Computational & Applied Mathematics
Volume 27, N. 3, pp. 305–318, 2008

ISSN 0101-8205
www.scielo.br/cam

# Exact solutions and conservation laws for Ibragimov-Shabat equation which describe pseudo-spherical surface

S.M. SAYED[1,2], A.M. ELKHOLY[1,2] and G.M. GHARIB[2]

[1]Mathematics Department, Faculty of Science, Beni-Suef University, Beni-Suef, Egypt
[2]Mathematics Department, P.O. Box 1144, Tabouk Teacher College, Deputy for Teacher College
Ministry of Higher Education, Tabouk, Kingdom of Saudi Arabia

E-mail: S_M_Sayed71@yahoo.com

**Abstract.** Travelling wave solution for Ibragimov-Shabat equation, is obtained by using an improved sine-cosine method and the Wu's elimination method. An infinite number of conserved quantities for the above equation are also obtained by solving a set of coupled Riccati equations.

**Mathematical subject classification:** 35, 53C, 58J, 58Z05.

**Key words:** nonlinear evolution equations, conservation laws, pseudo-spherical surfaces.

## 1 Introduction

Sine-cosine method and the Wu's elimination method have been useful in the calculation of soliton solutions of certain nonlinear evolution equations (NLEEs) of physical significance [1-6] restricted to one space variable $x$ and a time coordinate $t$.

Khater et al. [7, 8] used the notion of a differential equation (DE) for a function $u(x, t)$ that describes a pseudo-spherical surface (pss), and they derived some Bäcklund transformations and conservation laws for NLEEs which are integrability condition of $sl(2, R)$-valued linear problems [9-18].

It is well-known [19-23] that a DE for a real-valued function $u(x, t)$ or a differential system for a two-vector valued function $u(x, t)$, is said to describe

#727/07. Received: 25/V/07. Accepted: 02/IV/08.

pss if it is the necessary and sufficient condition for the existence of smooth real functions $f_{ij},\ 1 \le i \le 3,\ 1 \le j \le 2$, depending only on $u$ and a finite number of its derivatives, such that the one-forms

$$\omega_i = f_{i1}dx + f_{i2}dt, \quad 1 \le i \le 3, \tag{1}$$

satisfying the structure equations of a surface of constant Gaussian curvature $K = -1$,

$$d\omega_1 = \omega_3 \wedge \omega_2, \quad d\omega_2 = \omega_1 \wedge \omega_3, \quad d\omega_3 = \omega_1 \wedge \omega_2. \tag{2}$$

The inverse scattering method (ISM) was introduced first for the Korteweg-de Vries equation (KdVE) [24]. Later it was extended by Zakharov and Shabat [25] to a $2 \times 2$ scattering problem for the nonlinear Schrödinger equation (NLSE) and that was subsequently generalized by Ablowitz, Kaup, Newell and Segur (AKNS) [26] to include a variety of NLEEs. Khater et al. [7] generalized the results of Konno and Wadati [27] by considering $\nu$ as a three component vector and $\Omega$ as a traceless $3 \times 3$ matrix one-form. The above definition of a DE is equivalent to saying that the DE for $u$ is the integrability condition for the problem

$$d\nu = \Omega\nu, \quad \nu = \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}, \tag{3}$$

where $\nu$ is a vector and the $3 \times 3$ matrix $\Omega$ ($\Omega_{ij}, i, j = 1, 2, 3$) is traceless

$$tr\,\Omega = 0, \tag{4}$$

and consists of a one-paramter $(\eta)$, family of one-forms in the independent variables $(x, t)$, the dependent variable $u$ and its derivatives.

Khater et al. [7] introduced the inverse scattering problem (ISP):

$$\nu_{1x} = f_{31}\nu_2 - f_{11}\nu_3, \quad \nu_{2x} = -f_{31}\nu_1 - \eta\nu_3, \quad \nu_{3x} = -f_{11}\nu_1 - \eta\nu_2, \tag{5}$$

$$\nu_{1t} = f_{32}\nu_2 - f_{12}\nu_3, \quad \nu_{2t} = -f_{32}\nu_1 - f_{22}\nu_3, \quad \nu_{3t} = -f_{12}\nu_1 - f_{22}\nu_2. \tag{6}$$

The integrability condition for Eq. (3) is given by

$$d\Omega - \Omega \wedge \Omega = 0, \tag{7}$$

or in component form

$$f_{12,x} - f_{11,t} = f_{31}f_{22} - \eta f_{32}, \tag{8}$$

$$f_{22,x} = f_{11}f_{32} - f_{12}f_{31}, \tag{9}$$

$$f_{32,x} - f_{31,t} = f_{11}f_{22} - \eta f_{12}. \tag{10}$$

By various choices of the coefficients $f_{ij}$, it can be shown that the conditions (8)-(10) are equivalent to a large class of NLEEs. The procedure is clarified in the following example:

