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Stabilization of a locally damped thermoelastic system

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Abstract. We show that the solutions of a thermoelastic system with a localized nonlinear distributed damping decay locally with an algebraic rate to zero, that is, given an arbitrary $R > 0$, the total energy $E(t)$ satisfies for $t \geq 0$: $E(t) \leq C(1+t)^{-\gamma}$ for regular initial data such that $E(0) \leq R$, where C and γ are positive constants. In the two-dimensional case, we obtain an exponential decay rate when the nonlinear dissipation behaves linearly close to the origin.

Mathematical subject classification: 35B40, 35L70.

Key words: thermoelastic system, nonlinear localized damping, algebraic decay rate, exponential decay rate.

1 Introduction

In this work we study decay properties of the solutions of the following initial boundary-value problem associated with the thermoelastic system:

$$u_{tt} - a^2\Delta u - \left(b^2 - a^2\right) \nabla\text{div}\, u + \nabla\theta + \rho(x, u_t) = 0, \quad \text{in } \Omega \times (0, \infty), \tag{1.1}$$

$$\theta_t - \Delta\theta + \text{div}\, u_t = 0, \quad \text{in } \Omega \times (0, \infty), \tag{1.2}$$

$$u(x, 0) = u_o(x), \quad u_t(x, 0) = u_1(x), \quad \theta(x, 0) = \theta_o, \quad \text{in } \Omega, \tag{1.3}$$

$$u(x, t) = 0, \; \theta(x, t) = 0, \quad \text{on } \Gamma \times (0, \infty), \tag{1.4}$$

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where Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 2$. $u(x,t) = (u^1(x,t), \ldots, u^N(x,t))$ is the vector displacement, $\Delta u = (\Delta u^1(x,t), \ldots, \Delta u^N(x,t))$ is the Laplacian operator of u, $\operatorname{div} u$ is the divergence of u and ∇ is the gradient operator. θ represents the temperature distribution. The vector function ρ is a dissipative term, localized in a neighborhood of part of the boundary of Ω. The coefficients a and b are related to the Lamé coefficients of Elasticity Theory and $b^2 > a^2 > 0$.

In this paper, we show the uniform stabilization of the total energy for the system (1.1)-(1.4) with algebraic rates, where the dissipative term $\rho(x, u_t)$ is strongly nonlinear and effective only in a neighborhood of part of the boundary. When the nonlinear dissipative term $\rho(x,s)$ behaves linearly for small s and the dimension is two, we obtain an exponential decay rate for the total energy of the system. If the dimension $N \geq 3$ and $\rho(x,s)$ behaves linearly for all s, then the decay rate is also exponential. To prove these results we use ideas of [10], [8] and [13] to obtain some energy identities associated with localized multipliers in order to construct special difference inequalities for the associated energy. The main estimates in this work are obtained using Holmgren's Uniqueness Theorem and Nakao's Lemma.

Regarding works on the stabilization of thermoelastic systems, Dafermos [2] investigated the existence, uniqueness, regularity and stabilization (without rates) of the solution for the linear system in one dimension. Racke [17] considered a Cauchy problem for the three-dimensional nonlinear thermoelasticity equations and proved the global existence of smooth solutions for sufficiently small and smooth initial data. It was necessary to assume that certain nonlinear terms are quasilinear and have cubic nonlinearity.

Pereira-Perla Menzala [15] proved that the total energy of the linear thermoelastic system in an isotropic, non-homogeneous (bounded, n-dimensional) medium, with a linear dissipative term effective in the whole domain, decays to zero in an exponential rate. Rivera [23] later proved that the energy of the classical one-dimensional thermoelastic system decays to zero exponentially. Also, Henry-Lopes-Perisinotto [5] showed, using spectral analysis, that the three parts of the energy of that system decay exponentially to zero in the one-dimensional case, but such decay does not occur in higher dimensions.

Racke [18] used the energy method to prove the exponential decay to zero of displacement and temperature for the three-dimensional equations of linear thermoelasticity in bounded domains for inhomogeneous and anisotropic media assuming a linear damping force. Racke-Shibata-Zheng [20] considered the Dirichlet initial-boundary value problem in one-dimensional nonlinear thermoelasticity to prove that if the initial data are closed to the equilibrium then the problem admits a unique global smooth solution. They proved that as time tends to infinity, the solution is exponentially stable. They also used techniques based on the work of Rivera [23] to improve previous results of Racke-Shibata [19], which were based on spectral analysis to obtain decay rates for solutions of nonlinear thermoelasticity in one dimension. Rivera-Barreto [24] improved further the global in time unique existence result with exponential decay of the energy obtained by Racke-Shibata-Zheng [20] by assuming more general smoothness hypothesis on the initial data.

Rivera [22] considered the linear, inhomogeneous thermoelasticity equations in one dimension for a bounded domain with several boundary conditions considered. It is proved that the solution (u, θ) has some partial derivatives decaying exponentially to zero in the L^2-norm. Rivera [21] investigated the linear homogeneous and isotropic thermoelasticity equations with homogeneous Dirichlet boundary conditions in a general n-dimensional domain. The author shows that the curl-free part of the displacement and the thermal difference decay exponentially to zero as times goes to infinity. It is also proved that the divergence-free part of the displacement conserves its energy, which implies that if the divergence-free part of the initial data is not zero, then the total energy does not decay to zero uniformly.

Jiang, Rivera and Racke [7] proved the exponential decay for solutions of the linear isotropic system of thermoelasticity in bounded two- or three-dimensional domains with the hypothesis that the rotation of the displacement vanishes.

Lebeau-Zuazua [9] studied the linear system of thermoelasticity in two- and three-dimensional smooth bounded domain. They analyzed whether the energy of solutions decays exponentially to zero. They also prove that when the domain is convex, the decay rate is never uniform. Liu-Zuazua [12] established explicit formulas for the decay rate of the energy of a body in the framework of linear thermoelasticity when some part of the boundary of the body is clamped and on the rest there is some nonlinear velocity feedback. This result was obtained using the theory of semigroups, Lyapunov methods and multiplier techniques. Qin and Rivera ([16]) established the global existence, uniqueness and exponential stability of solutions to equations of one-dimensional nonlinear thermoelasticity with relaxation kernel and subject to Dirichlet boundary conditions for the displacement and to Neumann boundary conditions for the temperature difference.

Irmscher-Racke [6] obtained explicit sharp decay rates for solutions of the system of classical thermoelasticity in one dimension. They also considered the model of thermoelasticity with second sound and compared the results of both models with respect to the asymptotic behavior of solutions.

Our work generalizes, in the context of pure elasticity, the previous work of Bisognin-Bisognin-Charão ([1]), where it is proved a stabilization theorem with polynomial decay rate for the total energy of the system only in three-dimensions. We should mention that the work of Bisognin-Bisognin-Charão ([1]) generalized the work of Guesmia ([3]) that proved the stabilization of the total energy for pure system of elasticity with a nonlinear localized dissipation which does not couple the system of equations and behaves linearly far from the origin. Our work also generalizes the previously mentioned work of Pereira-Perla Menzala ([15]). Regarding the work of Jiang-Rivera-Racke ([7]), instead of assuming their condition that the rotation of the displacement vanishes and the dimension is two or three, we consider in the system a nonlinear localized weak dissipation in any dimension $N \geq 2$. Furthermore, according to the previous mentioned works, only part of the energy of the free thermoelastic system decays uniformly if $N \geq 2$. Thus, in this sense, our work also improves those results and other previous results for the thermoelastic system due to the fact that we have obtained exponential decay for the total energy when $N = 2$ and polynomial decay rate when $N \geq 3$ by including a weak localized nonlinear dissipative term $\rho(x, s)$ in the system.

2 Hypotheses and Notation

Throughout this work the dot ($\cdot$) will represent the usual inner product between two vectors in $\mathbb{R}^N$. Let $V_1 = H_0^1(\Omega)^N$, $V_2 = H_0^1(\Omega)$, $U_1 = V_1 \cap H^2(\Omega)^N$, $U_2 = V_2 \cap H^2(\Omega)$. $|\;|$ denotes the norm of a vector in $\mathbb{R}^N$, $|\nabla u(x,t)|^2 =$

$\sum_{i=1}^{N} |\nabla u^i(x,t)|^2$, $(\ ,\ )$ represents the inner product in $L^2(\Omega)^N$ and $\|\ \|$ denotes the corresponding norm. Also, $\|\nabla u\|^2 = \sum_{i=1}^{N} \left\|\nabla u^i\right\|^2$ and $\|u\|^2_{H^1(\Omega)} = \sum_{i=1}^{N} \left\|u^i\right\|_1^2$, where $\|\ \|_1$ denotes the usual norm in $H^1(\Omega)$. We use $|\Omega|$ to represent the measure of Ω.

Now, we list the hypotheses which we use to establish existence and uniqueness of solutions.

(H0) $\Omega \subset \mathbb{R}^N$ is a bounded open set with smooth boundary (at least of class C^2);

(H1) $u_o \in U_1,\ u_1 \in V_1; \theta_o \in U_2$

(H2) $\rho :\ \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}^N$ is a function such that:

(a) $\rho(x,\ s) \cdot s \geq 0,\ s \in \mathbb{R}^N,\ x \in \overline{\Omega}$;

(b) ρ and $\dfrac{\partial \rho}{\partial s_i}$ are continuous functions in $\overline{\Omega} \times \mathbb{R}^N$;

(c) $\sum_{i,\ k=1}^{N} u^i \dfrac{\partial \rho_i(x,s)}{\partial s_k} u^k \geq 0,\ \forall u \in \mathbb{R}^N, \forall x \in \overline{\Omega},\ \forall s \in \mathbb{R}^N$, i.e., $\dfrac{\partial \rho}{\partial s}$ is positive semi-definite.

(d) There exist positive constants $K_0,\ K_1,\ K_2$ and K_3 and numbers p, r, $-1 < r < +\infty,\ -1 < p \leq \dfrac{2}{N-2}$ if $N \geq 3$ and $-1 < p < +\infty$ if $N = 1$ or 2, such that: $K_2\ a(x)|s|^{r+2} \leq \rho(x,s) \cdot s$ and $|\rho(x,s)| \leq K_0\ a(x)\left(|s|^{r+1} + |s|\right)$, for $|s| \leq 1$, $K_3\ a(x)|s|^{p+2} \leq \rho(x,s) \cdot s$ and $|\rho(x,s)| \leq K_1\ a(x)\left(|s|^{p+1} + |s|\right)$, for $|s| \geq 1$, where the function $a = a(x)$ is such that $a : \overline{\Omega} \to \mathbb{R}^+$ belongs to $L^\infty(\Omega)$.

Remark 1. If $a(x)$ is a continuous function on $\overline{\Omega}$, then $\rho(x,s) = a(x)|s|^p s$, $s \in \mathbb{R}^N$, is an example of a function which satisfies (a)–(d) with $r = p$.

In order to study the stabilization of the total energy for this system, we specify where the damping is effective in the domain, that is, where the dissipative term is localized. We choose x_o in $\mathbb{R}^N$ and we define

$$\Gamma(x_o) = \left\{x \in \partial\Omega :\ (x - x_o) \cdot \eta(x) \geq 0\right\},$$

where $\eta(x)$ denotes the outward unit normal vector at $x \in \partial\Omega$.

Now, let $\omega \subset \overline{\Omega}$ be a neighborhood of $\Gamma\,(x_o)$. Then, in addition to the hypothesis (H2)(d) that the function $a = a(x)$ is nonnegative, we also assume that

$$a\,(x) \geq a_o > 0\,, \quad \text{in } \omega.$$

The energy $E(t)$ of the system (1.1) is defined by

$$E(t) = \frac{1}{2}\int_\Omega \left(|u_t|^2 + a^2\ |\nabla u|^2 + (b^2 - a^2)\ (\text{div}\, u)^2 + \theta^2 \right) dx, \qquad (2.1)$$

where $\{u = u(x,t),\ \theta = \theta(x,t)\}$ is the solution of (1.1)–(1.4). We denote the energy difference $E\,(t) - E\,(t+T)$ by ΔE. Also, let E_1 denote the part of E without the term $1/2\ \int_\Omega \theta^2$, and $E_2 = E - E_1$.

