting for pore and particle size distributions is derived in Section 2. The initial-boundary value problem for suspension injection is of Goursat type [23], yielding exact formulae for particle and pore populations at the inlet cross-section without solving the initial-boundary value problem (Section 3). Section 4 contains the linearized system for the injection of low suspended particle concentration. Exact analytical solutions and averaged equations for deep bed filtration are also derived in section 4. Finally, a section explaining the nomenclature used is included at the end of this paper.

For high injected concentration, the non-linear system derived allows finding asymptotic solutions, which are subject of a forthcoming publication [6]. The analysis of the averaged non-linear system for overall concentrations with analytical solutions for direct and inverse problems will appear in [5].

1 Classical deep bed filtration theory

Let us discuss the classical deep bed filtration model in order to compare it with the averaged deep bed filtration equations derived further in the paper.

The deep bed filtration model consists of equations for particle balance, for kinetics of the particle capture and of the modified Darcy's law [8, 10]:

$$\begin{cases} \dfrac{\partial}{\partial T} c\,(X,T) + \dfrac{\partial}{\partial X} c\,(X,T) = -\dfrac{1}{\phi}\dfrac{\partial}{\partial T}\sigma\,(X,T) \\ \dfrac{\partial}{\partial T}\sigma\,(X,T) = \lambda\,(\sigma)\,\phi\, c\,(X,T) \\ U = -\dfrac{k_0\, k\,(\sigma)}{\mu\, L}\dfrac{\partial p}{\partial X} \end{cases} \tag{1.1}$$

where dimensionless length and time are given by the following formulae:

$$X = \frac{x}{L}; \quad T = \frac{1}{L\phi}\int_o^t U(t)dt\ . \tag{1.2}$$

In the above, $c(X, T)$ is the suspended particle concentration (number of suspended particles per unit pore volume), $\sigma(X, T)$ is the deposited particle concentration (number of captured particles per unit rock volume), ϕ is the porosity, L is the core length. The filtration coefficient $\lambda(\sigma)$ expresses the particle capture rate. The formation damage function $k(\sigma)$ shows how permeability decreases due to particle retention.

The velocity U is independent of the linear coordinate X due to the aqueous suspension incompressibility.

The first two equations represent the kinematics of particle transport and capture; the third equation is a dynamical model that predicts pressure gradient increase due to permeability decline resulting from particle deposition. The third Eq. (1.1) decouples from the first and second Eqs. (1.1): the system of the two kinematics equations can be solved independently.

The initial and boundary conditions for particulate suspension injection into a porous media initially filled by particle-free fluid are:

$$T = 0 : c = 0,\ \sigma = 0; \quad X = 0 : c = c_{inj}\ . \tag{1.3}$$

For constant filtration coefficient, the solution of the initial-boundary value problem (1.1), (1.3) for $T > X$ (see [8, 14, 24]) is:

$$\begin{aligned} c\,(X,T) &= c_{inj}\exp\left(-\lambda\, X\right)\ , \\ \sigma\,(X,T) &= \lambda\,\phi\,(T - X)\; c_{inj}\exp\left(-\lambda\, X\right)\ . \end{aligned} \tag{1.4}$$

Ahead of the concentration front ($T < X$) both concentrations are zero.

The suspended concentration distribution reaches the steady state (first formula (1.4)) at the moment $T = 1$. Therefore, for constant filtration coefficient, the outlet concentration is equal to zero before the particle breakthrough $T < 1$, and it is constant afterwards. After $T = 1$, the deposition rate becomes constant. The deposit accumulates, and the deposited concentration grows proportionally to time (second formula (1.4)).

The explicit formula for suspended concentration (1.4) allows calculating the constant filtration coefficient from the constant outlet concentration [17, 24]. For general $\lambda(\sigma)$, the breakthrough curve $c(1, T)$ determines the filtration coefficient, and the solution of the inverse problem is unique and stable [3].

The solution (1.4) allows for the following physical interpretation of the filtration coefficient – the average dimensionless particle penetration depth is equal to $1/\lambda$:

$$\frac{\int_0^\infty X\, c\,(X, T)dX}{\int_0^\infty c\,(X, T)dX} = \frac{1}{\lambda}. \tag{1.5}$$

It is worth mentioning that particles move with the carrier water velocity according to the continuity equation (1.1). The analytical solution for one-dimensional deep bed filtration (1.4) contains a suspended concentration shock that moves with the carrier water velocity along the trajectory $X = T$ [8].

2 Stochastic equations for deep bed filtration

In this chapter we derive a population balance model for the transport of suspension with particle radius distribution through rock with pore radius distribution.

Figs. 1a,b present the geometry of the porous space adopted. The porous medium is represented by a bundle of parallel capillaries periodically intersected by mixing cameras. The radius of capillaries (r_p) is distributed. Mixing cameras alternate with capillaries (Fig. 1b). Size exclusion happens at the outlet of each camera: a particle enters a capillary if and only if the particle radius (r_s) does not exceed the pore radius.

A stochastic model for deep bed filtration should capture the feature that the larger the particle is, the higher its capture rate is. A particle can be captured only by a smaller pore, $r_p < r_s$ (the so called size exclusion mechanism of particle retention). Pore and particle sizes vary up to 2-3 orders of magnitude, and there is a significant overlap between pore and particle size distributions causing size exclusion capture [9, 18, 19]. Therefore, the distributions of pore and particle sizes should be taken into account in deep bed filtration models.

Let us consider the concentration $C(r_s, x, t)dr_s$ of suspended particles with radii in the interval $[r_s, r_s + dr_s]$:

$$C(r_s, x, t)dr_s = c(x, t) f_s(r_s, x, t)dr_s. \tag{2.1}$$

where $c(x, t)$ is the total suspended particle concentration (total number of suspended particles per unit volume) and $f_s(r_s, x, t)$ is the normalised suspended particle size distribution:

$$\int_0^\infty f_s(r_s, x, t)dr_s = 1 \ . \tag{2.2}$$

Integration of (2.1) considering (2.2) allows calculating the total suspended particle concentration:

$$\int_0^\infty C(r_s, x, t)dr_s = c(x, t)\,. \tag{2.3}$$

The concentration $S(r_s, x, t)\, dr_s$ of captured particles with radii in the interval $[r_s, r_s + dr_s]$ and the total captured particle concentration $\sigma(x, t)$ are related in the same way:

$$\int_0^\infty S(r_s, x, t)dr_s = \sigma(x, t) \ , \tag{2.4}$$

where

$$S(r_s, x, t)dr_s = \sigma(x, t) f_\sigma(r_s, x, t)dr_s. \tag{2.5}$$

and $f_\sigma(r_s, x, t)$ is the normalised captured particle size distribution:

$$\int_0^\infty f_\sigma(r_s, x, t)dr_s = 1. \tag{2.6}$$

The vacant pore concentration with radii in the interval $[r_p, r_p + dr_p]$ is $H(r_p, x, t)dr_p$. It is related to the total vacant pore concentration and pore size distribution as follows:

$$H(r_p, x, t)dr_p = h(x, t) f_p(r_p, x, t)dr_p, \tag{2.7}$$

where $f_p(r_p, x, t)$ is the normalised pore size distribution function:

$$\int_0^\infty f_p(r_p, x, t)dr_p = 1. \tag{2.8}$$

Integration of (2.7) accounting for (2.8) allows calculation of the total vacant pore concentration:

$$\int_0^\infty H(r_p, x, t)dr_p = h(x, t). \tag{2.9}$$

Let us consider the transport of an r_s-particle population $C(r_s, x, t)$ through a porous medium with a pore size distribution $H(r_p, x, t)$. In a time interval τ, the x-coordinates of individual particles will increase by an amount Δ due to the carrier water flow. The total r_s-particle population variation is equal to the sum of all "transitions" that accumulate r_s-particles at position x, minus the sum of all "transitions" that remove r_s-particles from the position x (Fig. 2):

$$\begin{aligned}\left[\phi C(r_s, x, t+\tau) + S(r_s, x, t+\tau)\right] - \left[\phi C(r_s, x, t) + S(r_s, x, t)\right] = \\ \int_0^\infty C(r_s, x-\Delta, t)\Psi(r_s, x-\Delta, t;\ \Delta, \tau)d\Delta + \\ - \int_0^\infty C(r_s, x, t)\Psi(r_s, x, t;\ \Delta, \tau)d\Delta\,,\end{aligned} \tag{2.10}$$

where ϕ is the porosity and $\Psi(r_s, x, t\,;\,\Delta, \tau)$ denotes the probability density that a suspended r_s-particle moves from x to $x + \Delta$ during time τ without being captured.

The first integral in (2.10) corresponds to all displacements that bring suspended particles to the position x and, hence, it represents a net *gain* to the function $[\phi C(r_s, x, t) + S(r_s, x, t)]$. Similarly, the second integral (2.10) corresponds to all transitions that remove suspended particles from x and, hence, it represents a net *loss* to the function $[\phi C(r_s, x, t) + S(r_s, x, t)]$.

The Eq. (2.10) is called the *Master equation* [11]. The integration on the right hand side of (2.10) is performed from zero to infinity because it is assumed that particles cannot perform negative displacements.

Let us consider the probability $\beta(\Delta, \tau)\,d\Delta$ that the carrier fluid experiences a shift between Δ and $\Delta + d\Delta$ during a time interval τ:

$$\int_0^\infty \beta(\Delta, \tau) d\Delta = 1\,. \tag{2.11}$$

The velocity U is an average fluid displacement per unit time:

$$U = \frac{1}{\tau} \int_0^\infty \Delta \beta(\Delta, \tau) d\Delta\,. \tag{2.12}$$

Let us assume that, due to the carrier fluid flow, particles perform advective "jumps" along the distance x without dispersion. In this case, the probability density $\beta(\Delta, \tau)$ for a fluid particle to perform a displacement Δ during the interval τ is:

$$\beta(\Delta, \tau) = \delta(\Delta - U\tau)\ , \tag{2.13}$$

where δ is Dirac's delta function (Fig. 3).

On the other hand, for size exclusion mechanism, only the fraction of the fluid flow through large pores will effectively transport suspended particles. So, the probability Ψ that an r_s-particle flows through pores larger than r_s is:

$$\Psi\left(r_s, x, t\,;\,\Delta, \tau\right)\, d\Delta = \beta\left(\Delta, \tau\right)\, d\Delta\, \frac{\int_{r_s}^{\infty} q\left(r_p, x, t\right) dr_p}{\int_0^{\infty} q\left(r_p, x, t\right) dr_p}\ , \tag{2.14}$$

where $q(r_p, x, t)dr_p$ is the flow through pores with radii in the interval $[r_p, r_p + dr_p]$ and the ratio on right hand side of (2.14) is the flow fraction through large pores (Fig. 1a). So, flow of large particles through small pores is not allowed.

Expanding the right hand side of Eq. (2.10) in Taylor series in Δ results in:

$$\begin{aligned} &C(r_s, x-\Delta, t)\,\Psi(r_s, x-\Delta, t\,;\Delta,\tau) \\ &\quad - C(r_s, x, t)\,\Psi(r_s, x, t\,;\Delta,\tau) \\ &= -\Delta\frac{\partial}{\partial x}\big[C(r_s, x, t)\,\Psi(r_s, x, t\,;\Delta,\tau)\big] \\ &\quad + \frac{\Delta^2}{2}\frac{\partial^2}{\partial x^2}\big[C(r_s, x, t)\,\Psi(r_s, x, t\,;\Delta,\tau)\big] + \cdots\,. \end{aligned} \tag{2.15}$$

Substituting (2.15) into (2.10) and considering (2.14) and (2.13) we obtain:

$$\begin{aligned} &\frac{1}{\tau}\big[\phi C(r_s, x, t+\tau) + S(r_s, x, t+\tau)\big] \\ &\quad - \big[\phi C(r_s, x, t) + S(r_s, x, t)\big] \\ &= -\frac{\partial}{\partial x}J(r_s, x, t) + U\tau\frac{\partial^2}{\partial x^2}J(r_s, x, t) + \cdots\,, \end{aligned} \tag{2.16}$$

where:

$$J(r_s, x, t) = UC(r_s, x, t)\,\frac{\int_{r_s}^{\infty} q(r_p, x, t)dr_p}{\int_0^{\infty} q(r_p, x, t)dr_p}. \tag{2.17}$$

The fluid velocity U is given by (2.12). Tending τ to zero in (2.16) and neglecting porosity variations due to particle retention yields:

$$\phi\frac{\partial}{\partial t}C(r_s, x, t) + \frac{\partial}{\partial x}J(r_s, x, t) = -\frac{\partial}{\partial t}S(r_s, x, t)\,. \tag{2.18}$$

Eq. (2.18) is the population balance equation for r_s-particles.

Let us discuss particle capture and pore plugging kinetics. The probability P that a particle with radius r_s meets a pore with radius r_p is proportional to the product of the number of particles with radii r_s by the fraction of flow through pores with radii r_p [8, 20]:

$$P(r_s, r_p, x, t)\, dr_s\, dr_p \quad \propto \quad U\, C(r_s, x, t) dr_s \frac{q(r_p, x, t) dr_p}{\int\limits_0^\infty q(r_p, x, t) dr_p}. \tag{2.19}$$

The concentration of particles with radii r_s captured by pores with radii r_p is $\underline{S}(r_s, r_p, x, t)$. The number of particles with radii r_s captured by pores r_p per unit time is called the particle-capture rate. This rate is proportional to the probability P:

$$\frac{\partial}{\partial t}\left[\underline{S}\left(r_s, r_p, x, t\right)\right] = \lambda'\left(r_s, r_p\right)\ UC\left(r_s, x, t\right) \frac{q(r_p, x, t) dr_p}{\int\limits_0^\infty q(r_p, x, t) dr_p}. \tag{2.20}$$