*(a) The Ibragimov-Shabat equation (ISE)*

$$u_t = u_{xxx} + 3u^2u_{xx} + 9uu_x^2 + 3u^4u_x, \tag{11}$$

$$\Omega = \begin{pmatrix} 0 & C_1 & -A_1 \\ -C_1 & 0 & -B_1 \\ -A_1 & -B_1 & 0 \end{pmatrix}, \tag{12}$$

where

$$\begin{aligned} A_1 &= \left(\frac{u_x}{u} + u^2\right)dx + \left(\frac{u_{xxx}}{u} + u^6 + 8u_x^2 + 5uu_{xx} + 9u^3u_x\right)dt, \\ B_1 &= \eta dx + \eta\left(\frac{u_{xx}}{u} + u^4 + 4uu_x\right)dt, \\ C_1 &= -\eta dx - \eta\left(\frac{u_{xx}}{u} + u^4 + 4uu_x\right)dt. \end{aligned} \tag{13}$$

The essence of the first step of the ISM is summarized as follows [7]. Find nine one-forms $\omega_i^j$, $i = 1, 2, 3,\ j = 1, 2, 3$ consisting of independent and dependent variables and their derivatives, such that the NLEE is given by

$$\Theta \equiv d\Omega - \Omega\wedge\Omega = 0, \quad \Omega = \begin{pmatrix} \omega_1^1 & \omega_1^2 & \omega_1^3 \\ \omega_2^1 & \omega_2^2 & \omega_2^3 \\ \omega_3^1 & \omega_3^2 & \omega_3^3 \end{pmatrix}, \quad Tr\Omega = 0. \tag{14}$$

It should be noted that the solution of these equations are of very special kind. In general, Eq. (14) gives three different equations, which cannot be satisfied simultaneously by one dependent variable $u$. It has been pointed out [7, 28] that $\Omega$ can be interpreted as a connection one-form for the principle $SL(3, R)$

bundle on $R^3$ and $\Theta$ as its curvature two form. The geometrical explanation of the $SL(3, R)$ structure is given in section 2.

The main aim of this paper is to extend the fundamental equations of pseudo-spherical surfaces in reference [28] by considering $\nu$ as a three component vector and $\Omega$ as a traceless $3 \times 3$ matrix one-form. In the present paper we use sine-cosine method and the Wu's elimination method derived in [1-6] in the construction of exact soliton solution for ISE describing pss. We also obtain an infinite number of conserved charges by solving a set of coupled Riccati equations and apply the geometrical method to ISE which describe pss.

The paper is organized as follows. In section 2 the geometry of pss is described and the correspondence between the soliton equations and their families of pss are established. In section 3 the sine-cosine method and the Wu's elimination method is used to obtain travelling wave solutions for the ISE. Section 4 contains the derivation of an infinite number of conserved charges from the Riccati equations. Finally, we give some conclusions in section 5.

## 2 On equations describing pss

In this section we shall show that the fundamental equations of pss, can be written in the form of Eq. (14). Let us start with the general description of a three-dimensional Riemannian manifold $S$ following reference [28]. An orthonormal basis is

$$\{e_i\}, \quad i = 1, 2, 3, \quad e_i . e_j = \delta_{ij}, \tag{15}$$

with respect to the Riemannian metric introduced on the tangent plane $T_p$ at each point $p \in S$. Then the structure equations for $S$ read

$$dp = \omega_i e_i, \quad i = 1, 2, 3 \tag{16}$$

$$de_i = \sum_{j=1}^{3} \omega_i^j e_j, \tag{17}$$

where $\omega_i$ are one forms dual to $\{e_i\}$ and $\omega_i^j$, $(i, j = 1, 2, 3)$ are called the connection one form. The integrability conditions are obtained by differentiating

Eq. (16) and using (17). These conditions are

$$d\omega_i = \sum_{j=1}^{3} \omega_i \wedge \omega_i^j, \tag{18}$$

$$d\omega_i^j = \sum_{k=1}^{3} \omega_i^k \wedge \omega_k^j. \tag{19}$$

To sum up, a set of one forms $(\omega_i,\ \omega_i^j)$ satisfying Eqs. (18) and (19) describes a pss locally through the structure Eqs. (16) and (17). It is easy to show that Eqs. (18) and (19) can be written in the form of Eq. (14) by choosing

$$\Omega = \begin{pmatrix} 0 & \omega_3 & -\omega_1 \\ -\omega_3 & 0 & -\omega_2 \\ -\omega_1 & -\omega_2 & 0 \end{pmatrix}, \tag{20}$$

from Eqs. (14) and (20) we obtain this relations

$$\begin{aligned} \omega_1^1 = \omega_2^2 = \omega_3^3 = 0, \qquad & \omega_1 = -\omega_1^3 = -\omega_3^1, \\ \omega_2 = -\omega_2^3 = -\omega_3^2, \qquad & \omega_3 = \omega_1^2 = -\omega_2^1. \end{aligned} \tag{21}$$