In order to reduce the size of the formulas, we use the following notation.

$$Q\,(t) = (t, t+T) \times \Omega, \qquad \int_{Q(t)} = \int_t^{t+T} \int_\Omega,$$

$$\sum_t = (t, t+T) \times \partial\Omega, \qquad \int_{\sum_t} = \int_t^{t+T} \int_{\partial\Omega},$$

$$Q\,(t)_{\hat{\omega}} = (t, t+T) \times \left(\hat{\omega} \cap \overline{\Omega}\right), \qquad \int_{Q(t)_{\hat{\omega}}} = \int_t^{t+T} \int_{\hat{\omega}\cap\overline{\Omega}},$$

$$Q\,(t)_\omega = (t, t+T) \times \omega \quad \text{and} \quad \int_{Q(t)_\omega} = \int_t^{t+T} \int_\omega .$$

Remark 2. The multiplication of (1.1) by u_t, (1.2) by θ followed by adding the resulting equations and integrating over $Q\,(t)$ produces:

$$\Delta E = E\,(t) - E\,(t+T) = \int_{Q(t)} |\nabla\theta|^2\ dx\ ds + \int_{Q(t)} \rho\,(x, u_t)\cdot u_t\ dx\ ds. \quad (2.2)$$

The hypothesis H2(a) implies that the energy is a nonincreasing function of t.

3 Results

Regarding the existence and uniqueness of solution for the problem (1.1)–(1.4), the following result holds.

Theorem 3.1 (Existence and Uniqueness). *Under the hypotheses of the previous section, the initial-boundary value problem* (1.1)–(1.4) *has a unique solution* $u = u(x,t)$, $\theta = \theta(x,t)$ *such that for each* $T > 0$, $u \in C([0,T], U_1)$, $u_t \in C([0,T], V_1)$, $u_{tt} \in C([0,T], L^2(\Omega)^N)$, $\theta \in C([0,T], U_2)$ *and* $\theta' \in C([0,T], V_2)$.

Proof. We sketch the proof based on Semigroups for the nonlinear case. We consider the operator A defined by

$$A = \begin{bmatrix} 0 & I & 0 \\ -A_1 & 0 & B \\ 0 & C & -A_2, \end{bmatrix} \tag{3.1}$$

where $A_1\, u = -a^2\, \Delta\, u - \left(b^2 - a^2\right)\, \nabla \mathrm{div}\, u$, $A_2\, \theta = -\Delta\theta$, $B\, \theta = -\nabla\theta$, and $C\, v = -\mathrm{div}\, v$. The domain of $A : D(A) \subset \mathbf{H} \to \mathbf{H}$ is given by

$$U_1 \times V_1 \times U_2,$$

where

$$\mathbf{H} = V_1 \times L^2(\Omega)^N \times L^2(\Omega)$$

is a Hilbert space with the inner product

$$\begin{aligned}\langle (z_1, z_2, z_3), (w_1, w_2, w_3) \rangle = \int_\Omega \Big\{ & z_2\, w_2 + a^2\, \nabla z_1 \cdot \nabla w_1 \\ & + (b^2 - a^2)\, \mathrm{div}\,(w_1)\, \mathrm{div}\,(z_1) + z_3\, w_3 \Big\}\, dx.\end{aligned}$$

Then, the original problem is equivalent to

$$\frac{dU}{dt} = A\, U + F(U), \tag{3.2}$$

$$U(0) = U_0 = (u_0, u_1, \theta_0), \tag{3.3}$$

where

$$F\big(w_1, w_2, w_3\big) = \big(0, \rho(\cdot,\ w_2), 0\big).$$

We use the theorem of Hille-Yosida to prove that the operator A (densely defined) generates a C^0-semigroup of contractions. The hypotheses of the Hille-Yosida are verified after proving that for each positive λ, there exists $(\lambda I - A)^{-1} \in \mathcal{B}(\mathbf{H})$ and $\|(\lambda I - A)^{-1}\| \le \frac{1}{\lambda}$.

Next, using hypothesis H2(b), it is easy to prove that $F : \mathbf{H} \to \mathbf{H}$ is Lipschitz continuous on bounded sets. As a consequence, one obtains a local (generalized) solution on a interval $[0, T_{\max}]$. Then, by proving that $\|U\|_X \leq C\ E(0)$, where is used the fact that the total energy of the system is decreasing due to hypothesis (H2)(a), it is possible to show that the solution can be extended to the interval $[0, \infty)$. Finally, by using hypothesis (H2)(b), one proves that F is continuously differentiable. It follows that the generalized solution of (3.2)–(3.3) is a classical solution on $[0,\ \infty)$. The uniqueness follows from the hypothesis (H2)(c) on the function ρ. $\square$

3.1 Boundedness of the Laplacian L^2-norm

In the previous section, we proved the existence of a unique solution (u, θ) for the thermoelastic system in the class

$$u \in C\big([0,\infty);\ H^2(\Omega)^N \cap H_0^1(\Omega)^N\big)\ \cap C^1\big([0,\infty);\ H_0^1(\Omega)^N\big)\ \cap C^2\big([0,\infty);\ L^2(\Omega)^N\big)$$

and

$$\theta \in C\big([0,\ \infty);\ H^2(\Omega) \cap H_0^1(\Omega)\big)\ \cap C^1\big([0\ \infty);\ H_0^1(\Omega)\big).$$

In order to prove that

$$\Delta u \in L^\infty\big(0,\ \infty;\ L^2(\Omega)^N\big) \tag{3.4}$$

we write

$$Lu = -u_{tt} - \nabla\theta - \rho\big(x, u_t(t)\big), \tag{3.5}$$

where Lu is the differential operator defined by

$$Lu = -a^2\ \Delta u - \big(b^2 - a^2\big)\ \nabla\ \mathrm{div}\ u.$$

To show (3.4), we only need to show that each term in the right-hand side of the equation (3.5) is in $L^\infty(0, \infty;\ \ L^2(\Omega)^N)$. This is proved in the next Lemmas. Then, the uniform boundedness of L^2-norm of the Laplacian of u follows from the elliptic regularity of the operator L.

Lemma 3.2. *The L^2-norm of $\rho(\cdot, u_1)$ is bounded by a constant that depends only on the initial data $u_0,\ u_1,\ \theta_0$.*

Proof. We have $\|\rho(x, u_1)\|^2 \leq \int_{\Omega_1} |\rho(x, u_1)|^2 \; dx + \int_{\Omega_2} |\rho(x, u_1)|^2 \; dx$, where

$$\Omega_1 := \left\{x \in \Omega : \; |u_1| \leq 1\right\}$$

and

$$\Omega_2 = \Omega - \Omega_1.$$

If $r \geq 0$, then, due to our hypothesis on the function ρ,

$$\begin{aligned}\int_{\Omega_1} |\rho(x, u_1)|^2 \, dx \; &\leq K_0^2 \int_{\Omega_1} a(x)^2 \; \left[|u_1|^{r+1} + |u_1|\right]^2 \; dx \\ &\leq C \; ||a||_\infty^2 \int_\Omega |u_1|^2 \, dx,\end{aligned}$$

which is bounded ($u_1 \in L^2(\Omega)^N$).

Let $N \geq 3$ and $0 \leq p \leq \frac{2}{N-2}$, then due to the hypotheses on the function ρ

$$\begin{aligned}\int_{\Omega_2(0)} |\rho(x, u_1)|^2 \, dx \; &\leq K_1^2 \int_{\Omega_1(0)} a(x)^2 \; \left[|u_1|^{p+1} + |u_1|\right]^2 \; dx \\ &\leq C \; ||a||_\infty^2 \int_\Omega |u_1|^{2\,p+2} \, dx \\ &\leq C||\nabla u_1||^{2\,p+2} \leq C \; E(0)^{p+1}.\end{aligned}$$

The last estimate is due to facts that $2 \leq 2\; p + 2 \leq \frac{2N}{N-2}$ and $H_0^1(\Omega)$ is continuously embedded in $L^q(\Omega)$ for $q \in \left[2, \frac{2N}{N-2}\right]$. Thus, the lemma holds for the case 1: $N \geq 3, r \geq 0$ and $0 \leq p \leq \frac{2}{N-2}$.

The proof of the other cases is similar. □

Now, we obtain an estimate for the L^2-norm of $||u_{tt}(0)||$.

Lemma 3.3. *There is a positive constant* $C = C(u_0, \; u_1, \; \theta_0)$ *such that* $||u_{tt}(0)|| \leq C$ *and* $||\theta_t(0)|| \leq C$.

Proof. We take L^2 inner-product between the first equation in the thermoelastic system with $u_{tt}(t)$ and evaluate at $t = 0$ to obtain

$$||u_{tt}(0)||^2 + (L\; u_0, u_{tt}(0)) + \; (\nabla\theta_0, u_{tt}(0)) + (\rho(x, u_1), u_{tt}(0)) = 0. \quad (3.6)$$

Since $u_{tt}(0) \in L^2(\Omega)^N$, $u_0 \in H^2(\Omega)^N$ and $\theta_0 \in H^2(\Omega)$, we have the following estimate

$$||u_{tt}(0)|| \leq a^2\ ||\Delta u_0||\ + (b^2 - a^2)\ ||\nabla \text{div}\ u_0|| + c^2\ ||\nabla\theta_0|| + ||\rho(x, u_1)|| \quad (3.7)$$

The result follows now from the previous lemma and our hypothesis on the initial data. The proof of the estimate for θ is similar. □

Lemma 3.4.

(i) *u_{tt} belongs to $L^\infty(0, \infty;\ L^2(\Omega)^N)$ and for each t the L^2-norm of $u_{tt}(t)$ is bounded by a constant that depends only on the initial data $u_0,\ u_1,\ \theta_0$.*

(ii) *For all $t \in (0, \infty)$,*

$$\begin{aligned} &||u_{tt}(t)||^2 + a^2\ ||\nabla u_t(t)||^2 + (b^2 - a^2)\ ||\text{div}\ u_t(t)||^2 + ||\theta_t(t)||^2 \\ &\quad + 2\int_0^t ||\nabla\theta_t(s)||^2\ ds \leq ||u_{tt}(0)||^2 \\ &\quad + a^2\ ||\nabla u_1 + (b^2 - a^2)\ ||\text{div}\ u_1||^2 + ||\theta_t(0)||^2 \end{aligned} \quad (3.8)$$

Proof. We differentiate once the first equation of the thermoelastic system with respect to t, multiply each member of the resulting equation by u_{tt} and integrate over Ω. Therefore, we have

$$\begin{aligned} &\frac{d}{dt}||u_{tt}||^2 + a^2\ \frac{d}{dt}||\nabla u_t||^2 + (b^2 - a^2)\ \frac{d}{dt}||\text{div}\ u_t||^2 \\ &\quad + 2\ \left(\frac{\partial\rho}{\partial s}(x, u_t(t)) \cdot u_{tt}, u_{tt}\right) + 2\ (\nabla\theta_t, u_{tt}) = 0. \end{aligned} \quad (3.9)$$

By the hypothesis on the initial data ($u_1 \in H_0^1(\Omega)^N$), it follows that $a^2\ \nabla u_1 + (b^2 - a^2)$ div u_1 belongs to $L^2(\Omega)^N$. So, after integrating this equation over $[0, t]$, we obtain

$$\begin{aligned} &||u_{tt}(t)||^2 + a^2\ ||\nabla u_t||^2 + (b^2 - a^2)\ ||\text{div}\ u_t||^2 \\ &\quad + 2\int_0^t \left(\frac{\partial\rho}{\partial s}(x, u_t(s)) \cdot u_{tt}(s), u_{tt}(s)\right)\ ds \\ &\quad + 2\int_0^t (\nabla\theta_t(s), u_{tt}(s))\ ds = ||u_{tt}(0)||^2 + a^2\ ||\nabla u_1||^2 \\ &\quad + (b^2 - a^2)\ ||\text{div}\ u_1||^2. \end{aligned} \quad (3.10)$$

Now, differentiating the second equation of the thermoelastic system with respect to t, multiplying by θ_t and integrating over Ω, we obtain

$$\frac{d}{dt}||\theta_t||^2 + 2\,||\nabla\theta_t||^2 - 2\,(\nabla\theta_t, u_{tt}) = 0.$$

Integrating over $[0, t]$, we obtain

$$||\theta_t(t)||^2 + 2\int_0^t ||\nabla\theta_t(s)||^2\,ds - 2\int_0^t (\nabla\theta_t, u_{tt}(s))\,ds = ||\theta_t(0)||^2. \quad (3.11)$$

Adding equations (3.10) and (3.11) and using $(H2)(c)$, we obtain

$$\begin{aligned}&||u_{tt}(t)||^2 + a^2\,||\nabla u_t(t)||^2 + (b^2 - a^2)\,||\text{div}\,u_t(t)||^2 + ||\theta_t(t)||^2\\ &+ 2\int_0^t ||\nabla\theta_t(s)||^2\,ds \le ||u_{tt}(0)||^2 + a^2\,||\nabla u_1||^2\\ &+ (b^2 - a^2)\,||\text{div}\,u_1||^2 + ||\theta_t(0)||^2.\end{aligned}$$

The result follows from the Lemma 3.3. □

Lemma 3.5. *The L^2-norm of $\rho(., u(t))$ is bounded by a constant that depends only on the initial data $u_0,\ u_1,\ \theta_0$.*

Proof. This proof is similar to the proof of Lemma 3.2, therefore we present the proof for the three-dimensional case only.