The proportionality coefficient is denoted by $\lambda'(r_s, r_p)$. It equals zero for $r_s > r_p$ when capture does not occur. Consequently,

$$\lambda'(r_s, r_p) = 0 \quad : \quad r_p > r_s. \tag{2.21}$$

Integration of (2.20) in r_p from zero to infinity, accounting for (2.21), yields the expression for the total number of captured particles with radii r_s per unit time:

$$\int\limits_0^{r_s} \frac{\partial}{\partial t}\left[\underline{S}\left(r_s, r_p, x, t\right)\right] dr_p = \frac{\partial}{\partial t} S\left(r_s, x, t\right). \tag{2.22}$$

Substitution of (2.20) in (2.22) results in:

$$\frac{\partial}{\partial\, t} S\left(r_s, x, t\right) = U\, C\left(r_s, x, t\right) \frac{\int\limits_0^{r_s} \lambda'\left(r_s, r_p\right)\, q\left(r_p, x, t\right) dr_p}{\int\limits_0^\infty q\left(r_p, x, t\right) dr_p}. \tag{2.23}$$

It is assumed that the capture of one particle plugs only one pore. Therefore, the change of the total number of pores with radii r_p is equal to the total amount of particles captured in the pores with size r_p:

$$H\left(r_p, x, 0\right) - H\left(r_p, x, t\right) = \int\limits_{r_p}^\infty \underline{S}\left(r_s, r_p, x, t\right) dr_s\ . \tag{2.24}$$

The integral in (2.24) represents the total number of particles captured in pores with radii r_p. The integration on the right hand side of (2.24) is performed from r_p to infinity because

$$\lambda'(r_s, r_p) = 0 \quad \text{for} \quad r_s < r_p \, .$$

Differentiation of (2.24) over t results in:

$$\frac{\partial}{\partial t} H\left(r_p, x, t\right) = -\frac{\partial}{\partial t} \int_{r_p}^{\infty} \underline{S}\left(r_s, r_p, x, t\right) dr_s \, . \tag{2.25}$$

Substituting (2.20) into (2.25), we obtain:

$$\frac{\partial}{\partial t} H\left(r_p, x, t\right) = -\frac{Uq\left(r_p, x, t\right)}{\int_0^{\infty} q\left(r_p, x, t\right) dr_p} \int_{r_p}^{\infty} \lambda'\left(r_s, r_p\right) C\left(r_s, x, t\right) dr_s \, . \tag{2.26}$$

Eqs. (2.18), (2.23) and (2.26) form a closed system for three unknowns $C(r_s, x, t)$, $S(r_s, x, t)$ and $H(r_p, x, t)$. Following [2] we assume that the flow through a pore with radii r_p is proportional to r_p^4, as in Hagen-Poiseuille flow [13]. This assumption allows to specify the flux expression in (2.17) and the closed system (2.18), (2.23), (2.26), takes the following form:

$$\begin{cases} \phi \frac{\partial}{\partial t} C\left(r_s, x, t\right) + U \frac{\partial}{\partial x} \left[C\left(r_s, x, t\right) \frac{\int_{r_s}^{\infty} r_p^4 H(r_p, x, t) dr_p}{\int_0^{\infty} r_p^4 H(r_p, x, t) dr_p} \right] \\ \qquad = -\frac{\partial}{\partial t} S\left(r_s, x, t\right) \\ \frac{\partial}{\partial t} S\left(r_s, x, t\right) = UC\left(r_s, x, t\right) \frac{\int_0^{r_s} \lambda'(r_s, r_p)\, r_p^4\, H(r_p, x, t) dr_p}{\int_0^{\infty} r_p^4 H(r_p, x, t) dr_p} \\ \frac{\partial}{\partial t} H\left(r_p, x, t\right) = -U \frac{r_p^4 H(r_p, x, t)}{\int_0^{\infty} r_p^4 H(r_p, x, t) dr_p} \int_{r_p}^{\infty} \lambda'\left(r_s, r_p\right) C\left(r_s, x, t\right) dr_s \end{cases} \tag{2.27}$$

Here it is assumed that the particulate suspension is incompressible, i.e. $U = U(t)$, so it was taken out of the brackets in the first equation (2.27). The system (2.27) takes the following form in dimensionless coordinates (1.2):

$$\begin{cases} \frac{\partial}{\partial T} C\left(r_s, X, T\right) + \frac{\partial}{\partial X}\left[C\left(r_s, X, T\right) \frac{\int_{r_s}^{\infty} r_p^4 H(r_p, X, T) dr_p}{\int_0^{\infty} r_p^4 H(r_p, X, T) dr_p}\right] \\ \qquad = -\frac{1}{\phi}\frac{\partial}{\partial T} S\left(r_s, X, T\right) \\ \frac{\partial}{\partial T} S\left(r_s, X, T\right) = \phi C\left(r_s, X, T\right) \frac{\int_0^{r_s} \lambda(r_s, r_p) r_p^4 H(r_p, X, T) dr_p}{\int_0^{\infty} r_p^4 H(r_p, X, T) dr_p} \\ \frac{\partial}{\partial T} H\left(r_p, X, T\right) = -\phi \frac{r_p^4 H(r_p, X, T)}{\int_0^{\infty} r_p^4 H(r_p, X, T) dr_p} \int_{r_p}^{\infty} \lambda\left(r_s, r_p\right) C\left(r_s, X, T\right) dr_s \end{cases} \tag{2.28}$$

where $\lambda = \lambda' L$.

The boundary condition at the core/reservoir inlet ($X = 0$) corresponds to the injection of particle suspension with a given size distribution. The initial conditions correspond to the absence of particles in the porous medium and to a given pore size distribution of the medium before the injection:

$$\begin{cases} X = 0 : C(r_s, 0, T) = C_{inj}(r_s) \\ T = 0 : C(r_s, X, 0) = 0;\ \ S(r_s, X, 0) = 0;\ \ H(r_p, X, 0) = H_0(r_p, X). \end{cases} \tag{2.29}$$

It is important to emphasize that the population balance equation (first Eq. (2.28)), compared with the model proposed in [20], contains the delay factor α in the advective term:

$$\alpha(r_s, x, t) = \frac{\int_{r_s}^{\infty} H\left(r_p, x, t\right) r_p^4\, dr_p}{\int_0^{\infty} H\left(r_p, x, t\right) r_p^4 dr_p} \ \leq\ 1. \tag{2.30}$$

The water carrying particles with a given size flows through the pores that are larger than the particles. So, the particles are carried by a fraction of the overall flux U (Fig. 1a). Consequently, the delay factor is smaller or equals one, i.e. the particle front lags behind the injected water front.

3 Dynamics of particle and pore populations at the inlet cross section

The second and third equations of the system (2.28) do not contain X-derivative, so the corresponding species concentrations should not be posed at the inlet boundary X=0 (this is a Goursat problem, [23]). This means that the concentration of the immobile species should not be specified at the inlet. Nevertheless, these values can be calculated using the boundary conditions for the mobile species (2.29) and the kinetic equations for the immobile species (second and third equation of the system (2.28)).