Let $M^2$ be a two – dimensional differentiable manifold parametrized by coordinates $x, t$. We consider a metric on $M^2$ defined by $\omega_1, \omega_2$. The first two equations in (2) are the structure equations which determine the connection form $\omega_3$, and the last equation in (2), the Gauss equation, determines that the Gaussian curvature of $M^2$ is $-1$, i.e. $M^2$ is a pss. Moreover, an evolution equation must be satisfied for the existence of forms (1) satisfying (2). This justifies the definition of a DE which describes a pss that we considered in the introduction.

It has been known, for a long time, that the sine-Gordon equation describes a pss. KdVE and mKdVE, were also shown to describe such surfaces [26]. Here we show that ISE equation also describe pss as well. The latter equation proved to be of great importance in many physical applications [7-14, 29, 30]. The procedure is clarified in the following example: Let $M^2$ be a differentiable surface, parametrized by coordinates $x, t$.

*(a) The ISE*

Consider

$$\begin{aligned}
\omega_1^3 = \omega_3^1 &= -\left(\frac{u_x}{u} + u^2\right)dx - \left(\frac{u_{xxx}}{u} + u^6 + 8u_x^2 + 5uu_{xx} + 9u^3u_x\right)dt,\\
\omega_2^3 = \omega_3^2 &= -\eta dx - \eta\left(\frac{u_{xx}}{u} + u^4 + 4uu_x\right)dt, \qquad (22)\\
\omega_1^2 = -\omega_2^1 &= -\eta dx - \eta\left(\frac{u_{xx}}{u} + u^4 + 4uu_x\right)dt.
\end{aligned}$$

Then $M^2$ is a pss iff $u$ satisfies ISE (11).

## 3 Exact solution for ISE

Now we shall find a travelling wave solution $u(x,t)$ for ISE

$$u_t = u_{xxx} + 3u^2u_{xx} + 9uu_x^2 + 3u^4u_x. \qquad (23)$$

Wang [1] has found some exact solutions for compound KdV-Burgers equations by using the homogenous balance method. In this section we obtain travelling wave solution class for IES by using an improved sine-cosine method [4, 5] and Wu's elimination method [2]. The main idea of the algorithm is as follows. Given a partial differential equation (PDE) of the form

$$f\left(u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, \cdots\right) = 0, \qquad (24)$$

where $f$ is a polynomial. By assuming travelling wave solutions of the form

$$u(x,t) = \phi(\rho), \quad \rho = \lambda(x - kt + c), \qquad (25)$$

where $k$, $\eta$ are constant parameters to be determined, and $c$ is an arbitrary constant, from the two Eqs. (24) and (25) we obtain an ordinary differential equation (ODE)

$$f\left(\phi', \phi'', \phi''', \cdots\right) = 0, \qquad (26)$$

where $\phi' = \dfrac{d\phi}{d\rho}$. According to the sine-cosine method [1-6], we suppose that Eq. (26) has the following formal travelling wave solution

$$\phi(\rho) = \sum_{i=1}^{n} \sin^{i-1}\psi\left(B_i \sin\psi + A_i \cos\psi\right) + A_0, \qquad (27)$$

and

$$\frac{d\psi}{d\rho} = \sin\psi \quad \text{or} \quad \frac{d\psi}{d\rho} = \cos\psi, \tag{28}$$

where $A_0, \cdots, A_n$ and $B_1, \cdots, B_n$ are constants to be determined. Then we proceed as follows:

(i) Equating the highest order nonlinear term and highest order linear partial derivative in (26), yield the value of $n$.

(ii) Substituting Eqs. (27), (28) into (26), we obtain a polynomial equation involving $\cos\psi \sin^j \psi$, $\sin^j \psi$ for $j = 0, 1, 2, \ldots, n$, (with $n$ being positive integer).

(iii) Setting the constant term and coefficients of $\sin\psi$, $\cos\psi$, $\sin\psi\cos\psi$, $\sin^2\psi, \ldots$, in the equation obtained in (ii) to zero, we obtain a system of algebraic equations about the unknown numbers $k, \lambda, A_0, A_i, B_i$ for $i = 1, 2, \ldots, n$.

(iv) Using the Mathematica and the Wu's elimination methods, the algebraic equations in (iii) can be solved.

