

## Two approximate methods of a Cauchy problem for the Helmholtz equation

XIANG-TUAN XIONG and CHU-LI FU

School of Mathematics and Statistics, Lanzhou University, Lanzhou 730000, P.R. China

E-mail: fuchuli@lzu.edu.cn

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**Abstract.** In this paper, we consider a Cauchy problem for the Helmholtz equation at fixed frequency, especially we give the optimal error bound for the ill-posed problem. Within the framework of general regularization theory, we present some spectral regularization methods and a modified Tikhonov regularization method to stabilize the problem. Moreover, Hölder-type stability error estimates are proved for these regularization methods. According to the regularization theory, the error estimates are order optimal. Some numerical results are reported.

**Mathematical subject classification:** 35R25, 35R30.

**Key words:** inverse problems, Helmholtz equation, spectral regularization, Tikhonov regularization, error estimate.

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### 1 Introduction

The Cauchy problem for Helmholtz equation arises from inverse scattering problems. Specific backgrounds can be seen in the existing literature, we can refer to [1]-[6] etc. A number of numerical methods for stabilizing this problem are developed. Several boundary element methods combined with iterative, conjugate gradient, Tikhonov regularization and singular value decomposition method are compared in [6]. However, these numerical methods are short of stability analysis and error analysis. Recently, in [5] many applications for a model of Helmholtz equation are introduced, a Fourier regularization method [7] (also is known as a method by cutting off high frequency directly) is applied for solving

a Cauchy problem for Helmholtz equation, some error estimates are also obtained. In [8], for sideways heat equation we connect the Fourier regularization method with general regularization theory, we discover that Fourier regularization method can be considered as the ‘generalized’ TSVD method [9]. Based on some ideas of [8], in this paper, we use some spectral methods [10] to solve the Cauchy problem. In addition, a revised Tikhonov regularization method is also considered. Since the numerical implementation for our methods is similar to the method provided by [5], we only give some numerical results.

This aim of this paper is to give some regularization methods within the framework of general regularization theory, which are different from “Approximate solution of a Cauchy problem for the Helmholtz equation” by T. Reginska and K. Reginski [5], where Fourier regularization method is only an ‘isolated’ method. Moreover, we find that for the Cauchy problem Fourier regularization method is one of the considered spectral methods.

## 2 Model problem and optimal error bound

Let  $\Omega = \mathbb{R}^2 \times (0, d) \subset \mathbb{R}^3$ . The first two variable are denoted by  $r = (x, y)$ . In addition, there are two boundaries:  $\Gamma_0 = \{(r, 0), r \in \mathbb{R}^2\} \subset \partial\Omega$ ,  $\Gamma = \{(r, d), r \in \mathbb{R}^2\} \subset \partial\Omega$ . Let us consider the following problem for Helmholtz equation

$$\begin{aligned} \Delta u + k^2 u &= 0 && \text{in } \Omega \\ u(r, d) &= g(r) && r \in \mathbb{R}^2, \\ \partial_z u(r, d) &= 0 && r \in \mathbb{R}^2, \\ u(\cdot, z) &\in L^2(\mathbb{R}^2) && z \in (0, d), \end{aligned} \tag{2.1}$$

where  $\Delta u = (\partial^2 u)/(\partial x^2) + (\partial^2 u)/(\partial y^2) + (\partial^2 u)/(\partial z^2)$  and  $k > 0$  is the wavenumber(real constant). We want to obtain the solution  $u(r, z)$  for  $0 < z < d$ . Since the data  $g(\cdot)$  are based on (physical) observations and are not known with complete accuracy, we assume that  $g(\cdot)$  and  $g^\delta(\cdot)$  satisfy

$$\|g(\cdot) - g^\delta(\cdot)\| \leq \delta, \tag{2.2}$$

where  $g(\cdot)$  and  $g^\delta(\cdot)$  belong to  $L^2(\mathbb{R}^2)$ ,  $g^\delta(\cdot)$  denotes the measured data and  $\delta$  denotes the noise level,  $\|\cdot\|$  denotes the  $L^2(\mathbb{R}^2)$ -norm.

In order to use Fourier transform technique with respect to variable  $r \in \mathbb{R}^2$ , we define the Fourier transform

$$\hat{u}(\xi, z) := \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{-i\xi \cdot \rho} u(r, z) dr, \quad \xi \in \mathbb{R}^2. \tag{2.3}$$

The problem (2.1) can be reformulated in the frequency space as

$$\begin{aligned} \hat{u}_{zz}(\xi, z) &= (|\xi|^2 - k^2)\hat{u}(\xi, z), & \xi \in \mathbb{R}^2, z \in (0, d) \\ \hat{u}(\xi, d) &= \hat{g}(\xi), & \xi \in \mathbb{R}^2, \\ \partial_z \hat{u}(\xi, d) &= 0, & \xi \in \mathbb{R}^2. \end{aligned} \tag{2.4}$$

If  $u$  is the solution of (2.1), then its Fourier transform  $\hat{u}$  is the solution to problem (2.4) and is given by

$$\hat{u}(\xi, z) = \cosh((d - z)\sqrt{|\xi|^2 - k^2})\hat{g}(\xi), \tag{2.5}$$

where  $\xi = (\xi_1, \xi_2)$ ,  $|\xi|^2 = \xi_1^2 + \xi_2^2$ .

From (2.5) and Parseval identity, we will see that if  $\hat{g}(\xi)$  decays rapidly at  $|\xi| \rightarrow \infty$  in frequency domain, then the solution of problem (2.1) lies in  $L^2(\Omega)$ .

If  $|\xi| \leq k$ , then  $|\xi|^2 - k^2 < 0$ ,  $\cosh((d - z)\sqrt{|\xi|^2 - k^2}) = \cos((d - z)\sqrt{k^2 - |\xi|^2})$ . Thus, (2.5) becomes

$$\hat{u}(\xi, z) = \cos((d - z)\sqrt{k^2 - |\xi|^2})\hat{g}(\xi). \tag{2.6}$$

If  $|\xi| > k$ , then  $|\xi|^2 - k^2 > 0$ , it is easy to see that if

$$\hat{u}_\delta(\xi, z) = (\hat{g}(\xi) + \delta) \cosh((d - z)\sqrt{|\xi|^2 - k^2}),$$

then

$$\hat{u}_\delta(\xi, z) - \hat{u}(\xi, z) = \delta \cosh((d - z)\sqrt{|\xi|^2 - k^2}).$$

Obviously  $\cosh((d - z)\sqrt{|\xi|^2 - k^2}) \rightarrow \infty$  as  $|\xi| \rightarrow \infty$ , this factor can amplify the error  $\delta$  arbitrarily.

In (2.5), setting  $z = 0$ , we have

$$\hat{u}(\xi, 0) = \cosh(d\sqrt{|\xi|^2 - k^2})\hat{g}(\xi). \tag{2.7}$$

As usual, in order to obtain convergence rate, we assume that there exists an a-priori bound for problem (2.1):

$$M = \{u(r, z) \mid \|u(\cdot, 0)\|_{L^2(\mathbb{R}^2)} \leq E\}, \quad (2.8)$$

where  $E$  is a given constant, throughout this paper, we use the same notation  $E$ .

Due to ill-posedness, problem (2.1) is unstable in numerical simulation and requires regularization methods. What is the optimal error bound for solving problem (2.1) by all regularization methods? In order to answer this question, we review some results on optimality theory for ill-posed problems.