Let us find the particle-capture and pore-plugging kinetics at the inlet cross section $X = 0$. We substitute the boundary condition (2.29) into the second and third equations of the system (2.28). This substitution yields a system of two ordinary integro-differential equations for captured particle and vacant pore concentrations at the inlet cross-section:

$$\begin{cases} \frac{d}{dT} S^{(0)}(r_s, T) = \phi C_{inj}(r_s) \dfrac{\int_0^{r_s} \lambda(r_s, r_p) r_p^4 H^{(0)}(r_p, T) dr_p}{\int_0^\infty r_p^4 H^{(0)}(r_p, T) dr_p} \\ \frac{d}{dT} H^{(0)}(r_p, T) = -\phi \dfrac{r_p^4 H^{(0)}(r_p, T)}{\int_0^\infty r_p^4 H^{(0)}(r_p, T) dr_p} \int_{r_p}^\infty \lambda(r_s, r_p) C_{inj}(r_s) dr_s \end{cases} \tag{3.1}$$

where: $H^{(0)}(r_p, T) = H(r_p, X = 0, T)$, $S^{(0)}(r_s, T) = S(r_s, X = 0, T)$.

The second equation (3.1) is independent of the first equation, and it can be solved separately. Afterwards, the first equation allows calculating the deposition kinetics.

There are no deposited particles and plugged pores at the beginning of deep bed filtration. This fact provides the initial conditions for the system of ordinary integro-differential equations (3.1):

$$T = 0: \ S^{(0)}(r_s, 0) = 0; \ H_0(r_p, T = 0) = H_0^{(0)}(r_p). \tag{3.2}$$

The solution of the system of ordinary integro-differential equations (3.1) determines the particle-capture and pore-plugging kinetics at the inlet cross section.

4 Low suspended particle concentration

In many deep bed filtration processes of practical interest, such as flooding of petroleum reservoirs by seawater or water filtration in aquifers, the injected suspended particle concentration is low:

$$c_{inj} << h(X, T = 0). \tag{4.1}$$

This assumption allows to linearize the basic system of governing equations (2.28) and to obtain exact analytical solutions for deep bed filtration of low concentration suspensions. It also permits derivation of averaged equations for the transport of low concentration suspensions in natural rocks.

4.1 Linearization of governing equations

Substituting the second equation (2.28) into the first one, we reduce the system of three equations (2.28) to two equations:

$$\begin{cases} \frac{\partial}{\partial T} C\left(r_s, X, T\right)+\frac{\partial}{\partial X}\left[C\left(r_s, X, T\right) \frac{\int_{r_s}^{\infty} r_p^4 H\left(r_p, X, T\right) d r_p}{\int_0^{\infty} r_p^4 H\left(r_p, X, T\right) d r_p}\right] \\ \qquad =-C\left(r_s, X, T\right) \frac{\int_0^{r_s} \lambda\left(r_s, r_p\right) r_p^4 H\left(r_p, X, T\right) d r_p}{\int_0^{\infty} r_p^4 H\left(r_p, X, T\right) d r_p} \\ \frac{\partial}{\partial T} H\left(r_p, X, T\right)=-\phi \frac{r_p^4 H\left(r_p, X, T\right)}{\int_0^{\infty} r_p^4 H\left(r_p, X, T\right) d r_p} \int_{r_p}^{\infty} \lambda\left(r_s, r_p\right) C\left(r_s, X, T\right) d r_s \end{cases} \tag{4.2}$$

Let us consider a suspension with particle radius distribution constant in time injected into a rock with homogeneous pore radius distribution. So, the boundary and initial distributions in (2.29) become:

$$\begin{cases} X=0: C=C_{inj}(r_s) \\ T=0: C=0; \;\; H(r_p, X, 0)=H_0(r_p). \end{cases} \tag{4.3}$$

For low concentration suspensions (4.1), the problem (4.2), (4.3) contains a small parameter

$$\varepsilon=\frac{c_{inj}}{h_0} << 1, \tag{4.4}$$

where: $c_{inj}=\int_0^{\infty} C_{inj}\left(r_s\right) d r_s$ and $h_0=\int_0^{\infty} H_0\left(r_p\right) d r_p$.

Let us find the first order expansion of the solution for the problem (4.2), (4.3) over the small parameter ε:

$$\begin{aligned} C\left(r_s, X, T\right) &= C^0\left(r_s, X, T\right)+\varepsilon C^1\left(r_s, X, T\right) \\ H\left(r_s, X, T\right) &= H^0\left(r_s, X, T\right)+\varepsilon H^1\left(r_s, X, T\right). \end{aligned} \tag{4.5}$$

Substituting the expansion (4.5) into (4.2) and (4.3), we obtain the zero order approximation system:

$$\begin{cases} \frac{\partial}{\partial T} C^0(r_s, X, T) + \frac{\partial}{\partial X}\left[C^0(r_s, X, T) \frac{\int_{r_s}^{\infty} r_p^4 H^0(r_p, X, T) dr_p}{\int_0^{\infty} r_p^4 H^0(r_p, X, T) dr_p} \right] \\ \qquad = -C^0(r_s, X, T) \frac{\int_0^{r_s} \lambda(r_s, r_p) r_p^4 H^0(r_p, X, T) dr_p}{\int_0^{\infty} r_p^4 H^0(r_p, X, T) dr_p} \\ \frac{\partial}{\partial T} H^0(r_p, X, T) = -\phi \frac{r_p^4 H^0(r_p, X, T)}{\int_0^{\infty} r_p^4 H^0(r_p, X, T) dr_p} \int_{r_p}^{\infty} \lambda(r_s, r_p) C^0(r_s, X, T) dr_s \end{cases} \tag{4.6}$$

subject to initial and boundary conditions

$$\begin{cases} X = 0 : C^0 = 0 \\ T = 0 : C^0 = 0; \ H^0(r_p, X, 0) = H_0(r_p). \end{cases} \tag{4.7}$$

The solution $C^0(r_s, X, T) = 0$ satisfies the first equation (4.6) and boundary conditions (4.7). From the second equation (4.6) and (4.7) it follows that

$$H^0(r_p, X, T) = H_0(r_p). \tag{4.8}$$

Therefore, if only the zero order approximation is considered, the solution shows that there is no particle transport or pore plugging.