$\|\rho\,(x, u)\|^2 \le \int_{\Omega_1(t)} |\rho\,(x, u)\,|^2\,dx + \int_{\Omega_2(t)} |\rho\,(x, u)\,|^2\,dx$, where for each t,

$$\Omega_1(t) := \left\{x \in \Omega :\ |u_t(t)| \le 1\right\}$$

and

$$\Omega_2(t) = \Omega - \Omega_1(t).$$

Let

$$I_1(t) := \int_{\Omega_1(t)} |\rho\,(x, u)\,|^2\,dx$$

and

$$I_2(t) := \int_{\Omega_2(t)} |\rho\,(x, u)\,|^2\,dx.$$

In order to prove the result, it is sufficient to estimate $I_1(t)$ considering the cases $r \geq 0$ and $-1 < r < 0$, and estimate $I_2(t)$ considering the cases $0 \leq p \leq 2$ and $-1 < p < 0$. If $r \geq 0$, then, due to H2(d),

$$\begin{aligned} I_1(t) &\leq K_0^2 \int_{\Omega_1(t)} a(x)^2 \left[|u_t|^{r+1} + |u_t|\right]^2 dx \\ &\leq C\, ||a||_\infty^2 \int_\Omega |u_t|^2\, dx \leq CE(t) \leq CE(0). \end{aligned}$$

The last inequalities are due to the fact that $||u_t||^2$ is part of the energy and the fact that the energy is decreasing.

If $0 \leq p \leq 2$, then due to H2(d),

$$\begin{aligned} I_2(t) &\leq K_1^2 \int_{\Omega_1(t)} a(x)^2 \left[|u_t|^{p+1} + |u_t|\right]^2 dx \\ &\leq C\, ||a||_\infty^2 \int_\Omega |u_t|^{2\,p+2}\, dx \leq C||u_t||_{H_0^1(\Omega)}^{2\,p+2} \leq C||\nabla u_t||^{2\,p+2}. \end{aligned}$$

The last term in this estimate is bounded for all t due to the previous lemma.

If $-1 < r < 0$, then

$$\begin{aligned} I_1(t) &\leq K_0^2 \int_{\Omega_1(t)} a(x)^2 \left[|u_t|^{r+1} + |u_t|\right]^2 dx \\ &\leq C \int_{\Omega_1(t)} |u_t|^{2r+2}\, dx \leq C \int_{\Omega_1(t)} |u_t|^2\, dx \leq C\; E(0), \end{aligned}$$

where the last inequality follows from Hölder's inequality. If $-1 < p < 0$, then

$$\begin{aligned} I_2(t) &\leq K_1^2 \int_{\Omega_2(t)} a(x)^2 \left[|u_t|^{p+1} + |u_t|\right]^2 dx \\ &\leq C \int_{\Omega_2(t)} |u_t|^2\, dx \leq C\; E(0). \end{aligned}$$

□

Lemma 3.6. $\nabla\theta \in L^\infty(0, \infty;\; L^2(\Omega)^N)$.

Proof. We use the fundamental identity:

$$||\nabla\theta||^2 + \int_\Omega \rho(x,\; u_t) \cdot\; u_t\, dx = -\frac{dE}{dt}. \tag{3.12}$$

Then, due to the hypothesis H2(a), we have

$$||\nabla\theta||^2 \leq \left|\frac{dE}{dt}\right|.$$

However,

$$\frac{dE}{dt}(t) = \int_\Omega \left[u_t \cdot u_{tt} + a^2\ \nabla u \cdot \nabla u_t + (b^2 - a^2)\ (\text{div}\ u)\ (\text{div}\ u_t) + \theta \cdot \theta_t\right] dx.$$

Therefore, using Cauchy-Schwartz's inequality, we obtain

$$\begin{aligned}\left|\frac{dE}{dt}\right| \ &\leq\ \frac{1}{2}\ \Big[||u_t||^2 + ||u_{tt}||^2 + a^2\ ||\nabla u||^2 + a^2\ ||\nabla u_t||^2 \\ &\quad + (b^2 - a^2)\ ||\text{div}\ u||^2 + (b^2 - a^2)\ ||\text{div}\ u_t||^2 + ||\theta||^2 + ||\theta_t||^2\Big]\end{aligned}$$

The terms $||u_t||^2,\ ||\nabla u||^2,\ ||\text{div}\ u||^2,\ ||\theta||^2$ are all bounded by a constant (independent of t) times the initial energy, since they are part of the energy and the energy is bounded by the initial energy. The remaining terms are bounded due to Lemma 3.4. We conclude that $||\nabla\theta||$ is bounded for all t, which concludes the proof. □

In summary: $||\Delta u(\cdot, t)||_{L^2(\Omega)} \leq C$ with C a positive constant which does not depend on t and depends only on the initial data, $||a(.)||_\infty$ and $|\Omega|$.

3.2 Theorem on stabilization

From now on, we will study the asymptotic behavior of the total energy of the system. Under certain hypothesis we will obtain exponential decay of the energy, which means that there exist positive constants M_0 and k_0 such that $E(t) \leq M_0\ \exp(-k_0\ t)$ if $t \geq 0$.

The main result of this paper is the following theorem of stabilization for the total energy of the system.

Theorem 3.7 (Stabilization). *Let $R > 0$ and initial data in $D(A)$ satisfying $E(0) < R$. Then, under the previous hypotheses, the total energy for the solution $u = u(x, t),\ \theta = \theta\,(x, t)$ of the problem* (1.1)–(1.4) *has the following asymptotic behavior in time*

$$E(t) \leq C\,(1 + t)^{-\gamma}, \tag{3.13}$$

where $C = C(R, r, p)$ is a positive constant (which depends on the initial data, $\|a\|_\infty$ and $|\Omega|$) and the decay rate γ is given according to the following cases:

Case 1. If $N = 2$ and

(a) $-1 < r < 0$ and $-1 < p < +\infty$ then $\gamma = \dfrac{-2(r+1)}{r}$;

(b) $r > 0$ and $-1 < p < +\infty$, then $\gamma = \dfrac{2}{r}$.

Case 2. If $N \geq 3$ and

(a) $r > 0$ and $0 < p \leq \dfrac{2}{N-2}$, then $\gamma = \min\left\{\dfrac{2}{r}, \dfrac{4(p+1)}{p\,(N-2)}\right\}$;

If $r = 0$ and $0 < p \leq \dfrac{2}{N-2}$, then $\gamma = \dfrac{4(p+1)}{p\,(N-2)}$;

If $r > 0$ and $p = 0$, then $\gamma = \dfrac{2}{r}$;

(b) $r > 0$ and $-1 < p < 0$, then $\gamma = \min\left\{\dfrac{2}{r}, \dfrac{4}{p(2-N)}\right\}$;

If $r = 0$ and $-1 < p < 0$, then $\gamma = \dfrac{4}{p(2-N)}$;

(c) $-1 < r < 0$ and $0 < p \leq \dfrac{2}{N-2}$, then

$$\gamma = \min\left\{\frac{-2(r+1)}{r}, \frac{4(p+1)}{p(N-2)}\right\};$$

If $-1 < r < 0$ and $p = 0$, then $\gamma = \dfrac{-2(r+1)}{r}$;

(d) $-1 < r < 0$ and $-1 < p < 0$, then

$$\gamma = \min\left\{\frac{-2(r+1)}{r}, \frac{4}{p(2-N)}\right\}.$$

The decay rate is exponential in the following cases:

(a) $N = 2$, $r = 0$ and $-1 < p < +\infty$,

(b) $N \geq 3$ and $r = p = 0$.

4 Energy identities

In the next lemma, we obtain energy identities that will be used in the following sections to obtain estimates in terms of energy differences.

Next, we use the notation

$$h : \nabla u = \big(h \cdot \nabla u_1, \cdots, h \cdot \nabla u_N\big),$$

where h is a vector field in $\mathbb{R}^N$.

Lemma 4.1 (Energy identities). *Let* $u(x,t),\ \theta(x,t)$ *be the solution of* (1.1)–(1.4)*,* $m \in W^{1,\infty}(\Omega)$ *and* $h : \overline{\Omega} \to \mathbb{R}^N$ *a vector field of class* C^1*. Then, the following identities hold.*

$$\begin{aligned}
&\int_{Q(t)} m\,(x)\ \Big\{-|u_t|^2 + a^2\ |\nabla u|^2\Big\}\ dx\ ds \\
&\quad = -\int_\Omega m\,(x)\ u_t \cdot u\ dx|_t^{t+T} - \int_{Q(t)} m\,(x)\ \rho\,(x, u_t) \cdot u\ dx\ ds \\
&\qquad - a^2 \int_{Q(t)} \sum_{i=1}^N \sum_{j=1}^N \frac{\partial u_i}{\partial x_j}\ \frac{\partial m}{\partial x_j}\ u_i\ dx\ ds - \int_{Q(t)} m\,(x)\ \nabla\theta \cdot u\ dx\ ds \\
&\qquad - (b^2 - a^2) \int_{Q(t)} \Big\{m(x)\ (\operatorname{div} u)^2 + \operatorname{div} u\ [\nabla m(x) \cdot u]\Big\}\ dx\ ds.
\end{aligned} \tag{4.1}$$

$$\begin{aligned}
&\frac{1}{2}\int_{\sum_t} (h \cdot \eta)\ \left[a^2\ \left|\frac{\partial u}{\partial \eta}\right|^2 + (b^2 - a^2)\ (\operatorname{div} u)^2\right]\ d\Gamma\ ds \\
&\quad = \int_\Omega (h : \nabla u) \cdot u_t|_t^{t+T}\ dx + \int_{Q(t)} (h : \nabla u) \cdot \rho\,(x, u_t)\ dx\ ds \\
&\qquad + \frac{1}{2}\int_{Q(t)} (\nabla \cdot h)\ \Big[|u_t|^2 - a^2\ |\nabla u|^2 - (b^2 - a^2)\ (\operatorname{div} u)^2\Big]\ dx\ ds \\
&\qquad + \int_{Q(t)} (h : \nabla u) \cdot (\nabla\theta)\ dx\ ds \\
&\qquad + a^2 \int_{Q(t)} \sum_{i,j,k=1}^N (\partial_k h_i)\ \big(\partial_i u_j\ \partial_k u_j\big)\ dx\ ds \\
&\qquad + (b^2 - a^2) \int_{Q(t)} \sum_{i,j,k=1}^N \big(\partial_j h_k\big)\ \big(\partial_i u_i\ \partial_k u_j\big)\ dx\ ds.
\end{aligned} \tag{4.2}$$