These yield the solitary wave solutions for the system (26). We remark that the above method yield solutions that includes terms sech$\rho$ or $\tanh\rho$, as well as their combinations. There are different forms of those obtained by other methods, such as the homogenous balance method [1-6]. We assume formal solutions of the form

$$u(x,t) = \phi(\rho), \quad \rho = \lambda(x - kt + c), \tag{29}$$

where $k$, $\eta$ are constant parameters to be determined later, and $c$ is an arbitrary constant. Substituting from (29) and (23), we obtain an ODE

$$k\phi' + 3\lambda\phi^2\phi'' + 9\lambda\phi\phi'^2 + 3\phi^4\phi' + \lambda^2\phi''' = 0. \tag{30}$$

(i) We suppose that equation (30) has the following formal solutions

$$\phi(\rho) = A_0 + B_1\sin\psi + A_1\cos\psi, \tag{31}$$

and

$$\frac{d\psi}{d\rho} = \sin\psi. \tag{32}$$

(ii) From two Eqs. (31) and (32), we get

$$
\begin{aligned}
&k\phi' + 3\lambda\phi^2\phi'' + 9\lambda\phi\phi'^2 + 3\phi^4\phi' + \lambda^2\phi''' \\
&\quad= \big[12A_1B_1A_0^3 + 12A_1^3B_1A_0 + 3\lambda B_1A_0^2 + 3\lambda A_1^2B_1\big]\sin\psi \\
&\qquad+ \big[3B_1A_0^4 + 18A_1^2B_1A_0^2 + 3A_1^4B_1 + (k+\lambda^2)B_1 \\
&\qquad+ 6\lambda A_1B_1A_0\big]\sin\psi\cos\psi + \big[-3A_1A_0^4 + 36A_1B_1^2A_0^2 - 18A_1^3A_0^2 \\
&\qquad+ 12A_1^3B_1^2 - 3A_1^5 - (k+4\lambda^2)A_1 + 6\lambda B_1^2A_0 - 12A_0A_1^2 \\
&\qquad+ 9\lambda B_1^2A_0\big]\sin^2\psi + \big[12B_1^2A_0^3 - 12A_1^2A_0^3 + 36A_1^2B_1^2A_0 + 6\lambda A_1B_1^2 \\
&\qquad- 12A_0A_1^4 - 6\lambda A_1A_0^2 - 6\lambda A_1^3 + 9\lambda A_1B_1^2\big]\cos\psi\sin^2\psi \\
&\qquad+ \big[-24A_1B_1A_0^3 + 36A_1B_1^3A_0 + 3\lambda B_1^3 - 60A_0B_1A_1^3 - 6\lambda B_1A_0^2 \\
&\qquad- 21\lambda A_1^2B_1 + 9\lambda B_1^3 - 18\lambda A_1^2B_1\big]\sin^3\psi + \big[18B_1^3A_0^2 + 18A_1^2B_1^3 \\
&\qquad- 54A_1^2B_1A_0^2 - 18A_1^4B_1 - 6\lambda^2B_1 - 24\lambda B_1A_1A_0 \\
&\qquad- 18\lambda A_1B_1A_0\big]\cos\psi\sin^3\psi + \big[12A_1B_1^4 - 54A_1B_1^2A_0^2 + 18A_1^3A_0^2 \\
&\qquad- 42A_1^3B_1^2 + 6A_1^5 + 6\lambda^2A_1 - 12\lambda B_1^2A_0 + 12A_1^2A_0 - 9\lambda A_0B_1^2 \\
&\qquad+ 9\lambda A_1^2A_0\big]\sin^4\psi + \big[12B_1^4A_0 - 72A_1^2B_1^2A_0 + 12A_0A_1^4 \\
&\qquad- 18\lambda A_1B_1^2 + 6\lambda A_1^3 - 27\lambda A_1B_1^2 + 9\lambda A_1^3\big]\cos\psi\sin^4\psi \\
&\qquad+ \big[-48A_1B_1^3A_0 + 48A_1^3B_1A_0 - 6\lambda B_1^3 + 18\lambda A_1^2B_1 - 9\lambda B_1^3 \\
&\qquad+ 27\lambda A_1^2B_1\big]\sin^5\psi + \big[3B_1^5 - 30A_1^2B_1^3 + 15B_1A_1^4\big]\cos\psi\sin^5\psi \\
&\qquad+ \big[30A_1^3B_1^2 - 3A_1^5 - 15A_1B_1^4\big]\sin^6\psi = 0.
\end{aligned}
\tag{33}
$$