In first order approximation, the initial-boundary value problem (4.2), (4.3) becomes:

$$\begin{cases} \frac{\partial}{\partial T} C^1(r_s, X, T) + \frac{\partial}{\partial X}\left[C^1(r_s, X, T) \frac{\int_{r_s}^{\infty} r_p^4 H_0(r_p) dr_p}{\int_0^{\infty} r_p^4 H_0(r_p) dr_p} \right] \\ \qquad = -C^1(r_s, X, T) \frac{\int_0^{r_s} \lambda(r_s, r_p) r_p^4 H_0(r_p) dr_p}{\int_0^{\infty} r_p^4 H_0(r_p) dr_p} \\ \frac{\partial}{\partial T} H^1(r_p, X, T) = -\phi \frac{r_p^4 H_0(r_p)}{\int_0^{\infty} r_p^4 H_0(r_p) dr_p} \int_{r_p}^{\infty} \lambda(r_s, r_p) C^1(r_s, X, T) dr_s \end{cases} \tag{4.9}$$

$$\begin{cases} X = 0 : C^1 = C_{inj}\left(r_s\right) \\ T = 0 : C^1 = 0;\ H^1(r_p, X, 0) = 0. \end{cases} \tag{4.10}$$

Let us consider the case of constant λ. From the first equation (4.9) it follows that the flux reduction factor α is independent of X and T. In this case, the first two equations (2.28), considering the first order approximation (4.5), become linear:

$$\begin{cases} \dfrac{\partial}{\partial T} C\left(r_s, X, T\right) + \alpha\left(r_s\right) \dfrac{\partial}{\partial X} C\left(r_s, X, T\right) = -\dfrac{1}{\phi}\dfrac{\partial}{\partial T} S\left(r_s, X, T\right) \\ \dfrac{\partial}{\partial T} S\left(r_s, X, T\right) = \lambda\phi\left[1 - \alpha\left(r_s\right)\right] C\left(r_s, X, T\right) \end{cases} \tag{4.11}$$

where:

$$\alpha\left(r_s\right) = \frac{\int\limits_{r_s}^{r_{p\max}} f_p\left(r_p, T = 0\right) r_p^4 dr_p}{\int\limits_{r_{p\min}}^{r_{p\max}} f_p\left(r_p, T = 0\right) r_p^4 dr}. \tag{4.12}$$

The carrier water velocity in dimensionless variables (1.2) is equal to one. The velocity of characteristic lines in the first equation (4.11) (i.e., the velocity of particles with size r_s) equals $\alpha(r_s) < 1$. The capture rate in the model (2.28) is proportional to the flow through small pores, so in (4.11) the term $[1 - \alpha(r_s)]$ does appear in the expression for the capture kinetics. Particles of different sizes undergo deep bed filtration independently of each other.

The effect of particle velocity reduction compared with carrier water velocity was observed in laboratory tests [15].

4.2 Analytical solution

The analytical solution for the linear system (4.11) subject to initial and boundary conditions (2.29) is obtained using the method of characteristics [23]:

$$C(r_s, X, T) = C_{inj} \exp\left[-\lambda \frac{1 - \alpha(r_s)}{\alpha(r_s)} X\right] \ :\ X < \alpha(r_s) T \tag{4.13}$$

$$C(r_s, X, T) = S(r_s, X, T) = 0 \quad : \quad X > \alpha(r_s) T. \tag{4.14}$$

$$S(r_s, X, T) = \lambda\phi\,[1 - \alpha(r_s)]\left(T - \frac{X}{\alpha(r_s)}\right)C(r_s, X, T). \tag{4.15}$$

The concentration front for particles with size r_s moves with velocity $\alpha(r_s)$ (see (4.13) and (4.15)). The decrement of the suspension concentration is also a function of particle size.

Let us consider the existence of minimum and maximum pore sizes. The flux reduction term $\alpha(r_s)$ for large particles ($r_s > r_{pmax}$) equals zero. From Eqs. (4.13)–(4.15) it follows that:

$$C(r_s, X, T) = S(r_s, X, T) = 0. \tag{4.16}$$

So, large particles ($r_s > r_{pmax}$) do not penetrate into the porous medium. All large injected particles are captured at the inlet cross-section ($X = 0$). The deposited concentration at the inlet cross section is obtained from the first equation (3.1):

$$S\,(r_s, X = 0, T)\; = \lambda\,\phi\,C_{inj}\,(r_s)\;T. \tag{4.17}$$

For small particles ($r_s < r_{pmin}$), the flux reduction term $\alpha(r_s)$ is equal to one, Eq. (4.12). For $\alpha(r_s) = 1$, Eqs. (4.13)–(4.15) become:

$$C(r_s, X, T) = C_{inj}\;\;,\quad X < T, \tag{4.18}$$

$$C(r_s, X, T) = 0\;\;,\quad X > T, \tag{4.19}$$

$$S(r_s, X, T) = 0. \tag{4.20}$$

Equations (4.18)–(4.20) show that small particles flow through porous media without being captured.

Fig. 4 exhibits the concentration waves for small, intermediate and large particles. Notice that the particle concentration for each particle size $C(r_s, X, T)$ is zero ahead of the propagation fronts: $X_f(r_s, T) = \alpha(r_s)T$ (see (4.15)). The concentration wave for small particles propagates through the reservoir without change (curve 1), while the concentration of large particles in the reservoir equals zero (curve 4). The smaller the particle is, the faster its concentration front propagates and the slower its capture is (curve 2 is located ahead and above curve 3).

The dynamics of suspended particle and vacancy distributions is shown in Fig. 5. The injected particle size distribution and the initial pore size distribution are presented in Fig. 5a. Fig. 5b shows the suspended particle and vacancy distributions for $T > 0$. In first order approximation, the flux reduction factor is not affected by deep bed filtration of low concentration suspension. In the interval $[r_{pmin}, r_{pmax}]$ the particle concentration decreases from $C_{inj}(r_s = r_{pmin})$ to zero for $r_s = r_{pmax}$.

The larger the particles are, the larger the decrement $(1 - \alpha(r_s))/\alpha(r_s)$ of the exponent in (4.13) is. So smaller particles have higher relative concentration $(C(r_s, X, T)/C_{inj}(r_s))$ and their concentration profile moves faster. Fig. 6 shows different particle size concentration history at the core outlet $(X = 1)$. The smaller the particle is, the earlier it arrives to the outlet and the higher its relative concentration is.

The simplified assumption that the filtration coefficient is constant was used in the current and previous sections in order to illustrate the effects of the flux reduction factor α on the velocity of the concentration front and on the concentration decrement. In reality, analytical solutions like (4.13)–(4.15) can be obtained for any $\lambda = \lambda(r_s, r_p)$. When the filtration coefficient is independent of pore radius, $\lambda = \lambda(r_s)$, the obtained system (4.11) is not changed. Therefore, the solution (4.13)–(4.15) is still valid. In the general case $\lambda = \lambda(r_s, r_p)$, the term $\lambda(1 - \alpha)$ in (4.11), (4.13) and (4.14) must be substituted by

$$\frac{\int_0^{r_s} \lambda\left(r_s, r_p\right) r_p^4 H\left(r_p\right) dr_p}{\int_0^{\infty} r_p^4 H\left(r_p\right) dr_p},$$

i.e., in general, the effects of capture and flux reduction cannot be separated.

For injection of low suspended particle concentration, the system (3.1) allows calculating immobile concentrations at the inlet cross-section:

$$S\left(r_s, X = 0, T\right) = \lambda\, \phi\, C_{inj}\, \left[1 - \alpha\left(r_s\right)\right] T, \tag{4.21}$$

$$H\left(r_p, X = 0, T\right) = \left[h(0, 0) - \int_{r_{p\min}}^{\infty} S\left(r_s, X = 0, T\right) dr_s\right] f\left(r_p\right). \tag{4.22}$$

The lower limit of the integral in (4.22) is r_{pmin} because $S(r_s, X, T)$ is zero for $r_s < r_{pmin}$.