$$
\begin{aligned}
&\frac{1}{2}\int_{\sum_t} ((x-x_o)\cdot\eta)\left[a^2\left|\frac{\partial u}{\partial\eta}\right|^2+(b^2-a^2)\,(\operatorname{div} u)^2\right] d\Gamma\, ds\\
&\quad=\int_\Omega ((x-x_o)\cdot\nabla u)\cdot u_t|_t^{t+T}\,dx\\
&\qquad+\int_{Q(t)} ((x-x_o)\cdot\nabla u)\cdot\rho\,(x,u_t)\;dx\;ds\\
&\qquad+\frac{N}{2}\int_{Q(t)}\left[|u_t|^2-a^2\;|\nabla u|^2-(b^2-a^2)\,(\operatorname{div} u)^2\right] dx\;ds\\
&\qquad+\int_{Q(t)}\left[a^2\;|\nabla u|^2+(b^2-a^2)\,(\operatorname{div} u)^2\right] dx\;ds\\
&\qquad+\int_{Q(t)} ((x-x_o)\cdot\nabla u)\cdot\nabla\theta\;dx\;ds.
\end{aligned}
\tag{4.3}
$$

$$
\begin{aligned}
&\int_{Q(t)}\left\{-\,|u_t|^2+a^2\,|\nabla u|^2+(b^2-a^2)\,(\operatorname{div} u)^2\right\}\,dxds\\
&\quad=-\int_\Omega u_t\cdot u|_t^{t+T}\,dx-\int_{Q(t)}[\nabla\theta+\rho\,(x,u_t)]\cdot u\;dx\;ds
\end{aligned}
\tag{4.4}
$$

where $\eta=\eta\,(x)$ *is the outward unit normal at* $x\in\partial\Omega$.

Proof. Equations (4.4), (4.1) and (4.2) are proved by multiplying (1.1), respectively, by the following multipliers $M\,(u)=u$, $M\,(u)=m\,(x)\,u$ and $M\,(u)=h:\nabla u$, followed by integration over $Q\,(t)$. Equation (4.3) is a special case of equation (4.2) for $h=x-x_o$. We used the fact that $u=0$ on $\partial\Omega\times[0,\infty)$ and, as a consequence, the following vector identity:

$$(h:\nabla u)\cdot\frac{\partial u}{\partial\eta}=(h\cdot\eta)\;\left|\frac{\partial u}{\partial\eta}\right|^2$$

on $\partial\Omega\times[0,\infty)$. □

5 Energy estimates

In this and the next sections, the symbol C may denote different positive constants. These constants depend on, at most, $|\Omega|$,$|a|_\infty$ and the initial data.

The idea to obtain the stabilization of the energy (to zero) is to show an estimate for the energy, as follows:

$$\sup_{t \leq s \leq t+T} E(s)^{1+\delta} \leq C \; (E(t) - E(t+T)) \qquad \text{for all} \quad t \geq 0,$$

for some positive δ and a fixed $T > 0$, which may be large.

Then the asymptotic behavior is obtained using the following Lemma:

Lemma 5.1 (Nakao [13]). *Let $\Phi(t)$ be a nonnegative function on $\mathbb{R}^+$ satisfying*

$$\sup_{t \leq s \leq t+T} \Phi(s)^{1+\delta} \leq C_1 \{\Phi(t) - \Phi(t+T)\}$$

with $T > 0$, $\delta > 0$ and C_1 a positive constant. Then $\Phi(t)$ has the decay property

$$\Phi(t) \leq C_1 \; \Phi(0) \; (1+t)^{-1/\delta}, \quad t \geq T$$

where C_1 is a positive constant.

If $\delta = 0$, then $\Phi(t)$ has the decay property (exponential decay)

$$\Phi(t) \leq C_1 \; \Phi(0) \; exp^{-\kappa \, t},$$

for positive constants κ and C_1.

We also include the following lemma, which will be used to estimate an integral involving the dissipative term ρ.

Lemma 5.2 (Gagliardo-Nirenberg). *Let $1 \leq r < p < \infty$, $1 \leq q \leq p$ and $0 \leq m$. Then,*

$$\|v\|_{W^{k,q}} \leq C_2 \, \|v\|^{\theta}_{W^{m,p}} \, \|v\|^{1-\theta}_{L^r}$$

for $v \in W^{m,p}(\Omega) \cap L^r(\Omega)$, $\Omega \subset \mathbb{R}^N$, where C_2 is a positive constant and

$$\theta = \left(\frac{k}{N} + \frac{1}{r} - \frac{1}{q} \right) \left(\frac{m}{N} + \frac{1}{r} - \frac{1}{p} \right)^{-1}$$

provided that $0 < \theta \leq 1$.

Now, we begin the estimates for the energy of (1.1).

Lemma 5.3. *Let $\beta > 0$ such that $\frac{N\beta}{2} - 1 > 0$ and $\gamma = 2\ \min\{\gamma_1,\ \gamma_2\}$, where $\gamma_1 = \left(\frac{N\beta}{2} - 1\right)$ and $\gamma_2 = \left(1 + \beta\left(1 - \frac{N}{2}\right)\right)$. Then,*

$$
\begin{aligned}
\gamma \int_t^{t+T} E_1(s)\ ds &\leq C\ \left[E_1(t) + E_1(t+T)\right] \\
&+ \frac{\beta\ M_o}{2} \int_t^{t+T} \int_{\Gamma(x_o)} \left[a^2 \left|\frac{\partial u}{\partial \eta}\right|^2 + (b^2 - a^2)\ (\operatorname{div} u)^2\right] d\Gamma\ ds \\
&+ \int_{Q(t)} \left[\beta\ M_o\ |\nabla u| + |u|\right]\ |\rho(x, u_t)|\ dx\ ds \\
&+ \int_{Q(t)} \left[\beta\ M_o\ |\nabla u| + |u|\right]\ |\nabla \theta|\ dx\ ds,
\end{aligned}
$$

for all $t \geq 0$, $T > 0$, where $M_o = \sup_{x \in \overline{\Omega}} |x - x_o|$.

The proof is obtained multiplying the identity (4.3) by β and adding with the identity (4.4). The details are similar to Oliveira-Charão ([14]) for an incompressible vector wave equation.

It is necessary to estimate the boundary integral which appear in the above lemma.

Lemma 5.4. *There exists a constant $C > 0$ such that*

$$
\begin{aligned}
&\int_t^{t+T} \int_{\Gamma(x_o)} \left[a^2 \left|\frac{\partial u}{\partial \eta}\right|^2 + (b^2 - a^2)\ (\operatorname{div} u)^2\right] d\Gamma\ ds \\
&\quad \leq C\ \{E_1(t) + E_1(t+T)\} + C\ \left\{\int_t^{t+T} \int_w \left[|u_t|^2 + |u|^2\right]\ dxds\right\} \\
&\qquad + C\left\{\int_{Q(t)} |\rho(x, u_t)|\ \left[|\nabla u| + |u|\right]\ dxds + \int_{Q(t)_\omega} |\nabla \theta|\ \left[|\nabla u| + |u|\right]\ dxds\right\}.
\end{aligned}
$$

Proof. Let $h : \mathbb{R}^N \to \mathbb{R}^N$ be a vector field of class C^1 over $\overline{\Omega}$ which satisfies:

$$h(x) = \eta(x) \qquad \text{on } \Gamma(x_o), \tag{5.1}$$

$$h(x) \cdot \eta(x) \geq 0 \qquad \text{on } \partial\Omega, \tag{5.2}$$

$$h(x) = 0 \qquad \text{in } \Omega \setminus \hat{\omega} \tag{5.3}$$

where $\eta = \eta(x)$ is the outward unit normal vector at $x \in \partial\Omega$, $\hat{\omega} \subset \mathbb{R}^N$ is an open set such that $\Gamma(x_o) \subset \hat{\omega} \cap \overline{\Omega} \subset w \subset \overline{\Omega}$.

Let $m \in W^{1,\infty}(\Omega)$ be a function such that

$$\frac{|\nabla m|^2}{m} \quad \text{is bounded,} \tag{5.4}$$

$$0 \leq m \leq 1 \quad \text{in } \Omega, \tag{5.5}$$

$$m = 1 \quad \text{in } \hat{\omega} \cap \overline{\Omega}, \tag{5.6}$$

$$m = 0 \quad \text{in } \overline{\Omega} \setminus \omega. \tag{5.7}$$

For the existence of such functions, see [4, 10].

Using identity (4.2), the properties of h, m and the summation convention, we have:

$$\begin{aligned}
&\int_t^{t+T} \int_{\Gamma(x_o)} \left[a^2 \left| \frac{\partial u}{\partial \eta} \right|^2 + (b^2 - a^2)\,(\text{div } u)^2 \right] d\Gamma\, ds \leq 2 \int_\Omega (h : \nabla u) \cdot u_t \, dx|_t^{t+T} \\
&\quad + \int_{Q(t)} (\nabla \cdot h) \left[|u_t|^2 - a^2\, |\nabla u|^2 - (b^2 - a^2)\,(\text{div } u)^2 \right] dxds \\
&\quad + 2 \int_{Q(t)} \rho\,(x, u_t) \cdot (h : \nabla u)\, dxds + 2 \int_{Q(t)} (h : \nabla u) \cdot (\nabla \theta)\, dxds \\
&\quad + 2 \int_{Q(t)} \left[a^2\, \partial_k h_i\, \partial_i u_j\, \partial_k u_j + (b^2 - a^2)\, \partial_i u_i\, \partial_j h_k\, \partial_k u_j \right] dxds \\
&\leq C \Big\{ E_1(t) + E_1(t+T) + \int_{Q(t)_{\hat{\omega}}} \left[|u_t|^2 + a^2\, |\nabla u|^2 + (b^2 - a^2)\,(\text{div } u)^2 \right] dxds \\
&\quad + \int_{Q(t)} |\rho\,(x, u_t)|\, |\nabla u|\, dxds + \int_{Q(t)_{\hat{\omega}}} \left| \partial_i u_j \right| \left| \partial_k u_j \right| dxds \\
&\quad + (b^2 - a^2) \int_{Q(t)_{\hat{\omega}}} |\partial_i u_i| \left| \partial_k u_j \right| dxds + \int_{Q(t)_{\hat{\omega}}} |\nabla u|\, |\nabla \theta|\, dxds \Big\}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\int_t^{t+T} \int_{\Gamma(x_o)} \left[a^2 \left| \frac{\partial u}{\partial \eta} \right|^2 + (b^2 - a^2)\,(\text{div } u)^2 \right] d\Gamma\, ds \\
&\quad \leq C \Big\{ E_1(t) + E_1(t+T) + \int_{Q(t)_w} |u_t|^2\, dxds \\
&\qquad + \int_{Q(t)} m(x) \left[a^2\, |\nabla u|^2 + (b^2 - a^2)\,(\text{div } u)^2 \right] dxds \\
&\qquad + \int_{Q(t)} |\rho\,(x, u_t)|\, |\nabla u|\, dxds + \int_{Q(t)_\omega} |\nabla u|\, |\nabla \theta|\, dxds \Big\}.
\end{aligned}$$

Since, by Poincare's inequality,

$$\left| \left[\int_\Omega m \ u \cdot u_t \ dx \right]_t^{t+T} \right| \leq C \left[E_1 (t) + E_1 (t+T) \right],$$

we estimate $\int_{Q(t)} m(x) \ \left[a^2 \ |\nabla u|^2 + (b^2 - a^2) \ (\operatorname{div} u)^2 \right] \ dxds$ using identity (4.1) and the properties of m:

$$\begin{aligned}
&\int_{Q(t)} m(x) \ \left[a^2 \ |\nabla u|^2 + (b^2 - a^2) \ (\operatorname{div} u)^2 \right] \ dxds \\
&\quad = \int_{Q(t)} m(x) \ |u_t|^2 \ dxds - \left[\int_\Omega m(x) \ u_t \cdot u \ dx \right]_t^{t+T} \\
&\quad - \int_{Q(t)} m(x) \ \rho(x, u_t) \cdot u \ dx \ ds - \int_\Omega m(x) \ \nabla\theta \cdot u \ dx \ ds \\
&\quad - a^2 \int_{Q(t)} \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{\partial u_i}{\partial x_j} \ \frac{\partial m}{\partial x_j} \ u_i \ dx \ ds \\
&\quad - (b^2 - a^2) \int_{Q(t)} \operatorname{div} u \left[\nabla m(x) \cdot u \right] \ dx \ ds.
\end{aligned} \tag{5.8}$$