(iii) Setting the coefficients of $\sin^j\psi\cos^i\psi$ for $i = 0, 1$ and $j = 1$ to $6$, we have the following set of over determined equations in the unknowns $A_0$, $B_1$, $A_1$, $\lambda$ and $k$:

$$
\begin{aligned}
&12A_1A_0^3 + 12A_1^3A_0 + 3\lambda A_0^2 + 3\lambda A_1^2 = 0, \\
&3A_0^4 + 18A_1^2A_0^2 + 3A_1^4 + (k+\lambda^2) + 6\lambda A_1A_0 = 0, \\
&-3A_1A_0^4 + 36A_1B_1^2A_0^2 - 18A_1^3A_0^2 + 12A_1^3B_1^2 - 3A_1^5 \\
&\quad- (k+4\lambda^2)A_1 + 6\lambda B_1^2A_0 - 12A_0A_1^2 + 9\lambda B_1^2A_0 = 0, \\
&12B_1^2A_0^3 - 12A_1^2A_0^3 + 36A_1^2B_1^2A_0 + 6\lambda A_1B_1^2 - 12A_0A_1^4 \\
&\quad- 6\lambda A_1A_0^2 - 6\lambda A_1^3 + 9\lambda A_1B_1^2 = 0, \\
&-24A_1A_0^3 + 36A_1B_1^2A_0 + 3\lambda B_1^2 - 60A_0A_1^3 - 6\lambda A_0^2 - 21\lambda A_1^2 \\
&\quad+ 9\lambda B_1^2 - 18\lambda A_1^2 = 0, \\
&18B_1^2A_0^2 + 18A_1^2B_1^2 - 54A_1^2A_0^2 - 18A_1^4 - 6\lambda^2 \\
&\quad- 24\lambda A_1A_0 - 18\lambda A_1A_0 = 0,
\end{aligned}
\tag{34a}
$$

$$\begin{aligned}
&12A_1B_1^4 - 54A_1B_1^2A_0^2 + 18A_1^3A_0^2 - 42A_1^3B_1^2 + 6A_1^5\\
&\quad + 6\lambda^2A_1 - 12\lambda B_1^2A_0 + 12A_1^2A_0 - 9\lambda A_0B_1^2 + 9\lambda A_1^2A_0 = 0,\\
&12B_1^4A_0 - 72A_1^2B_1^2A_0 + 12A_0A_1^4 - 18\lambda A_1B_1^2 + 6\lambda A_1^3\\
&\quad - 27\lambda A_1B_1^2 + 9\lambda A_1^3 = 0,\\
&-48A_1B_1^3A_0 + 48A_1^3B_1A_0 - 6\lambda B_1^3 + 18\lambda A_1^2B_1\\
&\quad - 9\lambda B_1^3 + 27\lambda A_1^2B_1 = 0,\\
&3B_1^4 - 30A_1^2B_1^2 + 15A_1^4 = 0,\\
&30A_1^2B_1^2 - 3A_1^4 - 15B_1^4 = 0.
\end{aligned} \tag{34b}$$

(iv) We now solve the above set of equations by using Mathematica and the Wu's elimination method, and obtain the following solution:

$$\begin{aligned}
A_1 &= \frac{-\lambda}{4A_0},\\
B_1 &= \frac{-\lambda}{12.3106A_0},\\
A_0 &= -\frac{\left[-(5\lambda^2+8k) \pm 4(\lambda^2+4k)^{1/2}(\lambda^2+k)^{1/2}\right]^{1/4}}{2\sqrt[4]{3}},
\end{aligned} \tag{35}$$

by integrating (32) and taking the integration constant equal zero, we obtain

$$\sin\psi = \operatorname{sech}\rho, \qquad \cos\psi = \pm\tanh\rho. \tag{36}$$

Substituting (35) and (36) into (31), we obtain

$$u(x,t) = A_0 + \frac{-\lambda}{12.3106A_0}\operatorname{sech}\rho \pm \frac{-\lambda}{4A_0}\tanh\rho, \tag{37}$$

where

$$\begin{aligned}
A_0 &= -\frac{\left[-(5\lambda^2+8k) \pm 4(\lambda^2+4k)^{1/2}(\lambda^2+k)^{1/2}\right]^{1/4}}{2\sqrt[4]{3}},\\
\rho &= \lambda(x-kt+c).
\end{aligned}$$

## 4 Infinite number of conserved charges for ISE

It will be found that the Riccati form of ISM is sometimes useful. Introducing the variables [31-33]

$$\Gamma_1 = \frac{\nu_1}{\nu_3}, \qquad \Gamma_2 = \frac{\nu_2}{\nu_3}, \tag{38}$$

which are related to the conserved charges $\alpha_{33}(x, t, \eta)$ in the following way, from Eq. (5)

$$\ln \alpha_{33}(\eta) = \ln \nu_3 = \int_{-\infty}^{\infty} (\eta \Gamma_2 - f_{11}\Gamma_1) dx. \tag{39}$$