4.3 Penetration depth

Let us calculate the average dimensionless penetration depth for particles with different sizes$\langle X(r_s, T)\rangle$:

$$\langle X(r_s, T)\rangle = \frac{\int_0^{\alpha T} XC(r_s, X, T)dX}{\int_0^{\alpha T} C(r_s, X, T)dX}, \tag{4.23}$$

where: $\alpha = \alpha(r_s)$.

The particle concentration $C(r_s, X, T)$ is zero ahead of the propagation front $X_f(r_s, T) = \alpha T$, so the integration in (4.23) is performed from zero to αT. Substituting (4.13) and (4.15) into (4.23), we calculate the average dimensionless penetration depth dynamics for particles with intermediate sizes:

$$\langle X(r_s, T)\rangle = \frac{\alpha}{\lambda(1-\alpha)}\left[1 - \frac{\lambda(1-\alpha)T\exp[-\lambda(1-\alpha)T]}{1-\exp[-\lambda(1-\alpha)T]}\right]. \tag{4.24}$$

The maximum dimensionless penetration depth distribution for an ensemble with a fixed particle size $\langle X(r_s)\rangle_{\max}$ is obtained by letting T tend to infinity in (4.24):

$$\langle X(r_s)\rangle_{\max} = \lim_{T\to\infty}\langle X(r_s, T)\rangle . \tag{4.25}$$

Substituting (4.24) into (4.25) we obtain the expression for maximum dimensionless penetration depth:

$$\langle X(r_s)\rangle_{\max} = \frac{1}{\lambda}\frac{\alpha(r_s)}{1-\alpha(r_s)} . \tag{4.26}$$

Fig. 7 shows how the maximum penetration depth depends of the particle radii. Three different probability distribution functions (PDF) were used for pore sizes: normal, lognormal and uniform. These three PDFs are bi-parametrical, so we assumed the same mean and standard deviation values ($\langle r_p\rangle = 11\mu m$, $s = 0.577\mu m$) for the three examples.

Particles with radii larger than r_{pmax} do not penetrate into porous medium, so $\langle X(r_s > r_{p\max})\rangle_{\max} = 0$. Particles with radii smaller than r_{pmin} flow without being captured, so $\langle X(r_s < r_{p\min})\rangle_{\max}$ tends to infinity when the time T tends to infinity. Finally, the smaller the particles are, the deeper they penetrate into the porous medium (Fig. 6).

4.4 Averaged concentration model

Let us derive the model for the average concentration and compare it with the classical model for deep bed filtration (1.1).

For small particles ($r_s < r_{pmin}$), the flux reduction factor equals one ($\alpha = 1$). Integration of both equations (4.11) in r_s from zero to r_{pmin}, yields the averaged equations for the population of small particles:

$$\begin{cases} \dfrac{\partial}{\partial T} c_1(X, T) + \dfrac{\partial}{\partial X} c_1(X, T) = -\dfrac{1}{\phi} \dfrac{\partial}{\partial T} \sigma_1(X, T) \\ \dfrac{\partial}{\partial T} \sigma_1(X, T) = 0 \,, \end{cases} \tag{4.27}$$

where c_1 is the total concentration of small particles:

$$c_1(X, T) = \int_0^{r_{p\min}} C(r_s, X, T) dr_s \,. \tag{4.28}$$

The solution for the system (4.27) subject to initial and boundary conditions (2.29) is:

$$c_1(X, T) = \begin{cases} c_{1inj}, & X < T \\ 0, & X > T \,, \end{cases} \tag{4.29}$$

$$\sigma_1(X, T) = 0 \,. \tag{4.30}$$

Equations (4.29) and (4.30) show that small particles move with the carrier water velocity without entrapment.

The equations for the total concentrations of intermediate size particles are obtained by integration of the first and second Eqs. (4.11) in r_s from r_{pmin} to

r_{pmax}:

$$\begin{cases} \dfrac{\partial}{\partial T} c_2(X,T) + \dfrac{\partial}{\partial X}\left[\langle\alpha\rangle c_2(X,T)\right] = -\dfrac{1}{\phi}\dfrac{\partial}{\partial T}\sigma_2(X,T) \\ \dfrac{\partial}{\partial T}\sigma_2(X,T) = \lambda\phi c_2\left[1-\langle\alpha\rangle\right] , \end{cases} \tag{4.31}$$

where c_2 is the total concentration of intermediate size particles:

$$c_2(X,T) = \int_{r_{p\min}}^{r_{p\max}} C(r_s, X, T)dr_s\,. \tag{4.32}$$

In Eq. (4.31), the average flux reduction factor $\langle\alpha\rangle$ for intermediate size particles is:

$$\langle\alpha\rangle = \frac{\int_{r_{p\min}}^{r_{p\max}} \alpha(r_s) f_s(r_s, X, T)\, dr_s}{\int_{r_{p\min}}^{r_{p\max}} f_s(r_s, X, T)\, dr_s}\,. \tag{4.33}$$

So, when compared with the classical deep bed filtration model (1.1), the advective term in the population balance equation (4.31) for intermediate size particles contains an average flux reduction term $\langle\alpha\rangle$. The second equation (4.31) shows that the filtration coefficient for the average concentration model equals the filtration coefficient for the classical model (1.1) times $(1-\langle\alpha\rangle)$. The term $(1-\langle\alpha\rangle)$ corresponds to the fraction of water flux through thin pores where the capture takes place.

For large particles, $\alpha(r_s)$ is equal to zero. Integration of system (4.11) in r_s from r_{pmax} to infinity allows obtaining the averaged model for large particles:

$$\begin{cases} \dfrac{\partial}{\partial T} c_3(X,T) = -\dfrac{1}{\phi}\dfrac{\partial}{\partial T}\sigma_3(X,T) \\ \dfrac{\partial}{\partial T}\sigma_3(X,T) = \lambda\phi\, c_3 , \end{cases} \tag{4.34}$$

where:

$$c_3(X,T) = \int_{r_{p\max}}^{\infty} C(r_s, X, T)dr_s\,. \tag{4.35}$$

The solution for the system (4.34) is:

$$c_3(X, T) = 0 : \quad X > 0 . \tag{4.36}$$

Equation (4.36) shows that there is no transport of large particles; all the suspended large particles are captured by the porous medium at the inlet cross section.

The solutions for systems (4.27), (4.31) and (4.34) show that only intermediate size particles perform deep bed filtration. This means that in the classical theory for deep bed filtration the concentration "c" used in system (1.1) must be equal to c_2. Fig. 8 illustrates the portions of overall particle concentration (dashed areas) that should be neglected in the classical theory for deep bed filtration (1.1).

Furthermore, the flux reduction factor (4.33) should be taken into account in the classical theory (1.1).