Now, using Young's inequality in the last integral, the fact that $\frac{|\nabla m|^2}{m}$ is bounded and absorbing the term with the divergence in the left-hand side, we obtain

$$\begin{aligned}
&\int_{Q(t)} m(x) \ \left[a^2 \ |\nabla u|^2 + (b^2 - a^2) \ (\operatorname{div} u)^2 \right] \ dxds \\
&\quad \leq C \int_{Q(t)_w} \left(|u_t|^2 + |u|^2 \right) \ dxds + C \left\{ E_1 (t) + E_1 (t+T) \right. \\
&\quad + \left. \int_{Q(t)} \left[|\rho(x, u_t)| + |\nabla\theta| \right] \ |u| \ dx \ ds \right\}
\end{aligned}$$

For the last estimate we have also used the inequality:

$$\begin{aligned}
\int_{Q(t)} \frac{\partial u_i}{\partial x_j} \frac{\partial m}{\partial x_j} u_i \ &\leq \int_{Q(t)} \frac{1}{2} \left[|u_i|^2 \ \frac{|\nabla m|^2}{m} + m \ |\nabla u_i|^2 \right] \ dxds \\
&\leq C \int_t^{t+T} \int_w |u|^2 \ dxds + \frac{1}{2} \int_{Q(t)} m \ |\nabla u|^2 \ dxds.
\end{aligned}$$

Combining the previous estimates, the proof of Lemma 5.4 follows. □

Lemma 5.5. *Let* $u\,(x,\ t)$ *be the solution of* (1.1)–(1.4). *There exists a constant* $T > 0$*, which depends on E(0), such that*

$$E\,(t) \leq C\ \Delta E + \int_{Q(t)_\omega} \left\{|u_t|^2 + |u|^2\right\}\ dx\ ds$$
$$+ \int_{Q(t)} |\rho\,(x, u_t)|\ \{|\nabla u| + |u|\}\ dx\ ds + \int_{Q(t)_\omega} |\nabla\theta|\ \{|\nabla u| + |u|\}\ dx\ ds$$

for all $t \geq 0$.

Proof. It follows from the identity in Remark 2 that

$$\int_{Q(t)} \theta^2\ dxds \leq C_\Omega \int_{Q(t)} |\nabla\theta|^2\ dxds \leq C_\Omega\ \Delta E.$$

This fact allows us to get an estimate for the total energy $E(t)$.

Because the energy decreases, we have: $T\ E\,(t+T) \leq \int_t^{t+T} E\,(s)\ ds$.

These facts and the last two lemmas imply that:

$$\begin{aligned} T\ E\,(t+T)\ &<\ C_1\,\{E\,(t) + E\,(t+T)\} + C_1 \int_{Q(t)_\omega} \left[|u_t|^2 + |u|^2\right]\ dxds \\ &+\ C_1 \int_{Q(t)} |\rho\,(x, u_t)|\ [|\nabla u| + |u|]\ dxds \\ &+\ C_1 \int_{Q(t)_\omega} |\nabla\theta|\ [|\nabla u| + |u|]\ dxds. \end{aligned}$$

Thus, if we choose a fixed T such that $T > 2\ C_1 + 1$, the lemma is proved. □

Lemma 5.6. *Let* $(u,\ \theta)$ *be the solution of* (1.1)–(1.4). *Then, for T given by the previous lemma, we have:*

If $r \geq 0,\ 0 \leq p \leq \frac{2}{N-2}$ *and* $N \geq 3$, *then*

$$\begin{aligned} \int_t^{t+T}\int_\Omega |\rho|\,[|\nabla u| + |u|]\,dx\,ds &\leq C\ (\Delta E)^{\frac{1}{r+2}}\sqrt{E(t)} \\ &+ C\ (\Delta E)^{\frac{p+1}{p+2}}\ E(t)^{\frac{4-p(N-2)}{4(p+2)}}. \end{aligned} \tag{5.9}$$

If $N = 2$, *the estimate* (5.9) *holds for* $r \geq 0,\ p \geq 0$.

If $r \geq 0,\ -1 < p < 0$ *and* $N \geq 3$*, then*

$$\int_t^{t+T}\int_\Omega |\rho|\,[|\nabla u| + |u|]\,dx\,ds \leq C\,(\Delta E)^{\frac{1}{r+2}}\sqrt{E(t)} + C\,(\Delta E)^{\frac{2}{4+(2-N)p}}\sqrt{E(t)}. \tag{5.10}$$

If $r \geq 0,\ -1 < p < 0$ *and* $N = 2$*, then*

$$\int_t^{t+T}\int_\Omega |\rho|\,[|\nabla u| + |u|]\,dx\,ds \leq C\,(\Delta E)^{\frac{1}{r+2}}\sqrt{E(t)} + C\,(\Delta E)^{\frac{1}{p+2}}\ E(t)^{\frac{p+1}{p+2}}. \tag{5.11}$$

If $-1 < r < 0,\ 0 \leq p \leq \frac{2}{N-2}$ *and* $N \geq 3$*, then*

$$\int_t^{t+T}\int_\Omega |\rho|\,[|\nabla u| + |u|]\,dx\,ds \leq C\,(\Delta E)^{\frac{r+1}{r+2}}\sqrt{E(t)} + C\,(\Delta E)^{\frac{p+1}{p+2}}\,E(t)^{\frac{4-p(N-2)}{4(p+2)}}. \tag{5.12}$$

If $N = 2$*, the estimate* (5.12) *holds for* $-1 < r < 0,\ p \geq 0$.

If $-1 < r < 0,\ -1 < p < 0$ *and* $N \geq 3$*, then*

$$\int_t^{t+T}\int_\Omega |\rho|\,[|\nabla u| + |u|]\,dx\,ds \leq C\,(\Delta E)^{\frac{r+1}{r+2}}\sqrt{E(t)} + C\,(\Delta E)^{\frac{2}{4+p(2-N)}}\sqrt{E(t)}. \tag{5.13}$$

If $-1 < r < 0,\ -1 < p < 0$ *and* $N = 2$*, then*

$$\int_t^{t+T}\int_\Omega |\rho|\,[|\nabla u| + |u|]\,dx\,ds \leq C\,(\Delta E)^{\frac{r+1}{r+2}}\sqrt{E(t)} + C\,(\Delta E)^{\frac{1}{p+2}}\,E(t)^{\frac{p+1}{p+2}}. \tag{5.14}$$

The numbers r *and* p *appear in hypothesis* (H2)(d) *on the growth of the function* ρ.

Proof. By hypotheses on the growth of ρ, we have

$$\begin{aligned}
&\int_t^{t+T}\int_\Omega |\rho(x,u_t)|\,[|\nabla u| + |u|]\,dx\,ds \\
&\leq \int_t^{t+T}\int_{\Omega_1} K_o\,a(x)\,\left\{|u_t|^{r+1} + |u_t|\right\}\,[|\nabla u| + |u|]\,dxds \\
&\quad + \int_t^{t+T}\int_{\Omega_2} K_1\,a(x)\,\left\{|u_t|^{p+1} + |u_t|\right\}\,[|\nabla u| + |u|]\,dxds =:\ I_1 + I_2,
\end{aligned}$$

for $t \geq 0$, where

$$\Omega_1 = \Omega_1(t) = \left\{x \in \Omega,\ |u_t(x,t)| \leq 1\right\}, \Omega_2 = \Omega \setminus \Omega_1.$$

We will use the following estimate as a consequence of Gagliardo-Nirenberg's Lemma, Poincare's inequality and the boundedness of the L^2-norm of Δu,

$$\begin{aligned} \|\nabla u\|_{L^{p+2}(\Omega)^N} &\leq C\,\|\nabla u\|^{\theta}_{H^1(\Omega)^N}\,\|\nabla u\|^{1-\theta}_{L^2(\Omega)^N} \\ &\leq C\,\|u\|^{\theta}_{H^2(\Omega)^N\cap H^1_o(\Omega)^N}\,\|\nabla u\|^{1-\theta}_{L^2(\Omega)^N} \\ &\leq C\,\|\Delta u\|^{\theta}_{L^2(\Omega)^N}\,\|\nabla u\|^{1-\theta}_{L^2(\Omega)^N} \\ &\leq C\,\|\nabla u\|^{1-\theta}_{L^2(\Omega)^N} \leq C E(t)^{\frac{1-\theta}{2}} \end{aligned} \tag{5.15}$$

with $\theta = \dfrac{N\ p}{2(p+2)}$.

Case 1: Estimating I_1 for $r \geq 0,\ N \geq 2$:

Using Poincare's inequality we obtain

$$\begin{aligned} I_1 &\leq \|\sqrt{a}\|_{L^\infty(\Omega)} 2\,K_o \int_t^{t+T}\int_{\Omega_1} \sqrt{a(x)}|u_t|\,(|\nabla u| + |u|)\,dx\,ds \\ &\leq C\left(\int_t^{t+T}\int_{\Omega_1} a(x)|u_t|^2\,dx\,ds\right)^{1/2}\left(\int_t^{t+T} E(s)\,ds\right)^{1/2} \\ &\leq C\left(\int_t^{t+T}\int_{\Omega_1} a(x)|u_t|^{r+2}\,dx\,ds\right)^{\frac{1}{r+2}}\sqrt{E(t)} \end{aligned}$$

because

$$\frac{2}{r+2} + \frac{r}{r+2} = 1,$$

where C depends on $|\Omega|$, $\|\sqrt{a}\|_{L^\infty(\Omega)}$ and the fixed number T. We have used the fact that $E(t)$ is a non increasing function of t.

Using the hypotheses H2(a), H2(d) and identity (2.2) in Remark 2, it follows that

$$\begin{aligned} I_1 &\leq C\left(\int_t^{t+T}\int_{\Omega_1} \rho(x,u_t)\cdot u_t\,dx\,ds\right)^{\frac{1}{r+2}}\sqrt{E(t)} \\ &\leq C\ (\Delta E)^{\frac{1}{r+2}}\sqrt{E(t)} \end{aligned}$$

Case 2: Estimating I_1 for $-1 < r < 0$, $N \geq 2$:

Using Hölder's inequality, Poincaré's inequality in $W_o^{1,r+2}(\Omega)^N$ and hypotheses H2(a), H2(d), we obtain

$$\begin{aligned}
I_1 &\leq C \int_t^{t+T} \int_{\Omega_1} a(x) |u_t|^{r+1} \left[|\nabla u| + |u| \right] dx\, ds \\
&\leq C \left(\int_t^{t+T} \int_{\Omega_1} a(x) |u_t|^{r+2}\, dx\, ds \right)^{\frac{r+1}{r+2}} \left(\int_t^{t+T} \int_{\Omega} \left(|\nabla u|^{r+2} + |u|^{r+2} \right) dx\, ds \right)^{\frac{1}{r+2}} \\
&\leq C \left(\int_t^{t+T} \int_{\Omega_1} \rho(x, u_t) \cdot u_t\, dx\, ds \right)^{\frac{r+1}{r+2}} \left(\int_t^{t+T} \int_{\Omega} |\nabla u|^{r+2}\, dx\, ds \right)^{\frac{1}{r+2}} \\
&\leq C\, (\Delta E)^{\frac{r+1}{r+2}} \sqrt{E(t)}
\end{aligned}$$

Case 3: Estimating I_2 for $0 \leq p \leq \frac{2}{N-2}$, $N \geq 3$ (or $p \geq 0$ if $N = 2$):

$$\begin{aligned}
I_2 &\leq 2K_1 \int_t^{t+T} \int_{\Omega_2} a(x) |u_t|^{p+1} \big(|\nabla u| + |u| \big)\, dx\, ds \\
&\leq C \left(\int_t^{t+T} \int_{\Omega_2} a(x)^{\frac{p+2}{p+1}} |u_t|^{p+2}\, dx\, ds \right)^{\frac{p+1}{p+2}} \left(\int_t^{t+T} \int_{\Omega_2} \left(|\nabla u| + |u|^{p+2} \right) dx\, ds \right)^{\frac{1}{p+2}} \\
&\leq C \left(\int_t^{t+T} \int_{\Omega_2} a(x)\, |u_t|^{p+2}\, dx\, ds \right)^{\frac{p+1}{p+2}} \left(\int_t^{t+T} \int_{\Omega} |\nabla u|^{p+2}\, dx\, ds \right)^{\frac{1}{p+2}}
\end{aligned}$$

where we have used Poincaré's inequality in $W_o^{1,p+2}(\Omega)^N$, Hölder's inequality and the hypothesis in (H2)(d) on the boundedness of $a(x)$.