Eqs. (5) and (6) can be rewritten as

$$\Gamma_{1x} = f_{31}\Gamma_2 + \eta \Gamma_1 \Gamma_2 + f_{11}\Gamma_1^2 - f_{11}, \tag{40}$$

$$\Gamma_{2x} = -f_{31}\Gamma_1 + f_{11}\Gamma_1\Gamma_2 + \eta \Gamma_2^2 - \eta, \tag{41}$$

and

$$\Gamma_{1t} = f_{32}\Gamma_2 + f_{22}\Gamma_1\Gamma_2 + f_{12}\Gamma_1^2 - f_{12}, \tag{42}$$

$$\Gamma_{2t} = -f_{32}\Gamma_1 + f_{12}\Gamma_1\Gamma_2 + f_{22}\Gamma_2^2 - f_{22}. \tag{43}$$

The set of coupled NLPDEs for $\Gamma_1$ and $\Gamma_2$ in (40)-(43) are known as Riccati equations. It is obvious from (39) that the solutions of Riccati equations eventually determine the conserved quantities. Now in order to solve (40)-(43), we assume a series solutions for $\Gamma_1$ and $\Gamma_2$ as

$$\Gamma_1(x, t, \eta) = \sum_{n=0}^{\infty} \phi_n^1(x, t)\eta^{-n}, \qquad \Gamma_2(x, t, \eta) = \sum_{n=0}^{\infty} \phi_n^2(x, t)\eta^{-n}. \tag{44}$$

Substituting (44) into (40) and (41), the following recursion relations are obtained:

$$\begin{aligned} &\phi_0^1 = \phi_1^2 = 0, \quad \phi_0^2 = 1, \quad \phi_1^1 = f_{11} - f_{31}, \\ &\phi_2^1 = (f_{11} - f_{31})_x, \quad \phi_2^2 = \frac{1}{2}(f_{11} - f_{31})^2, \end{aligned} \tag{45}$$

$$\phi_{kx}^1 = f_{31}\phi_k^2 + \sum_{m=0}^{k+1} \phi_m^1 \phi_{k-m+1}^2 + \sum_{m=0}^{k} f_{11}\phi_m^1\phi_{k-m}^1, \quad k = 1, 2, \ldots, \tag{46}$$

$$\phi_{kx}^2 = -f_{31}\phi_k^1 + \sum_{m=0}^{k+1} \phi_m^2 \phi_{k-m+1}^2 + \sum_{m=0}^{k} f_{11}\phi_m^1\phi_{k-m}^2, \quad k = 1, 2, \ldots \tag{47}$$

The infinite number of Hamiltonians (conserved quantities) may explicitly be determined in terms of smooth real functions $f_{ij}$ and their derivatives by expanding $\alpha_{33}(x, t, \eta)$ in the form

$$\ln \alpha_{33}(\eta) = \sum_{l=0}^{\infty} H_l \eta^{-l}, \tag{48}$$

and thus comparing (48) with (39) and (44), $H_l$ becomes

$$H_l = \int \big(\eta\phi_l^2 - f_{11}\phi_l^1\big)dx, \quad l = 0, 1, 2, \ldots . \tag{49}$$

The explicit expressions of the first few order Hamiltonians are

$$\begin{aligned}
&H_0 = \eta x, \qquad H_1 = -\int 2\big(qr + q^2\big)dx,\\
&H_2 = \int \Big[-2(r+q)q_x + \frac{\eta}{2}q^2\Big]dx,\\
&H_3 = 2\int (r+q)\big[q^2(q+r+2) - q_{xx}\big]dx,\\
&H_4 = \int \big[\eta q^3(2 - r - 3q) - \eta q q_{xx} - (r+q)\big(2q_{3x} - q^2(9q_x + r_x)\\
&\qquad - 4qq_x(r+4)\big)\big]dx,
\end{aligned} \tag{50}$$

where

$$f_{11} - f_{31} = 2q, \quad f_{11} + f_{31} = 2r,$$
$$2A = f_{22},\ 2B = f_{12} - f_{32},\ 2C = f_{12} + f_{32},$$

such that the functions $r$, $q$, $A$, $B$ and $C$ satisfy the equations [7]

$$A_x = qC - rB, \quad q_t - 2Aq - B_x + \eta B = 0, \quad C_x = r_t + 2Ar - \eta C. \tag{51}$$

This section ends with the following example:

*(a) The ISE*

For any solution $u$ of the ISE (11), we consider the functions

$$\begin{aligned}
&r = \frac{1}{2}\left(\frac{u_x}{u} + u^2 - \eta\right), \qquad q = \frac{1}{2}\left(\frac{u_x}{u} + u^2 + \eta\right),\\
&A = \frac{\eta}{2}\left(\frac{u_{xx}}{u} + u^4 + 4uu_x\right),\\
&B = \frac{1}{2}\Big[\Big(\frac{u_{xxx}}{u} + u^6 + 8u_x^2 + 5uu_{xx} + 9u^3u_x\Big)\\
&\qquad +\eta\left(\frac{u_{xx}}{u} + u^4 + 4uu_x\right)\Big],\\
&C = \frac{1}{2}\Big[\Big(\frac{u_{xxx}}{u} + u^6 + 8u_x^2 + 5uu_{xx} + 9u^3u_x\Big)\\
&\qquad -\eta\left(\frac{u_{xx}}{u} + u^4 + 4uu_x\right)\Big].
\end{aligned} \tag{52}$$

Eq. (49) implies that the first few order Hamiltonians are determined by the relation

$$\begin{aligned} H_1 &= -\frac{1}{2}\int\left[\left(\frac{u_x}{u}+u^2\right)^2+\left(\frac{u_x}{u}+u^2+\eta\right)^2-\eta^2\right]dx, \\ H_2 &= \int\left[\frac{\eta}{8}\left(\frac{u_x}{u}+u^2+\eta\right)^2-\left(\frac{u_x}{u}+u^2\right)\left(\frac{u_x}{u}+u^2\right)_x\right]dx, \\ &\vdots \end{aligned} \tag{53}$$

Whenever $u(x,t)$ is a solution (37) of the ISE. This Hamiltonians relations yields a sequence of conserved charges given by the coefficients of the series in $\eta$

$$\eta x+\sum_{l=1}^{\infty} H_l \eta^{-l}. \tag{54}$$

## 5 Conclusions

In this paper, the fundamental equations of pseudo-spherical surfaces in reference [28] may be rewritten by considering $\nu$ as a three component vector and $\Omega$ as a traceless $3\times 3$ matrix one-form [7]. The latter yields directly the curvature condition (Gaussian curvature equal to $-1$, corresponding to pseudo-spherical surfaces). This geometrical method is considered for the ISE.

We obtain travelling wave solution for ISE by using an improved sine-cosine method and Wu's elimination method. This geometrical method allows some further generalization of the work on conserved charges given by Wadati, Sanuki and Konno [34]. An infinite number of conserved charges for ISE mentioned above are derived in this way.

### REFERENCES

[1] M.L. Wang, *Exact solution for a compound KdV-Burgers equation.* Phys. Lett., **A213** (1996), 279–287.

[2] W.T. Wu, *Polynomial equations-solving and its applications.* Algorithms and Computation, (Beijing 1994), 1-9, Lecture Notes in Comput. Sci. 834, Springer-Verlag, Berlin, (1994).

[3] T.C. Xia, H.Q. Zhang and Z.Y. Yan, *New explicit exact travelling wave solution for a compound KdV-Burgers equation*. Chinese Phys., **8** (2001), 694–699.

[4] C.T. Yan, *A simple transformation for nonlinear waves*. Phys. Lett., **A224** (1996), 77–82.

[5] T.Z. Yan and H.Q. Zhang, *New explicit and exact travelling wave for a system variant Boussinesq equation in mathematical physics*. Phys. Lett., **A252** (1999), 291–296.

[6] X.D. Zhang, T.C. Xia and H.Q. Zhang, *New explicit exact travelling wave solution for compound KdV-Burgers equation in mathematical physics*. Applied Mathematics E-Notes, **2** (2002), 45–50.

[7] A.H. Khater, D.K. Callebaut and S.M. Sayed, *Conservation laws for some nonlinear evolution equations which describe pseudospherical surfaces*. J. of Geometry and Phys., **51** (2004), 332–352.

[8] A.H. Khater, D.K. Callebaut and S.M. Sayed, *Exact solutions for some nonlinear evolution equations which describe pseudo-spherical surfaces*. J. of Comput. and Appl. Math., **189** (2006), 387–411.

[9] A.H. Khater, D.K. Callebaut, A.A. Abdalla and S.M. Sayed, *Exact solutions for self-dual Yang-Mills Equations*. Chaos Solitons & Fractals, **10** (1999), 1309–1320.

[10] A.H. Khater, D.K. Callebaut, A.A. Abdalla, A.M. Shehata and S.M. Sayed, *Bäcklund Transformations and Exact solutions for self-dual $SU(3)$ Yang-Mills Equations.* Il Nuovo Cimento, **B114** (1999), 1–10.

[11] A.H. Khater, D.K. Callebaut and O.H. El-Kalaawy, *Bäcklund transformations and exact solutions for a nonlinear elliptic equation modelling isothermal magentostatic atmosphere.* IMA J. of Appl. Math., **65** (2000), 97–108.