Consider an ill-posed inverse operator equation [9], [11], [12], [13]

$$Ax^\dagger = y \quad (2.9)$$

where  $A : X \rightarrow Y$  is a bounded linear operator between infinite-dimensional Hilbert spaces  $X$  and  $Y$ . Let  $y^\delta \in Y$  be the available noisy data with  $\|y^\delta - y\| \leq \delta$ . Any operator  $\mathfrak{R} : Y \rightarrow X$  can be seen as a special method for solving (2.9), the approximate solution to (2.9) is then given by  $\mathfrak{R}y^\delta$ .

Let  $M \in X$  a bounded set. Define the worst case error according to

$$\Delta(\delta, \mathfrak{R}) := \sup \left\{ \|\mathfrak{R}y^\delta - x^\dagger\| \mid x^\dagger \in M, y^\delta \in Y, \|Ax^\dagger - y^\delta\| \leq \delta \right\}. \quad (2.10)$$

The worst case error characterizes the maximal error of the method  $\mathfrak{R}$  if the solution  $x^\dagger$  of (2.9) varies in the set  $M$ . A method  $\mathfrak{R}_0$  is called

- (i) *optimal on the set  $M$  if  $\Delta(\delta, \mathfrak{R}_0) = \inf_{\mathfrak{R}:Y \rightarrow X} \Delta(\delta, \mathfrak{R})$ ;*
- (ii) *order optimal on the set  $M$  if  $\Delta(\delta, \mathfrak{R}_0) \leq c \inf_{\mathfrak{R}:Y \rightarrow X} \Delta(\delta, \mathfrak{R})$  with  $c > 1$ .*

Let the “source set”  $M$  be given by

$$M_{\varphi, E} = \left\{ x^\dagger \in X \mid x^\dagger = [\varphi(A^*A)]^{1/2} v, \|v\| \leq E \right\}, \quad (2.11)$$

where the operator function  $\varphi(A^*A)$  is well defined via spectral representation  $\varphi(A^*A) = \int_0^a \varphi(\lambda) dE_\lambda$  where  $A^*A = \int_0^a \lambda dE_\lambda$  is the spectral decomposition of  $A^*A$ ,  $\{E_\lambda\}$  denotes the spectral family of the operator  $A^*A$  and  $a$  is a constant with  $\|A^*A\| \leq a$ . In addition,

**Assumption 2.1.** *The function  $\varphi : (0, a] \rightarrow (0, \infty)$  in (2.11) is continuous and satisfies*

- (i)  $\lim_{\lambda \rightarrow 0} \varphi(\lambda) = 0$
- (ii)  $\varphi(\lambda)$  is strong monotonically increasing on  $(0, a]$
- (iii)  $\rho(\lambda) = \lambda\varphi^{-1}(\lambda) : (0, \varphi(a)] \rightarrow (0, a\varphi(a)]$  is convex.

**Theorem 2.2.** [12] *Let  $M$  be given by (2.11), let Assumption 2.1 be satisfied and  $\delta^2/(E^2) \in \sigma(A^*A\varphi(A^*A))$  where  $\sigma(A^*A)$  denotes the spectrum of the operator  $A^*A$ . Then*

$$\omega(\delta, E) := \inf_{\mathfrak{H}} \Delta(\delta, \mathfrak{H}) = E \sqrt{\rho^{-1}\left(\frac{\delta^2}{E^2}\right)}. \tag{2.12}$$

Based on Theorem 2.2, we can obtain the following results:

**Theorem 2.3.** *Let  $\delta^2/E^2 \leq 1$ . Then under the assumption (2.8) we have the following optimal error bound for problem (2.1)*

$$\omega(\delta, E) = E^{1-\frac{\delta}{E}} \delta^{\frac{\delta}{E}} (1 + o(1)) \quad \text{for } \delta \rightarrow 0. \tag{2.13}$$

As for the proof of Theorem 2.3, we can refer to the Appendix.

Most regularization operators can be written in the form,

$$\mathfrak{H}_\alpha := g_\alpha(A^*A)A^* \tag{2.14}$$

with some function  $g_\alpha$  satisfying

$$\lim_{\alpha \rightarrow 0} g_\alpha(\lambda) = \frac{1}{\lambda}, \tag{2.15}$$

where  $\alpha > 0$  plays the role of regularization parameter. Then for the regularization solution with noisy data, we have  $x_\alpha^\delta := \mathfrak{H}_\alpha y^\delta$ . For example,

*Spectral method 1.*

$$g_\alpha(\lambda) = \begin{cases} \frac{1}{\lambda}, & \lambda \geq \alpha, \\ \frac{1}{\alpha}, & \lambda < \alpha. \end{cases}$$

*Spectral method 2.*

$$g_{\alpha}(\lambda) = \begin{cases} \frac{1}{\lambda}, & \lambda \geq \alpha, \\ \frac{1}{\sqrt{\alpha\lambda}}, & \lambda < \alpha. \end{cases}$$

*Spectral method 3 (TSVD method).*

$$g_{\alpha}(\lambda) = \begin{cases} \frac{1}{\lambda}, & \lambda \geq \alpha, \\ 0, & \lambda < \alpha. \end{cases}$$

*Tikhonov method.*

$$g_{\alpha}(\lambda) = \frac{1}{\alpha + \lambda}.$$

In general, the exact solution  $x^{\dagger} \in X$  is required to satisfy a so-called source condition, otherwise the convergence of the regularization method approximating the problem can be arbitrarily slow. For problem (2.1), the condition (2.8) is assumed for the above reason.

### 3 Error estimates on the Cauchy problem for the Helmholtz equation

In this section, we will analyze the error estimates by different regularization methods.

By the similar method in [14], (2.1) can be formulated as an operator equation in frequency domain

$$\hat{A}(z)\hat{u}(\xi, z) = \hat{g}(\xi), \quad (3.1)$$

obviously, the multiplication operator  $\hat{A}(z)$  is given by

$$\hat{A}(z) = \frac{1}{\cosh((d-z)\sqrt{|\xi|^2 - k^2})}, \quad (3.2)$$

and

$$\hat{A}^*(z) = \frac{1}{\cosh((d-z)\sqrt{|\xi|^2 - k^2})}, \quad \hat{A}^*(z)\hat{A}(z) = \frac{1}{|\cosh((d-z)\sqrt{|\xi|^2 - k^2})|^2}.$$

Now we give a new interpretation for the Fourier regularization method provided by [5]. Applying the TSVD method for solving problem (2.1), we have

$$\hat{u}_\alpha^\delta(\xi) = \begin{cases} \cosh((d-z)\sqrt{|\xi|^2 - k^2})\hat{g}^\delta(\xi), & \frac{1}{|\cosh((d-z)\sqrt{|\xi|^2 - k^2})|^2} \geq \alpha, \\ 0, & \frac{1}{|\cosh((d-z)\sqrt{|\xi|^2 - k^2})|^2} < \alpha. \end{cases} \tag{3.3}$$

The inequality  $\frac{1}{|\cosh((d-z)\sqrt{|\xi|^2 - k^2})|^2} < \alpha$  is equivalent to

$$|\cosh((d-z)\sqrt{|\xi|^2 - k^2})| > \frac{1}{\sqrt{\alpha}}. \tag{3.4}$$

If  $|\xi| > k$ , the (3.4) is equivalent to

$$|\xi|^2 > k^2 + \left[ \frac{1}{d-z} \operatorname{arccosh}\left(\frac{1}{\sqrt{\alpha}}\right) \right]^2.$$

Obviously, regularization method (3.3) stabilize problem (2.1) by cutting off high frequency.