Now, the hypothesis H2(d), identity (2.2) in Remark 2 and estimate (5.15) imply

$$\begin{aligned}
I_2 &\leq C \left(\int_t^{t+T} \int_{\Omega_2} \rho(x, u_t) \cdot u_t\, dx\, ds \right)^{\frac{p+1}{p+2}} E(t)^{\frac{1-\theta}{2}} \\
&\leq C\, (\Delta E)^{\frac{p+1}{p+2}} E(t)^{\frac{4-(N-2)p}{4(p+2)}}
\end{aligned}$$

Case 4a: Estimating I_2 for $-1 < p < 0$, $N \geq 3$:

$$
\begin{aligned}
I_2 &\leq K_1 \int_t^{t+T} \int_{\Omega_2} a(x)[|u_t|^{p+1} + |u_t|]\,(|\nabla u| + |u|)\,dx\,ds \\
&\leq C \left(\int_t^{t+T} \int_{\Omega_2} a(x)|u_t|^2\,dx\,ds \right)^{\frac{1}{2}} \left(\int_t^{t+T} \int_{\Omega_2} |\nabla u|^2\,dx\,ds \right)^{\frac{1}{2}} \qquad (5.16) \\
&\leq C \left(\int_t^{t+T} \int_{\Omega_2} a(x)|u_t|^2\,dx\,ds \right)^{\frac{1}{2}} \sqrt{T}\,\sqrt{E(t)}
\end{aligned}
$$

Thus,

$$
I_2 \leq C \left(\int_t^{t+T} \int_{\Omega_2} a(x)|u_t|^{\lambda\, l'}\,dx\,ds \right)^{\frac{1}{2\,l'}} \left(\int_t^{t+T} \int_{\Omega_2} |u_t|^{(2-\lambda)\,l}\,dx\,ds \right)^{\frac{1}{2\,l}} \sqrt{E(t)}
$$

where l' is the conjugate exponent of l.

Now, choosing

$$
\lambda = \frac{4(p+2)}{4 + p(2-N)} \quad \text{and} \quad l = \frac{2N}{(N-2)(2-\lambda)},
$$

we have $l' = \dfrac{p+2}{\lambda}$ and

$$
\begin{aligned}
I_2 &\leq C \left(\int_t^{t+T} \int_{\Omega_2} a(x)|u_t|^{p+2}\,dx\,ds \right)^{\frac{2}{4+p(2-N)}} \left(\int_t^{t+T} \int_{\Omega_2} |u_t|^{\frac{2N}{N-2}}\,dx\,ds \right)^{\frac{p(2-N)}{2(4+p(2-N))}} \sqrt{E(t)} \\
&\leq C \left(\int_t^{t+T} \int_{\Omega_2} \rho(x, u_t)\,u_t\,dx\,ds \right)^{\frac{2}{4+p(2-N)}} \sqrt{E(t)} \\
&\leq C\,(\Delta E)^{\frac{2}{4+p(2-N)}}\,\sqrt{E(t)}
\end{aligned}
$$

since $u_t \in L^\infty\big([0,\infty);\ H_0^1(\Omega)^N\big) \hookrightarrow L^\infty\big([0,\infty);\ L^{\frac{2N}{N-2}}(\Omega)^N\big)$ due to Lemma 3.4 and Sobolev's inequality. The positive constant C depends also on the initial data.

Case 4b: Estimating I_2 for $-1 < p < 0$, $N = 2$:

$$
I_2 \leq C \int_t^{t+T} \int_{\Omega_2} |u_t|^2\,(|\nabla u| + |u|)\,dx\,ds.
$$

Then, using Poincaré's inequality for $u \in W_0^{1,\frac{p+2}{p+1}}(\Omega)$ and the hypotheses on the function $\rho(x, s)$, it follows that

$$\begin{aligned} I_2 &\leq C \left(\int_t^{t+T} \int_\Omega |u_t|^{p+2}\, dx\, ds\right)^{\frac{1}{p+2}} \left(\int_t^{t+T} \int_\Omega [|\nabla u|^{\frac{p+2}{p+1}} + |u|^{\frac{p+2}{p+1}}]\, dx\, ds\right)^{\frac{p+1}{p+2}} \\ &\leq C \left(\int_t^{t+T} \int_\Omega \rho(x, u_t) \cdot u_t\, dx\, ds\right)^{\frac{1}{p+2}} \left(\int_t^{t+T} \int_\Omega |\nabla u|^{\frac{p+2}{p+1}}\, dx\, ds\right)^{\frac{p+1}{p+2}}. \end{aligned}$$

Now, using the inequality of Gagliardo-Nirenberg (Lemma 5.2) with $\theta = \frac{-p}{p+2}$, it follows that

$$||\nabla u||_{L^{\frac{p+2}{p+1}}} \leq C\, ||\nabla u||_{H^1}^{\theta}\, ||\nabla u||_{L^2}^{1-\theta} \leq C||\Delta u||_{L^2}^{\theta}\, E(t)^{\frac{1-\theta}{2}}.$$

Then, by Remark 2 and the boundedness of Δu (proved in a previous section), we obtain

$$I_2 \leq C\, (\Delta E)^{\frac{1}{p+2}}\, E(t)^{\frac{p+1}{p+2}}. \qquad \square$$

6 Main estimates for stabilization

Using Young's inequality and Lemmas 5.5, 5.6, we obtain the next result.

Proposition 6.1. *The energy for the solution of problem* (1.1)–(1.4) *satisfies*

$$\begin{aligned} E(t) \leq C\Bigg\{ & D_i(t)^2 + \int_{Q_\omega(t)} (|u|^2 + |u_t|^2)\, dx\, ds \\ & + \int_{Q_\omega(t)} |\nabla\theta|\, (|\nabla u| + |u|)\, dx\, ds \Bigg\}, \end{aligned} \tag{6.1}$$

$i = 1, 2, 3, 4$, *where*

$$\begin{aligned} & D_1(t)^2 = \Delta E + (\Delta E)^{\frac{2}{r+2}} + (\Delta E)^{\frac{4(p+1)}{4+p(N+2)}} \quad \textit{if} \;\; r \geq 0 \quad \textit{and} \\ & \quad 0 \leq p \leq \frac{2}{N-2} (0 \leq p < \infty,\ \textit{if}\, N = 2). \\ & D_2(t)^2 = \Delta E + (\Delta E)^{\frac{2}{r+2}} + (\Delta E)^{\frac{4}{4+p(2-N)}} \quad \textit{for the case} \\ & \quad r \geq 0,\ -1 < p < 0 \quad \textit{and} \quad N \geq 3. \end{aligned}$$

If $r \geq 0,\ -1 < p < 0$ *and* $N = 2$ *then* $D_2(t)^2 = \Delta E + (\Delta E)^{\frac{2}{r+2}}$.

$$D_3(t)^2 = \Delta E + (\Delta E)^{\frac{2(r+1)}{r+2}} + (\Delta E)^{\frac{4(p+1)}{4+p(N+2)}} \quad \text{for the case}$$

$$-1 < r < 0 \quad \text{and} \quad 0 \leq p \leq \frac{2}{N-2} \quad \text{(the same estimate holds if}$$

$$p \geq 0, -1 < r < 0 \quad \text{and} \quad N = 2).$$

$$D_4(t)^2 = \Delta E + (\Delta E)^{\frac{2(r+1)}{r+2}} + (\Delta E)^{\frac{4}{4+p(2-N)}} \quad \text{for the case}$$

$$-1 < r < 0,\ -1 < p < 0 \quad \text{and} \quad N \geq 3.$$

If $-1 < r < 0,\ -1 < p < 0$ *and* $N = 2$, *then* $D_4(t)^2 = \Delta E + (\Delta E)^{\frac{2(r+1)}{r+2}}$.

Now, using Poincare's and Young's inequalities, we obtain $\left(\text{with } \epsilon_0 = \frac{a^2}{4\,C\,(1+C_\Omega)}\right)$

$$\begin{aligned}
&\int_{Q_\omega(t)} |\nabla\theta|\ (|\nabla u| + |u|)\ dx\ ds \\
&\quad \leq \frac{1}{2\ \epsilon_0} \int_{Q_\omega(t)} |\nabla\theta|^2\ dx\ ds + \epsilon_0 \int_{Q_\omega(t)} (1 + C_\Omega)\ |\nabla u|^2\ dx\ ds \\
&\quad \leq 2\ C\ \frac{1 + C_\Omega}{a^2}\ \Delta E + \frac{1}{2\ C}\ E(t).
\end{aligned}$$

In the last inequality we have used the identity mentioned in Remark 2.

Using the above estimate in (6.1), we obtain

$$E(t) \leq C \left\{ D_i(t)^2 + \Delta E + \int_{Q_\omega(t)} (|u|^2 + |u_t|^2)\ dx\ ds \right\}. \tag{6.2}$$

Thus, the natural dissipation of the system reduced the proof of the theorem of stabilization to the following estimate.

Proposition 6.2. *Let* $R > 0$ *fixed and* $\{u,\ \theta\}$ *be the solution of the problem* (1.1)–(1.4). *Let* u_o, u_1 *and* θ_o *be such that* $E\,(0) \leq R$. *Then, there exists a constant* $C > 0$ *such that*

$$\int_t^{t+T} \int_\Omega |u|^2\ dx\ ds \leq C \left\{ D_i(t)^2 + \int_{Q_\omega(t)} |u_t|^2\ dx\ ds \right\}, \tag{6.3}$$

where $T > 0$ *is given by Lemma* 5.5, *the constant* C *depends on* R. *Here* $i = 1, 2, 3, 4$ *according to the cases previously described.*

Proof. We present the proof by contradiction for $N \geq 3$ (the proof for $N = 2$ is easier). Suppose there exists a sequence of solutions $\left\{(u^n)_{n\in\mathbb{N}}, (\theta^n)_{n\in\mathbb{N}}\right\}$ with the following corresponding initial data $\left\{(u_0^n)_{n\in\mathbb{N}}, (u_1^n)_{n\in\mathbb{N}}, (\theta_0^n)_{n\in\mathbb{N}}\right\}$ and a sequence $(t_n)_{n\in\mathbb{N}} \in \mathbb{R}^+$ such that

$$\lim_{n\longrightarrow\infty} \frac{\int_{t_n}^{t_n+T}\int_\omega |u^n|^2\,dxds}{D_i\,(t_n)^2 + \int_{t_n}^{t_n+T}\int_\omega |u_t^n|^2\,dxds} = \infty. \tag{6.4}$$

Let

$$\lambda_n^2 = \int_{t_n}^{t_n+T}\int_\omega |u^n|^2\,dxds \tag{6.5}$$

and

$$I_n(t_n) = \frac{1}{\lambda_n^2}\left[D_i\,(t_n)^2 + \int_{t_n}^{t_n+T}\int_\omega |u_t^n|^2\,dxds\right]. \tag{6.6}$$

Then, from (6.4), we obtain

$$\lim_{n\longrightarrow\infty} I_n(t_n) = 0. \tag{6.7}$$

Let $v^n(x,t) = \frac{u^n(x,t+t_n)}{\lambda_n}$ and $\eta^n(x,t) = \frac{\theta^n(x,t+t_n)}{\lambda_n}$, $0 \leq t \leq T$. Furthermore, from (6.5), we have that

$$\frac{1}{\lambda_n^2}\int_{t_n}^{t_n+T}\int_\omega |u^n|^2\,dxds = 1, \tag{6.8}$$

for all $n \in \mathbb{N}$.