[12] A.H. Khater, D.K. Callebaut and O.H. El-Kalaawy, *Bäcklund transformations and exact soliton solutions for nonlinear Schrödinger-type equations.* Il Nuovo Cimento, **B113** (1998), 1121–1136.

[13] A.H. Khater, D.K. Callebaut and R.S. Ibrahim, *Bäcklund transformations and Painlevé analysis: exact solutions for the unstable nonlinear Schrödinger equation modelling electron-beam plasma.* Phys. Plasmas, **5** (1998), 395–400.

[14] A.H. Khater, O.H. El-Kalaawy and D.K. Callebaut, *Bäcklund transformations for Alfvén solitons in a relativistic electron-positron plasma.* Physica Scripta, **58** (1998), 545–548.

[15] A.H. Khater, M.A. Helal and O.H. El-Kalaawy, *Two new classes of exact solutions for the KdV equation via Bäcklund transformations.* Choas, Solitons & Fractals, **8** (1997), 1901–1909.

[16] A.H. Khater and S.M. Sayed, *Exact Solutions for Self-Dual $SU(2)$ and $SU(3)$ Yang-Mills Fields.* International J. of Theoretical Physics, **41** (2002), 409–419.

[17] A.H. Khater, A.M. Shehata, D.K. Callebaut and S.M. Sayed, *Self-Dual Solutions for $SU(2)$ and $SU(3)$ Gauge Fields on Euclidean Space.* International J. of Theoretical Physics, **43** (2004), 151–159.

[18] C. Rogers and W. Shadwick, *Bäcklund transformations and their applications.* Academic Press, New York (1982).

[19] A.V. Bäcklund, *Einiges über Curven-und Flächentransformationen.* Lunds. Univ. Arsskr. Avd. X, For Ar, (1873); Afdelningen för Mathematik Och Naturetens kap, ibid, (1873-1874), 1–12.

[20] R. Beals and R.R. Coifman, *Inverse scattering and evolution equations.* Commun. Pure Appl. Math., **38** (1985), 29–42.

[21] R.K. Dodd and R.K. Bullough, *Bäcklund transformations for the AKNS inverse method.* Phys. Lett., **A62** (1977), 70–74.

[22] S. Lie, *Zür Theorie der Flächen Konstanter Krümmung III.* Arch. Math. Naturvidensk. V Heft 3, 282–306 (1880), 282–306; 3 (1880), 328–358.

[23] C.L. Terng, *Soliton equations and differential geometry.* J. of Diff. Geometry, **45** (1997), 407–445.

[24] C.S. Gardner, J.M. Greene, M.D. Kruskal and R.M. Miura, *Method for solving the korteweg-de Vries equation.* Phys. Lett., **19** (1967), 1095–1097.

[25] V.E. Zakharov and A.B. Shabat, *Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear Media.* Soviet Phys. JETP, **34** (1972), 62–69.

[26] M.J. Ablowitz, D.J. Kaup, A.C. Newell and H. Segur, *The inverse scattering transform-Fourier analysis for nonlinear problems.* Stud. Appl. Math., **53** (1974), 249–315.

[27] K. Konno and M. Wadati, *Simple derivation of Bäcklund transformation from Riccati Form of inverse method.* Progr. Theor. Phys., **53** (1975), 1652–1656.

[28] R. Sasaki, *Soliton equation and pseudospherical surfaces.* Nucl. Phys., **B154** (1979), 343–357.

[29] M.V. Foursov, P.J. Olver and E.G. Reyes, *On formal integrability of evolution equations and local geometry of surfaces.* Diff. Geometry and its Applications, **15** (2001), 183–99.

[30] S.S. Chern and K. Tenenblat, *Pseudospherical surfaces and evolution equations.* Stud. Appl. Math., **74** (1986), 55–83.

[31] S. Ghosh, A. Kundu and S. Nandy, *Soliton Solutions, Liouville Integrability and Gauge Equivalence of Sasa Satsume Equation.* J. Math. Phys., **40** (1999), 1993–2000.

[32] S. Ghosh and S. Nandy, *Inverse Scattering Method and Vector Higher Order Nonlinear Schrödinger Equation.* Nucl. Phys., **B561** (1999), 451–66.

[33] S. Nandy, *Inverse scattering approach to coupled higher-order nonlinear Schrödinger equation and $N$-soliton solutions.* Nucl. Phys., **B679** (2004), 647–58.

[34] M. Wadati, H. Sanuki and K. Konno, *Relationships among inverse method, Bäcklund transformation and an infinite number of conservation laws.* Progr. Theor. Phys., **53** (1975), 419–36.