If the spectral method 1 is devoted to solving problem (2.1), then we get

$$\hat{u}_\alpha^\delta(\xi, z) = \begin{cases} \cosh((d-z)\sqrt{|\xi|^2 - k^2})\hat{g}^\delta(\xi), & \frac{1}{|\cosh((d-z)\sqrt{|\xi|^2 - k^2})|^2} \geq \alpha, \\ \frac{1}{\alpha} \frac{1}{\cosh((d-z)\sqrt{|\xi|^2 - k^2})} \hat{g}^\delta(\xi), & \frac{1}{|\cosh((d-z)\sqrt{|\xi|^2 - k^2})|^2} < \alpha. \end{cases} \tag{3.5}$$

Because the three spectral methods in Section 2 are very similar, we only give the approximation properties of the first spectral method.

**Theorem 3.1.** *Supposed that  $u(\rho, z)$  is exact solution with exact data  $g$  and that  $u_\alpha^\delta(\rho, z)$  is approximate solution by spectral method 1 with noisy data  $g^\delta$ . If we have an a-priori bound  $\|u(\cdot, 0)\| \leq E$  and the data functions satisfy  $\|g - g^\delta\| \leq \delta$ , and if we choose*

$$\sqrt{\alpha} = 1/\cosh\left(\frac{d-z}{d} \operatorname{arccosh}\left(\frac{E}{\delta}\right)\right), \tag{3.6}$$

then we can obtain the following error estimate for  $\delta \rightarrow 0$ :

$$\|u(\cdot, z) - u_\alpha^\delta(\cdot, z)\| \leq 3 \cdot \left(\frac{1}{2}\right)^{\frac{z}{d}} E^{1-\frac{z}{d}} \delta^{\frac{z}{d}} (1 + o(1)) \rightarrow 0. \tag{3.7}$$

**Proof.** For the subsequent analysis, we define two sets:  $W := \{\xi : |\xi| \leq k\}$  and  $I := \{\xi : |\xi| > k\}$ . Due to Parseval identity, we have

$$\|u(\cdot, z) - u_\alpha^\delta(\cdot, z)\|_{L^2(\mathbb{R}^2)} = \|\hat{u}(\cdot, z) - \hat{u}_\alpha^\delta(\cdot, z)\|_{L^2(\mathbb{R}^2)}.$$

According to  $\mathbb{R}^2 = W \cup I$ , there holds

$$\begin{aligned} \|\hat{u}(\cdot, z) - \hat{u}_\alpha^\delta(\cdot, z)\|_{L^2(\mathbb{R}^2)} &\leq \|\hat{u}(\cdot, z) - \hat{u}_\alpha^\delta(\cdot, z)\|_{L^2(I)} \\ &\quad + \|\hat{u}(\cdot, z) - \hat{u}_\alpha^\delta(\cdot, z)\|_{L^2(W)}. \end{aligned} \quad (3.8)$$

**Case I:**  $|\xi|^2 - k^2 > 0$ , i.e.,  $\xi \in I$ .

First define two sets

$$A = \left\{ \xi \in I \mid \frac{1}{|\cosh((d-z)\sqrt{|\xi|^2 - k^2})|^2} \geq \alpha \right\}$$

and

$$B = \left\{ \xi \in I \mid \frac{1}{|\cosh((d-z)\sqrt{|\xi|^2 - k^2})|^2} < \alpha \right\},$$

due to the inequality  $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$ , for  $a, b \geq 0$ , we have

$$\begin{aligned} \|\hat{u}(\cdot, z) - \hat{u}_\alpha^\delta(\cdot, z)\|_{L^2(I)} &= \left( \int_A |\cosh((d-z)\sqrt{|\xi|^2 - k^2})(\hat{g} - \hat{g}^\delta)|^2 d\xi \right. \\ &\quad + \int_B \left| \frac{1}{\alpha \cosh((d-z)\sqrt{|\xi|^2 - k^2})} \hat{g}^\delta \right. \\ &\quad \left. \left. - \cosh((d-z)\sqrt{|\xi|^2 - k^2}) \hat{g} \right|^2 d\xi \right)^{1/2} \\ &\leq I_1 + I_2, \end{aligned} \quad (3.9)$$

where

$$\begin{aligned} I_1 &:= \left( \int_A |\cosh((d-z)\sqrt{|\xi|^2 - k^2})(\hat{g} - \hat{g}^\delta)|^2 d\xi \right)^{1/2}; \\ I_2 &:= \left( \int_B \left| \frac{1}{\alpha \cosh((d-z)\sqrt{|\xi|^2 - k^2})} \hat{g}^\delta - \cosh((d-z)\sqrt{|\xi|^2 - k^2}) \hat{g} \right|^2 d\xi \right)^{1/2}. \end{aligned}$$



For  $I_1$ , by noting the definition of the set  $A$ , we obtain

$$\begin{aligned}
 I_1 &\leq \sup_A \left| \cosh((d-z)\sqrt{|\xi|^2 - k^2}) \left( \int_A |(\hat{g} - \hat{g}^\delta)|^2 d\xi \right)^{1/2} \right| \\
 &\leq \frac{1}{\sqrt{\alpha}} \times \delta
 \end{aligned}
 \tag{3.10}$$

For  $I_2$ , according to the triangle inequality

$$\|a - c + c - b\| \leq \|a - c\| + \|b - c\|$$

and the formula (2.7), we have

$$\begin{aligned}
 I_2 &\leq \left( \int_B \left| \frac{1}{\alpha \cosh((d-z)\sqrt{|\xi|^2 - k^2})} \hat{g}^\delta - \frac{1}{\alpha \cosh((d-z)\sqrt{|\xi|^2 - k^2})} \hat{g} \right|^2 d\xi \right)^{1/2} \\
 &\quad + \left( \int_B \left| \frac{1}{\alpha \cosh((d-z)\sqrt{|\xi|^2 - k^2})} \hat{g} - \cosh((d-z)\sqrt{|\xi|^2 - k^2}) \hat{g} \right|^2 d\xi \right)^{1/2} \\
 &\leq \sup_B \left| \frac{1}{\alpha \cosh((d-z)\sqrt{|\xi|^2 - k^2})} \right| \cdot \delta \\
 &\quad + \left( \int_B \left| \left( \frac{1}{\alpha \cosh((d-z)\sqrt{|\xi|^2 - k^2})} - \cosh((d-z)\sqrt{|\xi|^2 - k^2}) \right) \hat{g} \right|^2 d\xi \right)^{1/2} \\
 &\leq \sup_B \left| \frac{1}{\alpha \cosh((d-z)\sqrt{|\xi|^2 - k^2})} \right| \cdot \delta \\
 &\quad + \left( \int_B \left| \left( \frac{1}{\alpha \cosh((d-z)\sqrt{|\xi|^2 - k^2})} \right. \right. \right. \\
 &\quad \left. \left. \left. - \cosh((d-z)\sqrt{|\xi|^2 - k^2}) \right) \frac{\hat{u}(\cdot, 0)}{\cosh(d\sqrt{|\xi|^2 - k^2})} \right|^2 d\xi \right)^{1/2} \\
 &\leq I_3 + I_4,
 \end{aligned}
 \tag{3.11}$$

where

$$I_3 := \sup_B \left| \frac{1}{\alpha \cosh((d-z)\sqrt{|\xi|^2 - k^2})} \right| \cdot \delta;$$