The estimate (6.1) together com (6.7) and (6.8) imply that

$$\begin{aligned}
E(v^n(t),\eta^n(t)) &= E\left(\frac{u^n(x,t+t_n)}{\lambda_n}, \frac{\theta^n(x,t+t_n)}{\lambda_n}\right)\\
&= \frac{1}{\lambda_n^2}E\big(u^n(x,t+t_n),\theta^n(x,t+t_n)\big)\\
&\leq \frac{1}{\lambda_n^2}E\big(u^n(t_n),\theta^n(t_n)\big)\\
&\leq \frac{C}{\lambda_n^2}\left\{D_i\,(t_n)^2 + \int_{t_n}^{t_n+T}\int_\omega |u_t^n|^2 + |u^n|^2\right\}\\
&= \frac{C}{\lambda_n^2}\left\{D_i\,(t_n)^2 + \int_{t_n}^{t_n+T}\int_\omega |u_t^n|^2 dxds\right\} + C\\
&= C\,I_n(t_n) + C.
\end{aligned}$$

But, $I_n(t_n)$ is bounded due to (6.7). Therefore, we obtain that

$$E(v^n(t), \eta^n(t)) \leq C, \quad \text{i.e.}$$

$$\int_\Omega |v_t^n|^2 + a^2\, |\nabla v^n|^2 + \left(b^2 - a^2\right)\, \left(\operatorname{div} v^n\right)^2 + \left|\eta^n\right|^2 dx \leq C,$$

for all $0 \leq t \leq T$ and for all $n \in \mathbb{N}$, where $C > 0$ does not depend on t and n. Therefore,

$$\|v_t^n(t)\|_{L^2(\Omega)} \leq C, \quad \|\nabla v^n(t)\|_{L^2(\Omega)} \leq C \quad \text{and} \quad \|\eta^n(t)\|_{L^2(\Omega)} \leq C, \tag{6.9}$$

for all $0 \leq t \leq T$ and for all $n \in \mathbb{N}$. However, from Poincaré's inequality and from the estimate (6.9), it follows that

$$\begin{aligned} \|v^n(t)\|^2_{[L^2(\Omega)]^N} &= \int_\Omega |v^n(x,t)|^2\, dx = \int_\Omega \frac{1}{\lambda_n^2} |u^n(x, t+t_n)|^2\, dx \\ &\leq C_1 \int_\Omega \frac{1}{\lambda_n^2} |\nabla u^n(x, t+t_n)|^2\, dx = C_2 \int_\Omega |\nabla v^n(x,t)|^2\, dx \leq C, \end{aligned}$$

for all $0 \leq t \leq T$ and for all $n \in \mathbb{N}$.

Thus, there exists a constant $C > 0$ such that

$$\int_\Omega |v^n(x,t)|^2\, dx \leq C, \tag{6.10}$$

for all $0 \leq t \leq T$ and for all $n \in \mathbb{N}$.

We conclude that

$$\begin{aligned} &v^n \text{ is bounded in } \; W^{1,\infty}(0, T; [L^2(\Omega)]^N) \cap L^\infty(0, T; [H_0^1(\Omega)]^N), \\ &\eta^n \text{ is bounded in } L^\infty(0, T; L^2(\Omega)), \end{aligned} \tag{6.11}$$

for all $n \in \mathbb{N}$.

Now, we claim that

$$\lim_{n \longrightarrow \infty} \frac{1}{\lambda_n} \rho\left(x, u_t^n\,(t+t_n)\right) = 0 \;\text{ in }\; L^1((0,T) \times \Omega). \tag{6.12}$$

First we prove this claim for the case where $r \geq 0$ and $0 \leq p \leq \frac{2}{N-2}$.

For the case $r \geq 0$ and $0 \leq p \leq \frac{2}{N-2}$, proceeding as we did previously when we defined regions $\Omega_1(t)$, $\Omega_2(t)$ to estimate I_1 and I_2, we obtain

$$\int_{tn}^{tn+T} \int_{\Omega} |\rho(x, u_t^n)| \, dx \, ds \leq C \left\{ (\Delta E)^{\frac{1}{r+2}} + (\Delta E)^{\frac{p+1}{p+2}} \right\}, \tag{6.13}$$

where C depends on T, $|\Omega|$ and $||a||_\infty$. Now, using the definition of $D_1(t)$

$$\int_{tn}^{tn+T} \int_{\Omega} |\rho(x, u_t^n)| \, dx \, ds \leq C \; \left\{ D_1\,(t) + D_1\,(t)^{\frac{4+p(N+2)}{2(p+2)}} \right\}.$$

By (6.6),

$$\frac{1}{\lambda_n} \int_{tn}^{tn+T} \int_{\Omega} |\rho(x, u_t^n)| \, dx \, ds \leq C \left\{ I_n(t_n) + \lambda_n^{\frac{Np}{2(p+2)}} \; I_n(t_n)^{\frac{4+p(N+2)}{2(p+2)}} \right\}$$

where

$$I_n^2\,(t_n) = \frac{1}{\lambda_n^2} \left\{ D_i\,(t_n)^2 + \int_{t_n}^{t_n+T} |u_t^n|^2 \, dxds \right\}.$$

However, $\{\lambda_n\}_{n \geq 1}$ is a bounded sequence:

$$\begin{aligned} \lambda_n &= \left\{ \int_{t_n}^{t_n+T} \int_{\omega} |u^n|^2 \, dxds \right\}^{1/2} \\ &\leq \left\{ C_2 \int_{t_n}^{t_n+T} E\left(u^n\,(s)\right) \, ds \right\}^{1/2} \\ &\leq \left\{ C_2 \; T \; E\big(u_n(0), \theta_n\,(0)\big) \right\}^{1/2} \leq C = C(R). \end{aligned}$$

Hence, (6.7) and the definition of I_n imply that

$$\begin{aligned} &\frac{1}{\lambda_n} \int_{tn}^{tn+T} \int_{\Omega} |\rho(x, u_t^n)| \, dx \, ds \\ &\leq C \left\{ I_n(t_n) + \lambda_n^{\frac{N\,p}{2(p+2)}} \; I_n(t_n)^{\frac{4+p(N+2)}{2(p+2)}} \right\} \to 0, \end{aligned} \tag{6.14}$$

as m goes to ∞.

The remaining cases are treated similarly. We have proved that

$$\frac{1}{\lambda_n} \rho\big(x, u_t^n(t + t_n)\big) \to 0 \quad \text{in} \quad L^1\big([0, T] \times \Omega\big). \tag{6.15}$$

Thefore, (6.12) is proved.

At this point, we shall prove that

$$\lim_{n\longrightarrow\infty}\frac{1}{\lambda_n}\big(\nabla\theta^n(t+t_n)\big)=0 \ \text{ in } \ L^1\big((0,T)\times\Omega\big). \tag{6.16}$$

To prove (6.16), we use Cauchy-Schwarz's inequality to obtain

$$\begin{aligned}\int_0^T\int_\Omega\left|\frac{\nabla\theta^n(t+t_n)}{\lambda_n}\right|dxdt &\le C\sqrt{T}\left[\frac{1}{\lambda_n^2}\int_0^T\int_\Omega\left|\nabla\theta^n(t+t_n,x)\right|^2dxdt\right]^{\frac{1}{2}}\\ &\le C\sqrt{T}\left[\frac{1}{\lambda_n^2}\int_{t_n}^{t_n+T}\int_\Omega\left|\nabla\theta^n(s,x)\right|^2dxds\right]^{\frac{1}{2}}\\ &\le C\sqrt{T}\left[\frac{1}{\lambda_n^2}D_i\,(t_n)^2\right]^{\frac{1}{2}}\\ &\le C\sqrt{I_n}\end{aligned} \tag{6.17}$$

Then,

$$\frac{1}{\lambda_n}\int_0^T\int_\Omega|\nabla\theta^n(t+t_n)|\,dxds\le C\sqrt{I_n(t_n)}\longrightarrow 0 \tag{6.18}$$

when $n\longrightarrow\infty$.

Then, (6.16) holds.

Also, we need to show that

$$\int_0^T\int_\omega|v_t|^2\,dxds=0. \tag{6.19}$$

From (6.7) we have that

$$I_n(t_n)=\frac{1}{\lambda_n^2}\left[D_i\,(t_n)^2+\int_{t_n}^{t_n+T}\int_\omega|u_t^n|^2\,dxds\right]\longrightarrow 0,$$

then

$$\begin{aligned}0=\lim_{n\longrightarrow\infty}\int_{t_n}^{t_n+T}\int_\omega\left|\frac{u_t^n(s)}{\lambda_n}\right|^2dxds &=\lim_{n\longrightarrow\infty}\int_0^T\int_\omega\left|\frac{u_t^n(t+t_n)}{\lambda_n}\right|^2dxds\\ &=\lim_{n\longrightarrow\infty}\int_0^T\int_\omega|v_t^n(t)|^2\,dxdt\\ &=\int_0^T\int_\omega|v_t(t)|^2\,dxdt,\end{aligned}$$

since $v^n\longrightarrow v$ weak-$\star$ in $L^2\big(0,T;[L^2(\Omega)]^N\big)$.

So, (6.19) is proved.

Finally, we prove that

$$\lim_{n\longrightarrow\infty}\int_0^T\int_\omega \nabla\eta^n\big[|v^n(t)|+|\nabla v^n(t)|\big]\,dxdt \;=\; 0. \tag{6.20}$$

From Cauchy-Schwarz's inequality, we have that

$$\begin{aligned}
&\int_0^T\int_\omega \nabla\eta^n\big[|v^n(t)|+|\nabla v^n(t)|\big]\,dxdt\\
&\le \bigg(\int_0^T\int_\omega |\nabla\eta^n|^2\,dxdt\bigg)^{\frac{1}{2}}\bigg(\int_0^T\int_\omega |\nabla v^n|^2 \;+\; |v^n|^2\,dxdt\bigg)^{\frac{1}{2}}\\
&\le\; C\bigg(\frac{1}{\lambda_n^2}\int_{t_n}^{t_n+T}\int_\Omega |\nabla\theta^n|^2\,dxds\bigg)^{\frac{1}{2}}\bigg(\frac{1}{\lambda_n^2}\int_{t_n}^{t_n+T}\int_\Omega |\nabla u^n|^2\,dxdt\bigg)^{\frac{1}{2}}\\
&\le\; C\bigg(\frac{1}{\lambda_n^2}D_i\,(t_n)^2\bigg)^{\frac{1}{2}}\bigg(\int_0^T\int_\Omega |\nabla v^n|^2\,dxdt\bigg)^{\frac{1}{2}}\\
&\le\; C\sqrt{T}\sqrt{I_n}\;\longrightarrow\; 0.
\end{aligned}$$

Therefore, (6.20) holds.