5 Summary and conclusions

A population balance model for deep bed filtration with size exclusion mechanism was derived from the Master equation. The model accounts for particle flux reduction due to motion of particles with a given size in larger pores only.

For low concentration suspensions, the change of flux reduction factor due to pore plugging can be neglected, allowing linearization of the governing equations. The linearized deep bed filtration problem has exact analytical solution for any particle and pore size distributions. The analysis of the solution leads to the following conclusions:

1. The absence of r_s-particle flow in pores smaller than r_s results in particle flux reduction when compared with the carrier water flux. Consequently, a flux reduction factor α appears in the advective term of the population balance equation.

2. The capture rate is proportional to the flux through small pores. So, the factor $(1 - \alpha)$ appears in the expression for the capture kinetics.

3. Populations of particles with different sizes filtrate independently. The velocity of each population is $\alpha(r_s)$ times lower than the carrier water

velocity. The capture rate for each population is also a function of the particle radius.

4. The penetration depth for the intermediate size particles equals the flux reduction factor α divided by the filtration coefficient times $(1-\alpha)$. The smaller the particles are, the deeper they penetrate.

5. The averaged model includes three systems of equations: one for small particles, another for large particles and the third for intermediate size particles.

6. Deep bed filtration occurs only when there is an overlap between pore and particle size distributions. Particles larger than the maximum pore radii do not enter the porous media and particles smaller than the minimum pore radii are transported without being captured. Consequently, only intermediate size particle concentration should be taken into account in the classical deep bed filtration theory (1.1) for size exclusion mechanism.

7. The averaged model for intermediate size particles differs from the traditional deep bed filtration model by the flux reduction factor α that appears in the advective term of population balance equation. This difference is due to particle motion through large pores only. The two models also differ from each other by the filtration coefficient expression, which in the averaged model is proportional to $(1-\alpha)$. This is due to particle capture by small pores only.

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7 Nomenclature

c	total suspended particle concentration, $no.\ L^{-3}$
C	concentration distribution of suspended particles with radius r_s, $no.\ L^{-4}$
f	probability distribution function, L^{-1}
h	total vacant pore concentration, $no.\ L^{-3}$
H	concentration distribution of vacant pores with radius r_p, $no.\ L^{-4}$
J	particle flux, $L^{-3}.T^{-1}$
L	core length, L
p	pressure, $ML^{-1}.T^{-2}$
P	probability density that a particle with radius r_s meets a pore with radius r_p, L^{-2}
r_p	pore radius, L
r_s	particle radius, L
s	standard deviation, L
S	concentration distribution of retained particles with radius r_s, $no.\ L^{-4}$
t, T	dimensional and dimensionless times
U	fluid velocity, $L.T^{-1}$
x, X	dimensional and dimensionless linear co-ordinates
$\langle X \rangle$	dimensionless average penetration depth

Greek Letters

α	flux reduction factor
δ	Dirac's delta function
ϕ	porosity
λ	dimensionless filtration coefficient
μ	viscosity, $ML^{-1}.T^{-1}$
σ	total captured particle concentration, $no.\ L^{-3}$

Subscripts

F	front
inj	injected
0	initial
p	pore
s	suspended particle
σ	captured particles

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Size exclusion during particle suspension transport in porous media: stochastic and averaged equations

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Abstract. A pore scale population balance model is formulated for deep bed filtration of stable particulate suspensions in porous media. The particle capture from the suspension by the rock occurs by the size exclusion mechanism. The equations for particle and pore size distributions have been derived from the stochastic *Master equation*. The model proposed is a generalization of stochastic Sharma-Yortsos equations – it accounts for particle flux reduction due to restriction for large particles to move via small pores. Analytical solution for low particle concentration is obtained for any particle and pore size distributions. The averaged macro scale equations, derived from the stochastic pore scale model, significantly differ from the traditional deep bed filtration model.

Mathematical subject classification: 76S05, 60H15.

Key words: deep bed filtration, particle size distribution, pore size distribution, size exclusion mechanism, stochastic model, averaging.

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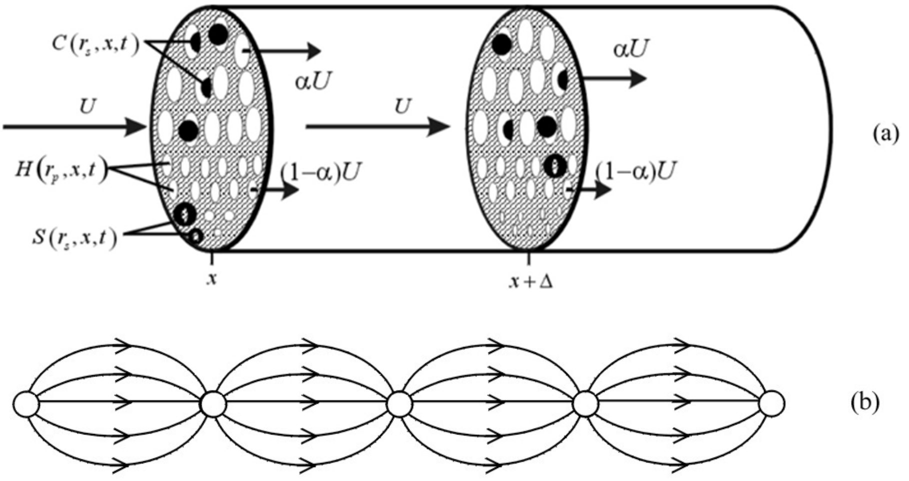

Figure 1 – Geometric model of the porous medium (a) Sketch for particulate suspension transport in porous media: particles are captured due to size exclusion. (b) Connectivity in the porous medium model.

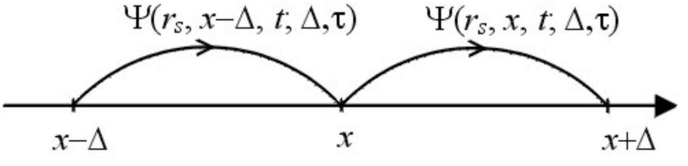

Figure 2 – Transition probabilities for particle displacement in *Master* equation.

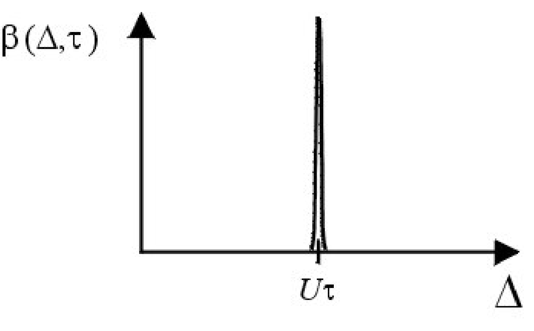

Figure 3 – Shape of the fluid displacement probability function β.