$$I_4 := \sup_B \left| \frac{\left[ \frac{1}{\alpha \cosh((d-z)\sqrt{|\xi|^2 - k^2})} - \cosh((d-z)\sqrt{|\xi|^2 - k^2}) \right]}{\cosh(d\sqrt{|\xi|^2 - k^2})} \right|$$

$$\times \left( \int_B |\hat{u}(\cdot, 0)|^2 d\xi \right)^{1/2}.$$

Noting that the set  $B$  is equivalent to  $\{\eta \mid 1/\cosh((d-z)\eta) < \sqrt{\alpha}\}$ , where  $\eta := \sqrt{|\xi|^2 - k^2}$ , we have

$$I_3 \leq \frac{\delta}{\sqrt{\alpha}}. \quad (3.12)$$

As for  $I_4$ , via the formula (2.8), there holds

$$I_4 \leq \sup_B \left| \frac{\left[ \frac{1 - \alpha |\cosh((d-z)\sqrt{|\xi|^2 - k^2})|^2}{\alpha \cosh((d-z)\sqrt{|\xi|^2 - k^2})} \right]}{\cosh(d\sqrt{|\xi|^2 - k^2})} \right| \times E. \quad (3.13)$$

It is easy to see that the elements in the set  $B$  satisfy

$$1 - \alpha |\cosh((d-z)\sqrt{|\xi|^2 - k^2})|^2 < 0.$$

So by neglecting the negative term  $-\frac{1}{\alpha \cosh((d-z)\sqrt{|\xi|^2 - k^2})}$ , we have

$$I_4 < \sup_B \left[ \frac{\alpha |\cosh((d-z)\sqrt{|\xi|^2 - k^2})|^2}{\alpha |\cosh((d-z)\sqrt{|\xi|^2 - k^2})|} \right] / \cosh(d\sqrt{|\xi|^2 - k^2}) \times E$$

$$= \sup_B \cosh((d-z)\sqrt{|\xi|^2 - k^2}) / \cosh(d\sqrt{|\xi|^2 - k^2}) E. \quad (3.14)$$

On the one hand  $f(\eta) := \cosh((d-z)\eta) / \cosh(d\eta)$  is decreasing with respect to  $\eta$  for  $0 < z < d$ , on the other hand, the set  $B$  is equivalent to

$$\left\{ \eta \mid \cosh((d-z)\eta) > \frac{1}{\sqrt{\alpha}} \right\}, \quad \text{or} \quad \left\{ \eta \mid \eta > \frac{1}{d-z} \operatorname{arccosh} \left( \frac{1}{\sqrt{\alpha}} \right) =: \eta_0 \right\},$$

hence

$$f(\eta) \leq f(\eta_0) = \frac{1}{\sqrt{\alpha}} / \left[ \cosh \left( \frac{d}{d-z} \operatorname{arccosh} \left( \frac{1}{\sqrt{\alpha}} \right) \right) \right]. \tag{3.15}$$

According to the selection (3.6) of  $\alpha$ ,  $\cosh \left( \frac{d}{d-z} \operatorname{arccosh} \left( \frac{1}{\sqrt{\alpha}} \right) \right) = E/\delta$  holds. Thus

$$I_4 \leq f(\eta)E \leq \frac{\frac{1}{\sqrt{\alpha}}}{(E/\delta)} E = \frac{\delta}{\sqrt{\alpha}}. \tag{3.16}$$

Combination of (3.12) and (3.16) gives

$$I_2 \leq 2 \frac{\delta}{\sqrt{\alpha}}. \tag{3.17}$$

Hence by (3.9) (3.10) and (3.17), we get

$$\|\hat{u}(\cdot, z) - \hat{u}_\alpha^\delta(\cdot, z)\|_{L^2(I)} \leq 3\delta/\sqrt{\alpha} = 3\delta \cosh \left( \frac{d-z}{d} \operatorname{arccosh} \left( \frac{E}{\delta} \right) \right). \tag{3.18}$$

According to the asymptotic expression  $\delta \cosh \left( t \operatorname{arccosh} \frac{E}{\delta} \right) = E^t \left( \frac{\delta}{2} \right)^{1-t} (1 + o(1))$ , for  $\delta \rightarrow 0$ ,  $0 < t < 1$  in [14], we have

$$\|\hat{u}(\cdot, z) - \hat{u}_\alpha^\delta(\cdot, z)\|_{L^2(I)} \leq 3E^{1-\frac{z}{d}} \left( \frac{\delta}{2} \right)^{\frac{z}{d}} (1 + o(1)), \quad \text{for } \delta \rightarrow 0. \tag{3.19}$$

**Case II:**  $|\xi|^2 - k^2 \leq 0$ , i.e.,  $\xi \in W$ .

In this case, we note that  $\cosh((d-z)\sqrt{|\xi|^2 - k^2}) = \cos((d-z)\sqrt{k^2 - |\xi|^2})$ . It is easy to establish the following error estimate

$$\|\hat{u}(\cdot, z) - \hat{u}_\alpha^\delta(\cdot, z)\|_{L^2(W)} \leq \delta = o\left(E^{1-z/d} \delta^{z/d}\right). \tag{3.20}$$

Therefore, via (3.8), (3.19), (3.20), we get the error estimate (3.7). □

**Remark 3.2.** For the other two spectral methods, if the regularization parameter  $\alpha$  is properly chosen, we can establish the error estimates similarly. We formulate it as follows:

$$\|u(\cdot, z) - u_\alpha^\delta(\cdot, z)\| \leq CE^{1-\frac{z}{d}} \delta^{\frac{z}{d}} (1 + o(1)) + \delta, \quad \text{for } \delta \rightarrow 0, \tag{3.21}$$

where  $C$  is a positive constant without depending on  $\delta$  and  $E$ . From the theory of Section 2, we can conclude that these three spectral regularization methods are order optimal.

Now we will devote to the Tikhonov regularization method. According to Section 2, similarly we have the following regularized solution in the frequency domain:

$$\hat{u}_{\alpha, Tik}^{\delta}(\cdot, z) = \frac{\cosh((d - z)\sqrt{|\xi|^2 - k^2})}{1 + \alpha |\cosh((d - z)\sqrt{|\xi|^2 - k^2})|^2} \hat{g}^{\delta}. \tag{3.22}$$

If the  $1 + \alpha |\cosh((d - z)\sqrt{|\xi|^2 - k^2})|^2$  is replaced by a faster filter  $1 + \alpha |\cosh(d\sqrt{|\xi|^2 - k^2})|^2$ , we have the revised Tikhonov approximate solution [15, 16]

$$\hat{u}_{\alpha, RTik}^{\delta}(\cdot, z) = \frac{\cosh((d - z)\sqrt{|\xi|^2 - k^2})}{1 + \alpha |\cosh(d\sqrt{|\xi|^2 - k^2})|^2} \hat{g}^{\delta}. \tag{3.23}$$

**Lemma 3.3.** *Let  $0 < z < d$ ,  $|\xi| > k$ ,  $\alpha > 0$ . Then*

$$\sup_{|\xi| > k} \frac{\cosh((d - z)\sqrt{|\xi|^2 - k^2})}{1 + \alpha |\cosh(d\sqrt{|\xi|^2 - k^2})|^2} \leq \left(\frac{2}{\sqrt{\alpha}}\right)^{\frac{d-z}{d}}. \tag{3.24}$$

$$\sup_{|\xi| > k} \frac{\cosh((d - z)\sqrt{|\xi|^2 - k^2}) \cosh(d\sqrt{|\xi|^2 - k^2})}{1 + \alpha |\cosh(d\sqrt{|\xi|^2 - k^2})|^2} \leq \left(\frac{2}{\sqrt{\alpha}}\right)^{\frac{2d-z}{d}}. \tag{3.25}$$