Now, we pass to the limit of $\big\{(v^n(t))_{n\in\mathbb{N}},\,(\eta^n(t))_{n\in\mathbb{N}}\big\}$. From (6.11) it follows that there are function $v(t)$, $\eta(t)$ and subsequences of the sequences v^n and η^n, which we continue to represent by v^n and η^n such that

$$\begin{aligned}
&v^n(t) \rightharpoonup v(t) \quad \text{weak} \star \text{ in } \; W^{1,\infty}\big(0,T;[L^2(\Omega)]^N\big)\cap L^\infty\big(0,T;[H_0^1(\Omega)]^N\big)\\
&\eta^n(t) \rightharpoonup \eta(t) \quad \text{weak} \star \text{ in } \; L^\infty\big(0,T;L^2(\Omega)\big).
\end{aligned}$$

Therefore, the functions $v(t)$ and $\eta(t)$ satisfy:

i) $v \in W^{1,\infty}(0,T;[L^2(\Omega)]^N)\cap L^\infty(0,T;[H_0^1(\Omega)]^N)$ and $\eta \in L^\infty(0,T; L^2(\Omega))$;

ii) $v_{tt} - a^2\,\Delta v - \big(b^2 - a^2\big)\,\nabla\,(\text{div}\,v) = 0$,

iii) $\eta_t - \Delta\eta + \text{div}\,v_t = 0$,

iv) $\displaystyle\int_0^T\int_\omega |v_t|^2\,dxds = 0$,

v) $\int_0^T \int_\omega |v|^2 dx ds = 1.$

We observe that $\{\eta_t,\ v_t\}$ is also solution of equations ii), iii). Since $v_t = 0$ a.e. in $\omega \times (0, T)$ and $(v_t)'' - a^2\ \Delta\,(v_t) - \left(b^2 - a^2\right)\ \nabla\,(\mathrm{div}\ v_t) = 0$ with homogeneous Dirichlet boundary conditions, it follows from a consequence of Holmgren's Uniqueness Theorem ([11], p. 88) that $v_t = 0$ in $\Omega \times (0, T)$, therefore $v\,(x, t) = F\,(x)$ a.e. in $\Omega \times (0, T)$. Thus, v satisfies $-a^2\ \Delta v - \left(b^2 - a^2\right)\ \nabla \mathrm{div}\ v = 0$ with homogeneous Dirichlet boundary conditions. From the uniqueness of this Dirichlet problem, it results that $v = 0$ a.e. in $(0, T) \times \Omega$. This contradicts the item (v). □

The following estimate is a consequence of the previous proposition.

Proposition 6.3.

$$E(t) \leq C \left\{ D_i(t)^2 + \int_{Q_\omega(t)} |u_t|^2\ dx\, ds \right\} \tag{6.21}$$

where $D_i(t)$ $(i = 1, 2, 3, 4)$ *are given in Proposition* 6.1.

7 Proof of the theorem of stabilization

Now, it remains to estimate the integral $\int_{Q_\omega(t)} |u_t|^2\ dx\, ds$ in terms of ΔE (energy difference).

Proof. Case a: $r \geq 0$ and $0 \leq p \leq \frac{2}{N-2}$ if $N \geq 3$ $(r \geq 0,\ p \geq 0$ if $N = 2)$:

Using the hypothesis on ω in (H2)(d), Hölder's inequality and the definitions of $\Omega_1,\ \Omega_2$, we obtain

$$\begin{aligned}
\int_{Q_\omega(t)} |u_t|^2\ dx\, ds &\leq C \int_t^{t+T} \int_\Omega a(x)|u_t|^2\ dx\, ds \\
&\leq C \left\{ \left[\int_t^{t+T} \int_{\Omega_1} a(x)|u_t|^{r+2}\ dx\, ds \right]^{\frac{2}{r+2}} + \int_t^{t+T} \int_{\Omega_2} a(x)|u_t|^{p+2}\ dx\, ds \right\} \\
&\leq C \left\{ \left[\int_t^{t+T} \int_{\Omega_1} \rho(x, u_t) \cdot u_t\ dx\, ds \right]^{\frac{2}{r+2}} + \int_t^{t+T} \int_{\Omega_2} \rho(x, u_t) \cdot u_t\ dx\, ds \right\}
\end{aligned}$$

where the last C depends on $|\Omega|$, T and $||a||_\infty$.

Then, due to Remark 2, we have

$$\int_{Q_\omega(t)} |u_t|^2\, dx\, ds \le C\left\{\Delta E + (\Delta E)^{\frac{2}{r+2}}\right\} \tag{7.1}$$

From estimates (6.21) and (7.1), and the expression for $D_1(t)$ we obtain

$$E(t) \le C\left\{\Delta E + (\Delta E)^{\frac{2}{r+2}} + (\Delta E)^{\frac{4(p+1)}{4+p(N+2)}}\right\}$$

Then, using the fact that $E(t)$ is nonincreasing function of t, we obtain that

$$E(t) \le C\,(\Delta E)^{K_1}$$

where

$$K_1 = \min\left\{\frac{2}{r+2},\ \frac{4(p+1)}{4+p\,(N+2)}\right\} \tag{7.2}$$

is such that $0 < K_1 \le 1$ and C is a positive constant which depends on the initial data.

We have obtained the following inequality

$$\sup_{t\le s\le t+T} E(s)^{\frac{1}{K_1}} \le E(t)^{\frac{1}{K_1}} \le C\Delta E \tag{7.3}$$

If we set $1+\gamma = \frac{1}{K_1}$, then $\gamma = \frac{1-K_1}{K_1}$ and applying Nakao's Lemma to (7.3) we obtain for $N \ge 3$ that

$$E(t) \le C_1(1+t)^{-\gamma_1} \tag{7.4}$$

with

$$\gamma_1 = \min\left\{\frac{2}{r},\ \frac{4(p+1)}{p\,(N-2)}\right\}, \quad \text{if} \quad r > 0,\ p > 0.$$

We see from (7.2) that if $r = p = 0$, then according to Nakao's Lemma, the decay rate is exponential. If $r = 0,\ p > 0$, then the decay rate depends only on $p : \gamma_1 = \frac{4(p+1)}{p(N-2)}$; and if $r > 0,\ p = 0$, then the decay rate depends only on $r : \gamma_1 = \frac{2}{r}$.

If $N = 2$, $r > 0$ and $p \ge 0$ then $\gamma_1 = \frac{2}{r}$. If $N = 2$, $r = 0$ and $p \ge 0$ then the decay rate is exponential.

Case b: $r \geq 0$ and $-1 < p < 0$ and $N \geq 2$:

In this case we have

$$\int_t^{t+T} \int_{\Omega_2} a(x)|u_t|^2 \, dx \, ds \leq C_1 \, (\Delta E)^{\frac{4}{4+p(2-N)}} .$$

Then, in the same way as in case (a), we get

$$\int_{Q_\omega(t)} |u_t|^2 \, dx \, ds \leq C_1 \left\{ (\Delta E)^{\frac{2}{r+2}} + (\Delta E)^{\frac{4}{4+p(2-N)}} \right\}. \tag{7.5}$$

Now, assuming that $N \geq 3$ and considering (6.21), (7.5) and the definition of $D_2(t)$ we have

$$E(t) \leq C \left\{ \Delta E + (\Delta E)^{\frac{2}{r+2}} + (\Delta E)^{\frac{4}{4+p(2-N)}} \right\}.$$

If we define

$$K_2 = \min \left\{ \frac{2}{r+2}, \; \frac{4}{4+p(2-N)} \right\}, \tag{7.6}$$

then we have $0 < K_2 < 1$ and

$$E(t) \leq C \, (\Delta E)^{K_2}.$$

Similarly to the case (a), using Nakao's Lemma, we obtain

$$E(t) \leq C(1+t)^{-\gamma_2}$$

with

$$\gamma_2 \; = \; \min \left\{ \frac{2}{r}, \; \frac{4}{p(2-N)} \right\} \quad \text{if} \quad r > 0.$$

From (7.6), it follows that if $r = 0$ then $\gamma_2 = \dfrac{4}{p(2-N)}$.

When $N = 2$, (6.21), (7.5) and the definition of $D_2(t)$ imply that

$$E(t) \leq C \left\{ \Delta E + (\Delta E)^{\frac{2}{r+2}} \right\}.$$

If $r = 0$, then $E(t) \leq C \; \Delta E$. Thus, by Nakao's Lemma, we obtain exponential decay of the energy.

If $r > 0$, we obtain

$$E(t) \leq C\ (\Delta E)^{K_2}$$

with

$$K_2 = \frac{2}{r+2}, \qquad 0 < K_2 < 1. \tag{7.7}$$

Applying Nakao's Lemma, we obtain

$$E(t) \leq C(1+t)^{-\gamma_2}$$

with $\gamma_2 = \dfrac{2}{r}$.

Case c: $-1 < r < 0$ and $0 \leq p \leq \frac{2}{N-2}$ $(-1 < r < 0,\ p \geq 0$ if $N = 2)$:

We have

$$\begin{aligned}\int_t^{t+T}\int_{\Omega_1} a(x)|u_t|^2\,dx\,ds &\leq C\int_t^{t+T}\int_{\Omega_1} a(x)|u_t|^{r+2}\,dx\,ds\\ &\leq C\int_t^{t+T}\int_{\Omega_1} \rho(x,u_t)\cdot u_t\,dx\,ds \leq C\Delta E\end{aligned}$$

and it follows that

$$\int_{Q_\omega(t)} |u_t|^2\,dx\,ds \leq C\Delta E \tag{7.8}$$

Thus, from (6.21), (7.8) and the definition of $D_3(t)$, we get

$$E(t) \leq C\left\{\Delta E + (\Delta E)^{\frac{2(r+1)}{r+2}} + (\Delta E)^{\frac{4(p+1)}{4+p(N+2)}}\right\}. \tag{7.9}$$

Therefore

$$E(t)^{\frac{1}{K_3}} \leq C\Delta E$$

with

$$K_3 = \min\left\{\frac{2(r+1)}{r+2},\ \frac{4(p+1)}{p\,(N+2)+4}\right\} \tag{7.10}$$

is such that $0 < K_3 < 1$.

We conclude by Nakao's Lemma,

$$E(t) \leq C(1+t)^{-\gamma_3}$$

with

$$\gamma_3 = \min\left\{\frac{-2(r+1)}{r}, \frac{4(p+1)}{p(N-2)}\right\} \quad \text{if} \quad 0 < p \leq \frac{2}{N-2} \quad \text{and} \quad N \geq 3.$$

From (7.10) it follows that $\gamma_3 = \dfrac{-2(r+1)}{r}$ if $p = 0$ and $N \geq 3$ or $N = 2$ and $p \geq 0$.

Case d: $-1 < r < 0$, $-1 < p < 0$ and $N \geq 2$:

We have

$$\int_{Q_\omega(t)} |u_t|^2 \, dx\, ds \leq C \left\{\Delta E + (\Delta E)^{\frac{4}{4+p(2-N)}}\right\}.$$

Assuming that $N \geq 3$, it follows from (6.21), the definition of $D_4(t)$ and the previous inequality that

$$E(t) \leq C \left\{\Delta E + (\Delta E)^{\frac{2(r+1)}{r+2}} + (\Delta E)^{\frac{4}{4+p(2-N)}}\right\}.$$

Thus, we get

$$E(t)^{\frac{1}{K_4}} \leq C \Delta E = C\, [E(t) - E(t+T)],\ t \geq 0,$$

with

$$K_4 = \min\left\{\frac{2(r+1)}{r+2}, \frac{4}{4+p(2-N)}\right\}.$$

Then, using Nakao's Lemma, we obtain

$$E(t) \leq C(1+t)^{-\gamma_4},\ t \geq 0$$

with

$$\gamma_4 = \min\left\{\frac{-2(r+1)}{r}, \frac{4}{p(2-N)}\right\}.$$

Now we consider $N = 2$. It follows from (6.21), the definition of $D_4(t)$ and the inequality at the beginning of this case that

$$E(t) \leq C \left\{\Delta E + (\Delta E)^{\frac{2(r+1)}{r+2}}\right\}.$$

Thus,

$$E(t)^{\frac{1}{K_4}} \leq C \Delta E = C\, [E(t) - E(t+T)],\ t \geq 0,$$

with $K_4 = \dfrac{2(r+1)}{r+2}$.

Then, using Nakao's Lemma, we obtain

$$E(t) \leq C(1+t)^{-\gamma_4}, \ t \geq 0$$

with $\gamma_4 = \dfrac{2(r+1)}{-r}$.

Now, the proof of the Theorem 3.7 is complete. □

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