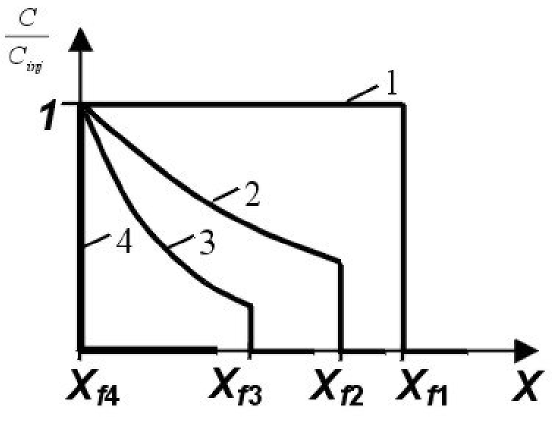

Figure 4 – Concentration waves for different particle sizes. Each front moves with velocity $\alpha(r_s)$. Line 1 corresponds to transport of small particles ($r_{s1} < r_{pmin}$). Lines 2 and 3 are related to filtration of intermediate size particles, $r_{s2} < r_{s3}$. Line 4 shows the large particle concentration ($r_{s4} > r_{pmax}$).

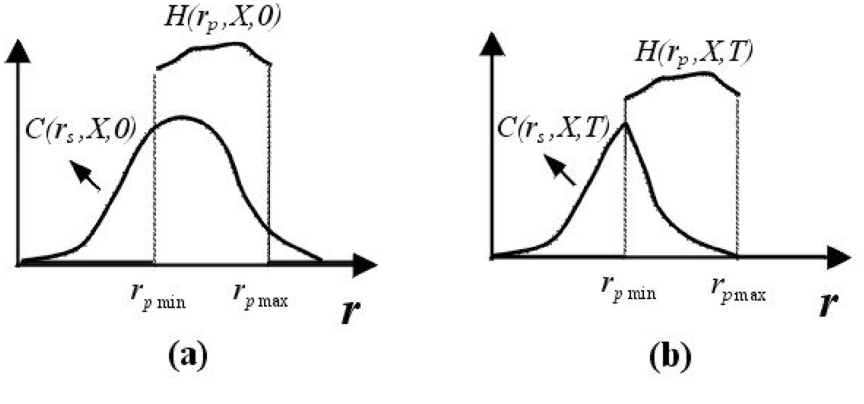

Figure 5 – Concentration distributions for suspended particles and vacant pores during filtration: (a) initial distributions; (b) distributions behind the concentration front for $T > 0$.

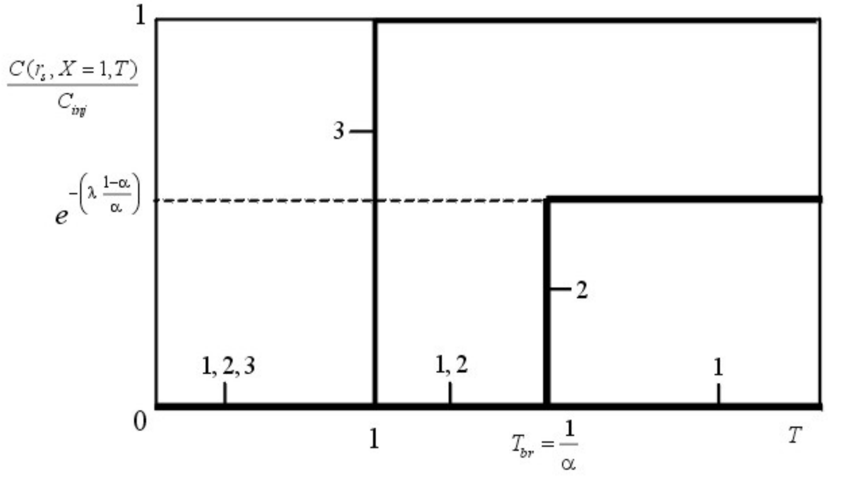

Figure 6 – Suspended concentration at the core outlet. Lines 1 and 3 are related to large and small particles, respectively. Line 2 corresponds to intermediate size particle concentration.

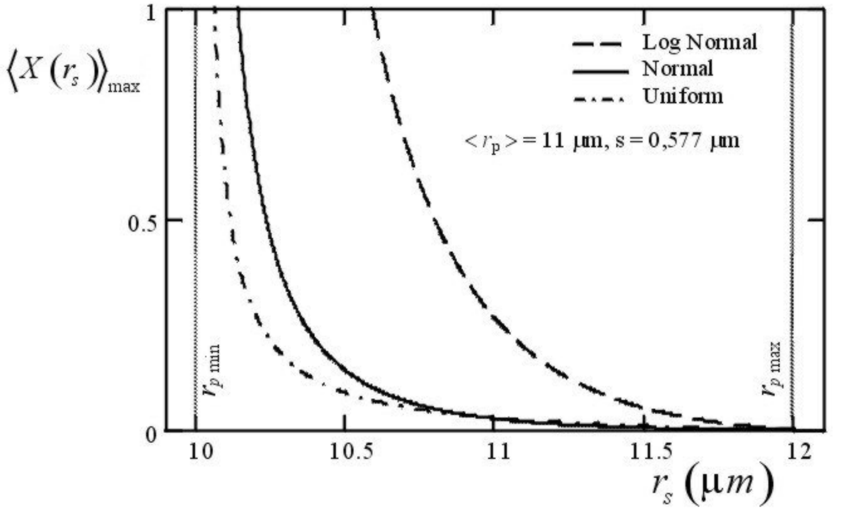

Figure 7 – Effect of particle size on dimensionless penetration depth $\langle X(rs) \rangle_{\max}$ for intermediate size particles. Three curves were calculated for three different pore size distributions assuming the same average pore size ($\langle rp \rangle = 11\mu m$), standard deviation ($s = 0; 577\mu m$) and dimensionless filtration coefficient ($\lambda = 50$).

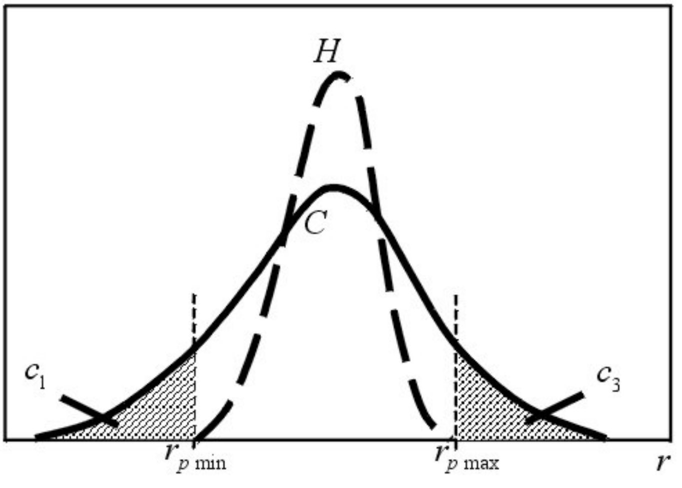

Figure 8 – Shapes of suspended particle and vacant pore size distributions. Areas c_1 and c_3 represent the fraction of particles that do not perform deep bed filtration (small and large particles, respectively).

This article has received corrections in agreement with the ERRATUM published in Volume 24 Number 3.