**Proof.** First we prove (3.24). Let  $\eta := \sqrt{|\xi|^2 - k^2}$  and note that  $\cosh((d - z)\eta) \leq \exp((d - z)\eta)$ , we have

$$\frac{\cosh((d - z)\eta)}{1 + \alpha |\cosh(d\eta)|^2} \leq \frac{\exp((d - z)\eta)}{1 + \frac{\alpha \exp(2d\eta)}{4}} = 4 \frac{\exp[(d - z)\eta]}{4 + \alpha \exp[2d\eta]}. \tag{3.26}$$

Denote  $\zeta(\eta) := \frac{4 \exp[(d - z)\eta]}{4 + \alpha \exp[2d\eta]}$ . Differentiating  $\zeta$  and setting  $\zeta'(\eta_0) = 0$ , we find

$$\exp(2d\eta_0) = \frac{4}{\alpha} \cdot \frac{d - z}{d + z} \leq \frac{4}{\alpha}, \tag{3.27}$$

moreover, for

$$\eta > \eta_0, \zeta'(\eta) < 0; \quad \text{for } \eta < \eta_0, \zeta'(\eta) > 0,$$

hence  $\zeta(\eta)$  attains its unique maximum  $\zeta_{\max} = \zeta(\eta_0)$  at the point  $\eta_0$ . Since (3.27),

$$e^{(d-z)\eta_0} = (e^{2d\eta_0})^{\frac{d-z}{2d}} \leq \left(\frac{4}{\alpha}\right)^{\frac{d-z}{2d}}.$$

So

$$\zeta(\eta_0) = \frac{4 \exp((d-z)\eta_0)}{4 + \alpha \exp(2d\eta_0)} \leq \exp((d-z)\eta_0) \leq \left(\frac{4}{\alpha}\right)^{\frac{d-z}{2d}} = \left(\frac{2}{\sqrt{\alpha}}\right)^{\frac{d-z}{d}}. \quad (3.28)$$

Due to (3.26), we have

$$\frac{\cosh((d-z)\eta)}{1 + \alpha |\cosh(d\eta)|^2} \leq \zeta(\eta) \leq \zeta(\eta_0).$$

Noting (3.28), we have the inequality (3.24).

As for (3.25), noting  $\cosh(d\eta) \leq \exp(d\eta)$  and (3.26), we obtain

$$\frac{\cosh((d-z)\eta) \cosh(d\eta)}{1 + \alpha |\cosh(d\eta)|^2} \leq \zeta(\eta) e^{d\eta} = \frac{4e^{(2d-z)\eta}}{4 + \alpha e^{2d\eta}} =: \beta(\eta). \quad (3.29)$$

Repeating the process in the case of (3.24) for  $\beta(\eta)$  yields the inequality (3.25). □

**Theorem 3.4.** *Supposed that  $u(\rho, z)$  is exact solution with exact data  $g$  and that  $u_{\alpha, RTik}^{\delta}(\rho, z)$  is approximate solution by revised Tikhonov method with noisy data  $g^{\delta}$ . If we have an a-priori bound  $\|u(\cdot, 0)\| \leq E$  and the data functions satisfy  $\|g - g^{\delta}\| \leq \delta$ , and if we choose  $\alpha = (\frac{\delta}{E})^2$ , then we can obtain the following error estimate for  $\delta \rightarrow 0$ :*

$$\|u(\cdot, z) - u_{\alpha, RTik}^{\delta}(\cdot, z)\| \leq 3 \cdot 2^{1-\frac{z}{d}} E^{1-\frac{z}{d}} \delta^{\frac{z}{d}} \rightarrow 0. \quad (3.30)$$

**Proof.** Due to Parseval identity, we have

$$\|u(\cdot, z) - u_{\alpha}^{\delta}(\cdot, z)\|_{L^2(\mathbb{R}^2)} = \|\hat{u}(\cdot, z) - \hat{u}_{\alpha}^{\delta}(\cdot, z)\|_{L^2(\mathbb{R}^2)}.$$

Similar to the proof of Theorem 3.1, there holds

$$\begin{aligned} \|\hat{u}(\cdot, z) - \hat{u}_{\alpha}^{\delta}(\cdot, z)\|_{L^2(\mathbb{R}^2)} &\leq \|\hat{u}(\cdot, z) - \hat{u}_{\alpha}^{\delta}(\cdot, z)\|_{L^2(I)} \\ &\quad + \|\hat{u}(\cdot, z) - \hat{u}_{\alpha}^{\delta}(\cdot, z)\|_{L^2(W)}. \end{aligned} \quad (3.31)$$

**Case I:**  $|\xi|^2 - k^2 > 0$ , i.e.,  $\xi \in I$ .

In this case, noting the inequalities (3.24) and (3.25), we have

$$\begin{aligned}
 & \|\hat{u}(\cdot, z) - \hat{u}_{\alpha, RTik}^{\delta}(\cdot, z)\|_{L^2(I)} \\
 &= \left( \int_I \left| \frac{\cosh((d-z)\sqrt{|\xi|^2 - k^2})}{1 + \alpha |\cosh(d\sqrt{|\xi|^2 - k^2})|^2} (\hat{g}^{\delta} - \hat{g}) \right|^2 d\xi \right)^{1/2} \\
 &+ \left( \int_I \left| \frac{\cosh((d-z)\sqrt{|\xi|^2 - k^2})}{1 + \alpha |\cosh(d\sqrt{|\xi|^2 - k^2})|^2} \hat{g} - \cosh((d-z)\sqrt{|\xi|^2 - k^2}) \hat{g} \right|^2 d\xi \right)^{1/2} \\
 &\leq \sup_{|\xi|>k} \frac{\cosh((d-z)\sqrt{|\xi|^2 - k^2})}{1 + \alpha |\cosh(d\sqrt{|\xi|^2 - k^2})|^2} \left( \int_I |\hat{g} - \hat{g}^{\delta}|^2 d\xi \right)^{1/2} \\
 &+ \frac{\alpha \cosh((d-z)\sqrt{|\xi|^2 - k^2}) |\cosh(d\sqrt{|\xi|^2 - k^2})|^2}{1 + \alpha |\cosh(d\sqrt{|\xi|^2 - k^2})|^2} \left( \int_I |\hat{g}|^2 d\xi \right)^{1/2} \\
 &\leq \left( \frac{2}{\sqrt{\alpha}} \right)^{\frac{d-z}{d}} \cdot \delta \\
 &+ \alpha \sup_{|\xi|>k} \frac{\cosh((d-z)\sqrt{|\xi|^2 - k^2}) \cosh(d\sqrt{|\xi|^2 - k^2})}{1 + \alpha |\cosh(d\sqrt{|\xi|^2 - k^2})|^2} \left( \int_I |\hat{u}(\xi, 0)|^2 d\xi \right)^{1/2} \\
 &\leq \left( \frac{2}{\sqrt{\alpha}} \right)^{\frac{d-z}{d}} \cdot \delta + \alpha \cdot \left( \frac{2}{\sqrt{\alpha}} \right)^{\frac{2d-z}{d}} E
 \end{aligned} \tag{3.32}$$

Thus we take

$$\alpha = \left( \frac{\delta}{E} \right)^2.$$

Hence we have

$$\|\hat{u}(\cdot, z) - \hat{u}_{\alpha, RTik}^{\delta}(\cdot, z)\|_{L^2(I)} \leq 2^{1-\frac{z}{d}} E^{1-\frac{z}{d}} \delta^{\frac{z}{d}} + 2^{2-\frac{z}{d}} E^{1-\frac{z}{d}} \delta^{\frac{z}{d}}. \tag{3.33}$$

**Case II:**  $|\xi|^2 - k^2 \leq 0$ , i.e.,  $\xi \in W$ .

In this case, we have

$$\|\hat{u}(\cdot, z) - \hat{u}_{\alpha, RTik}^{\delta}(\cdot, z)\|_{L^2(W)} \leq \delta + \delta^2/E = o(E^{1-\frac{z}{d}} \delta^{\frac{z}{d}}), \quad \text{for } \delta \rightarrow 0. \tag{3.34}$$

Thus, according to (3.31), (3.33), (3.34), it yields (3.30). □

### 4 Numerical experiments

In this section, some numerical results are reported in order to show how the regularization method works. In section 3, we have seen that the Fourier regularization method in [5] is also the Spectral method 3 (TSVD). Since some numerical results on Fourier regularization method has been investigated, here we only provide numerical results for the revised Tikhonov regularization method. For sake of comparison, we take the same numerical example as the one in [5]. The numerical experiments are accomplished by Matlab.

In our test problem  $d = 1$  and  $k = 4$  are fixed. We take the function

$$g(x, y) = \exp\left(-\frac{\pi^2}{\pi^2 - x^2}\right) \exp\left(-\frac{\pi^2}{\pi^2 - y^2}\right) \tag{4.1}$$

as an exact data function on  $\Gamma := (-\pi, \pi)^2$ . For creating  $g^\delta$ , a normally distributed noise of variance  $\epsilon$  is added to  $g$ . A matrix  $G^\delta$  containing samples from  $g^\delta$  on an equidistant grid  $\{r_{s,l}\}_{s,l=0}^n$  for  $n = 64$  is created firstly. Using the 2D fast Fourier transform (FFT2) to the matrix  $G^\delta$ , we can obtain a new matrix approximating the Fourier transform of  $g^\delta$ . Next, this new matrix is multiplied by the matrix

$$\left[ \frac{\cosh((d - z)\sqrt{|m|^2 - k^2})}{1 + \alpha |\cosh(d\sqrt{|m|^2 - k^2})|^2} \right]_{m \in (0,n) \times (0,n)},$$

finally the 2D inverse FFT (IFFT2) is applied. The results are shown as follows:

From the above results, we can see that the revised Tikhonov regularization works well. Furthermore from Fig. 1 to Fig. 2, from Fig. 3 to Fig. 4, we find that the regularization parameter  $\alpha$  cannot be too small. Certainly  $\alpha$  can not be too large. This accords with the regularization theory.

### 5 Concluding remark

In this paper, we obtained order optimal error estimates by spectral regularization methods and a revised Tikhonov regularization method for a Cauchy problem for the Helmholtz equation. The Fourier regularization method in [5] can be considered as the spectral method 3 (TSVD). Numerical results show that the methods work well.

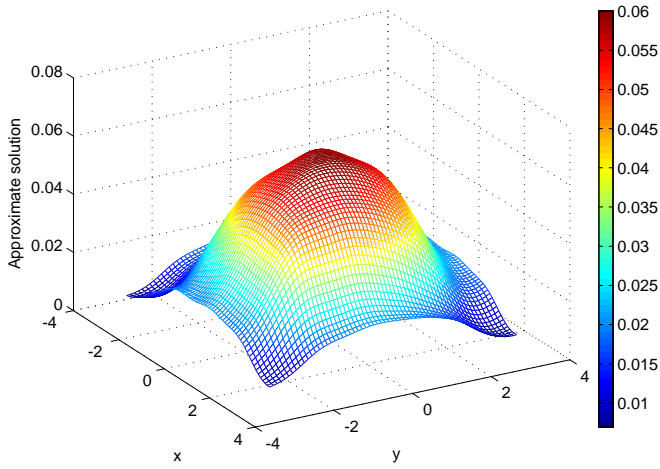


Figure 1:  $z = 0.8$ ,  $\alpha = 0.01$ ,  $\epsilon = 10^{-2}$ .

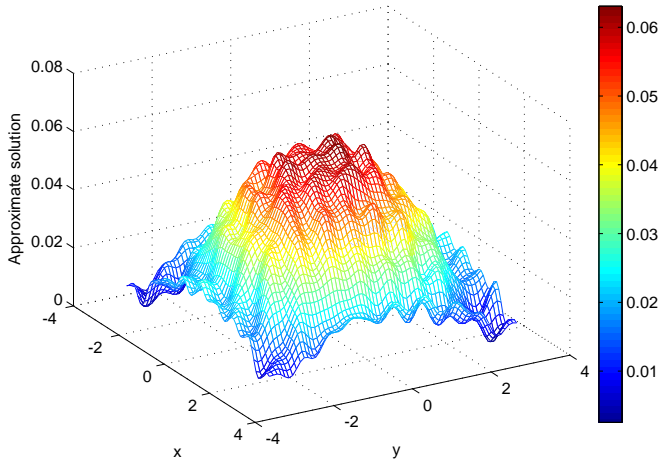


Figure 2:  $z = 0.8$ ,  $\alpha = 1 * 10^{-8}$ ,  $\epsilon = 10^{-2}$ .

### Appendix: proof of theorem 2.3

In order to prove the Theorem 2.3, we formulate the problem of identifying  $u(r, z)$  from (unperturbed) data  $g(r)$  as an operator equation

$$A(z)u(r, z) = g(r) \quad (\text{A.1})$$



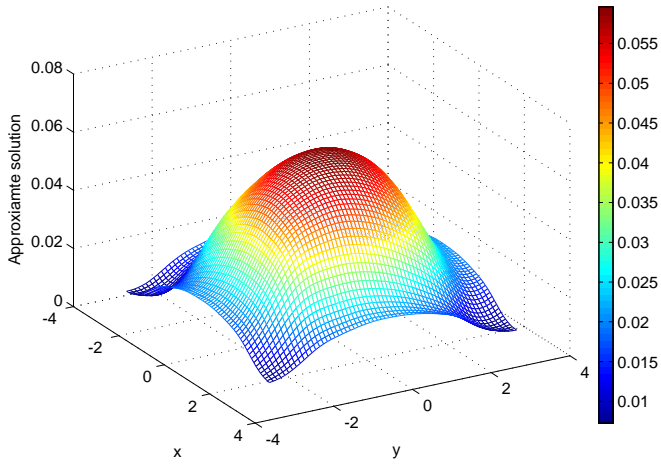


Figure 3:  $z = 0.8$ ,  $\alpha = 0.01$ ,  $\epsilon = 10^{-3}$ .

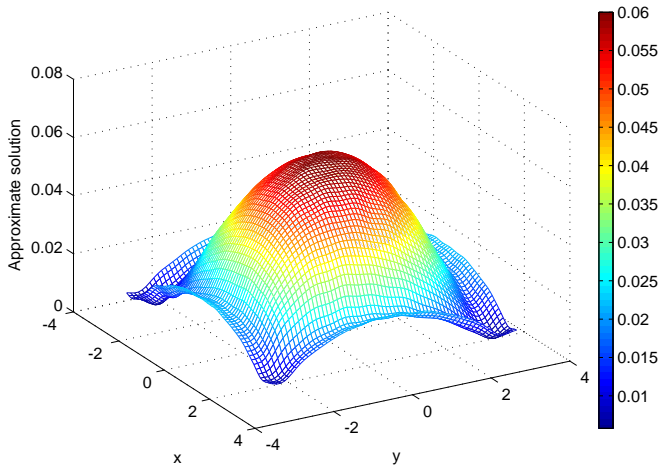


Figure 4:  $z = 0.8$ ,  $\alpha = 1 \times 10^{-8}$ ,  $\epsilon = 10^{-3}$ .

with a linear operator  $A(z) \in L^2(\mathbb{R}^2) \rightarrow L^2(\mathbb{R}^2)$ . Equation (A.1) is equivalent to the operator equation

$$\hat{A}(z)\hat{u}(\xi, z) = \hat{g}(\xi) \quad \text{with} \quad \hat{A}(z) = FA(z)F^{-1} \quad (\text{A.2})$$

where  $F : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$  is the Fourier operator. From (2.5), the multiplication operator  $\hat{A}(z)$  is given by

$$\hat{A}(z) = \frac{1}{\cosh((d-z)\eta)}, \quad \text{with } \eta = \sqrt{|\xi|^2 - k^2}, \quad (\text{A.3})$$

i.e.,

$$\hat{A}(z) = \begin{cases} \frac{1}{\cos((d-z)\sqrt{k^2 - |\xi|^2})}, & |\xi| < k, \\ \frac{1}{\cosh((d-z)\eta)}, & |\xi| \geq k, \end{cases} \quad (\text{A.4})$$

with

$$\hat{A}^*(z) = \begin{cases} \frac{1}{\cos((d-z)\sqrt{k^2 - |\xi|^2})}, & |\xi| < k, \\ \frac{1}{\cosh((d-z)\eta)}, & |\xi| \geq k. \end{cases} \quad (\text{A.5})$$

Thus,

$$\hat{A}^{-1}(z) = \begin{cases} \cos((d-z)\sqrt{k^2 - |\xi|^2}), & |\xi| < k, \\ \cosh((d-z)\eta), & |\xi| \geq k. \end{cases} \quad (\text{A.6})$$

Obviously,  $\hat{A}^{-1}(z)$  is an unbounded multiplication operator in that  $\cosh((d-z)\eta) \rightarrow \infty$  as  $\eta \rightarrow \infty$ . Hence we can call the the set  $I := \{\xi \mid |\xi| \geq k\}$  is the ill-posed part of problem (A.2). For treating the ill-posed part, we need to transform the source condition (2.8) into an equivalent condition in the frequency. First the condition (2.8) is equivalent to

$$\begin{aligned} \hat{u}(\xi, z) \in \hat{M} &= \left\{ \hat{u}(\xi, z) \in L^2(\mathbb{R}^2) \mid \|\hat{u}(\xi, 0)\| \right. \\ &= \left. \frac{\cosh(d\eta)}{\cosh((d-z)\eta)} \|\hat{u}(\xi, z)\| \leq E \right\}. \end{aligned} \quad (\text{A.7})$$

**Proposition A.1.** *Consider the operator equation (A.2). Then the set  $M$  given by (A.7) is equivalent to the general source set*

$$\hat{M}_{\varphi, E} = \left\{ \hat{u}(\xi, z) \in L^2(\mathbb{R}^2) \mid \|[\varphi(\hat{A}^*(z)\hat{A}(z))]^{-1/2}\hat{u}(\xi, z)\| \leq E \right\} \quad (\text{A.8})$$

where  $\varphi(\lambda)$  is given by (in parameter representation)

$$\lambda(\eta) = 1/\cosh^2((d-z)\eta), \quad \varphi(\eta) = \cosh^2((d-z)\eta)/\cosh^2(d\eta), \quad (\text{A.9})$$

where  $1 \leq \eta < \infty$ .

**Proof.** Noting that (A.7) and (A.8), we can obtain (A.9) easily.

**Proposition A.2.** *The function  $\varphi(\lambda)$  given by (A.9) is continuous and satisfies the properties:*

(i)  $\lim_{\lambda \rightarrow 0} \varphi(\lambda) = 0;$

(ii)  $\varphi(\lambda)$  is strong monotonically increasing;

(iii)  $\rho(\lambda) = \lambda\varphi^{-1}(\lambda)$  is strong monotonically and possesses the parameter representation

$$\lambda(\eta) = \cosh^2((d - z)\eta) / \cosh^2(d\eta), \rho(\eta) = 1 / (\cosh^2(d\eta)); \quad (\text{A.10})$$

(iv)  $\rho^{-1}(\lambda)$  is strong monotonically increasing and possesses the parameter representation

$$\lambda(\eta) = 1 / (\cosh^2(d\eta)), \rho^{-1}(\eta) = \cosh^2((d - z)\eta) / \cosh^2(d\eta); \quad (\text{A.11})$$

(v) for the inverse function  $\rho^{-1}$  of  $\rho$  there holds for any fixed  $z \in (0, d)$

$$\rho^{-1}(\lambda) = (\lambda)^{z/d}(1 + o(1)), \text{ for } \lambda \rightarrow 0. \quad (\text{A.12})$$

**Proof.** Consider  $\lambda(\eta)$  given by (A.9), we have

$$\dot{\lambda}(\eta) = -\frac{(d - z) \sinh((d - z)\eta)}{\cosh^3((d - z)\eta)} < 0. \quad (\text{A.13})$$

We can see that  $\lambda(\eta)$  is strong monotonically decreasing with  $\lim_{\eta \rightarrow \infty} \lambda(\eta) = 0$ . So,

$$\lim_{\lambda \rightarrow 0} \varphi(\lambda) = \lim_{\eta \rightarrow \infty} \cosh^2((d - z)\eta) / \cosh^2(d\eta) = 0, \text{ for } d > z.$$

Thus conclusion (i) holds. Noting that the function  $f(x) = x \tanh(x)$  is a strong monotonically increasing, we have

$$\dot{\varphi}(\eta) = -\frac{\cosh^2((d - z)\eta)}{\cosh^2(d\eta)} [d \tanh(d\eta) - (d - z) \tanh((d - z)\eta)] < 0. \quad (\text{A.14})$$

From (A.13) and (A.14), it is easy to see  $\varphi'(\lambda) = \frac{\dot{\varphi}(\eta)}{\lambda(\eta)} > 0$ . Thus, we proved (ii). Obviously  $\varphi^{-1}(\lambda)$  is strong monotonically increasing, consequently,  $\rho(\lambda)$  is strong monotonically increasing. (A.10) can be obtained by

$$\lambda(\eta) = \cosh^2((d - z)\eta) / \cosh^2(d\eta), \quad \varphi^{-1}(\eta) = 1 / (\cosh^2((d - z)\eta)).$$

Now (iv) is a direct conclusion of (iii). In order to prove (v), we need to prove that  $\lim_{\lambda \rightarrow 0} F(\lambda) = 1$  where  $F(\lambda)$  is given by

$$F(\lambda) = \rho^{-1}(\lambda)(1/\lambda)^{z/d}.$$

We use (A.11), note that  $\lambda(\eta)$  is strong monotonically decreasing with  $\lim_{\eta \rightarrow \infty} \lambda(\eta) = 0$ , we have

$$\lim_{\lambda \rightarrow 0} F(\lambda) = \lim_{\eta \rightarrow \infty} [\cosh^2((d - z)\eta) / \cosh^2(d\eta)] (\cosh^2(d\eta))^{z/d} = 1.$$

Thus, we proved (v).

**Proposition A.3.** *The function  $\rho(\lambda)$  given by (A.10) is strong convex.*

**Proof.** From  $\rho'' = \frac{\ddot{\rho}\dot{\lambda} - \dot{\rho}\ddot{\lambda}}{\dot{\lambda}^3}$  and  $\dot{\lambda} < 0$  we get that  $\rho'' > 0$  is equivalent to  $\ddot{\rho}\dot{\lambda} < \dot{\rho}\ddot{\lambda}$ . Noting that  $\lambda(\eta) = \rho(\eta)\phi(\eta)$  with  $\phi(\eta) = \cosh^2((d - z)\eta)$ , hence  $\rho'' > 0$  is equivalent to the inequality

$$\rho\ddot{\rho} - 2\dot{\rho}^2 < \rho\dot{\rho}\frac{\ddot{\phi}}{\dot{\phi}}. \tag{A.15}$$

Let  $x = d\eta$  and  $\tau = (d - z)/d$ , by elementary calculations we find that (A.15) is equivalent to the inequality

$$\psi(x) > 0, \tag{A.16}$$

where

$$\psi(x) = x \tanh(x) [x \coth(2x) - \tau x \coth(2\tau x)]. \tag{A.17}$$

Since  $h(x) = x \coth(2x)$  is strong monotonically increasing on  $\mathbb{R}^+$ , we find that  $\psi(x) > 0$  holds for any  $x > 0$  and  $0 < \tau < 1$ . Hence (A.15) always holds, i.e.,  $\rho$  is strong convex.

Now we apply Theorem 2.1 to our problem in the frequency domain. According to Proposition A.1-Proposition A.3, the conditions in Assumption 2.1 are satisfied for the operator  $\hat{A}(z)$ . Due to (A.9),

$$\sigma(\hat{A}^* \hat{A} \varphi(\hat{A}^* \hat{A})) = \sigma(1/\cosh^2(d\eta)) = (0, 1]$$

holds. If  $\delta^2/E^2 \leq 1$ , then obviously  $\delta^2/E^2 \in \sigma(\hat{A}^* \hat{A} \varphi(\hat{A}^* \hat{A}))$  holds. Noting

$$\omega(\delta, \mathfrak{H}) := \inf_{\mathfrak{H}} \Delta(\delta, \mathfrak{H}) \quad \text{with} \quad \Delta(\delta, \mathfrak{H}) = \sup \|\mathfrak{H}g^\delta(r) - u(r, z)\|_{L^2(\mathbb{R}^2)},$$

where the supremum is taken over  $g^\delta(r) \in L^2(\mathbb{R}^2)$ ,  $\|g^\delta - g\|_{L^2(\mathbb{R}^2)} \leq \delta$  and  $u(r, z) \in M$ , we define

$$\hat{\omega}(\delta, \hat{\mathfrak{H}}) := \inf_{\hat{\mathfrak{H}}} \Delta(\delta, \hat{\mathfrak{H}}) \quad \text{with} \quad \Delta(\delta, \hat{\mathfrak{H}}) = \sup \|\hat{\mathfrak{H}}\hat{g}^\delta(\xi) - \hat{u}(\xi, z)\|_{L^2(\mathbb{R}^2)},$$

where the supremum is taken over  $\hat{g}^\delta(\xi) \in L^2(\mathbb{R}^2)$ ,  $\|\hat{g}^\delta - \hat{g}\|_{L^2(\mathbb{R}^2)} \leq \delta$  and  $\hat{u}(\xi, z) \in \hat{M}$ . According to Parseval identity, we have  $\omega(\delta, \mathfrak{H}) = \hat{\omega}(\delta, \hat{\mathfrak{H}})$ . Moreover, according to the definition of  $\hat{\omega}(\delta, \hat{\mathfrak{H}})$ ,  $\hat{\omega}(\delta, \hat{\mathfrak{H}}) = \hat{\omega}_I(\delta, \hat{\mathfrak{H}}) + \hat{\omega}_W(\delta, \hat{\mathfrak{H}})$ , where  $\hat{\omega}_I(\delta, \hat{\mathfrak{H}})$  and  $\hat{\omega}_W(\delta, \hat{\mathfrak{H}})$  denotes the optimal error bounds on the sets  $I$  and  $W$ , respectively.

For  $I \subset \mathbb{R}^2$ , the best possible error bound in the frequency domain

$$\begin{aligned} \hat{\omega}_I(\delta, \hat{\mathfrak{H}}) &= E\sqrt{\rho^{-1}(\delta^2/E^2)} = E\left(\frac{\delta}{E}\right)^{z/d} (1 + o(1)) \\ &= E^{1-\frac{z}{d}}\delta^{\frac{z}{d}}(1 + o(1)), \quad \text{for } \delta \rightarrow 0. \end{aligned}$$

For  $W \subset \mathbb{R}^2$ , a regularized multiplication operator satisfies  $\|\hat{\mathfrak{H}}\| \leq c$  where  $c$  is a constant depending on regularization parameter. Moreover, there holds

$$\begin{aligned} \|\hat{\mathfrak{H}}\hat{g}^\delta - \hat{u}(\cdot, z)\|_{L^2(W)} &\leq \|\hat{\mathfrak{H}}\hat{g}^\delta - \hat{u}(\cdot, z)\|_{L^2(\mathbb{R}^2)} \\ &\leq \|\hat{\mathfrak{H}}\hat{g}^\delta - \hat{\mathfrak{H}}\hat{g}\|_{L^2(\mathbb{R}^2)} + \|\hat{\mathfrak{H}}\hat{g} - \hat{u}(\cdot, z)\|_{L^2(\mathbb{R}^2)} \\ &\leq c\delta + \|\hat{\mathfrak{H}}\hat{A}\hat{u}(\cdot, z) - \hat{u}(\cdot, z)\|_{L^2(\mathbb{R}^2)}, \end{aligned}$$

but

$$\|\hat{\mathfrak{H}}\hat{A}\hat{u}(\cdot, z) - \hat{u}(\cdot, z)\|_{L^2(\mathbb{R}^2)} \leq \sup_{\mathbb{R}^2} \left\{ |(\hat{\mathfrak{H}}\hat{A} - 1) [\varphi(\hat{A}^* \hat{A})]^{-1/2}| \right\} E,$$

from Section 3, for spectral cut-off method, the error estimates for  $\|\hat{\mathfrak{N}}\hat{g}^\delta - \hat{\mathfrak{N}}\hat{g}\|_{L^2(\mathbb{R}^2)}$  and  $\|\hat{\mathfrak{N}}\hat{g}^\delta - \hat{u}(\cdot, z)\|_{L^2(\mathbb{R}^2)}$  is  $\delta$  and 0, respectively. Hence in  $W$  the best possible error bound  $\hat{\omega}_W(\delta, \hat{\mathfrak{N}}) = \delta$ .

Obviously  $\delta = o(E^{1-\frac{\alpha}{d}}\delta^{\frac{\alpha}{d}})$ , for  $\delta \rightarrow 0$ , hence we obtain Theorem 2.3.

**Acknowledgements.** The authors would like to thank the referees for their carefully reading and valuable comments and suggestions on the manuscript, which led to an improvement version. The work is supported by the National Natural Science Foundation of China (Nos. 10671085, 10571079), the Natural Science Foundation of Gansu Province of China (No. 3ZS051-A25-015) and the Fundamental Research Fund for Physics and Mathematics of Lanzhou University (No. Lzu05005).

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