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Some derivative free quadratic and cubic convergence iterative formulas for solving nonlinear equations

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Abstract. Finding the zeros of a nonlinear equation $f(x) = 0$, is a classical problem which has nice applications in various branches of science and engineering. In this paper, we introduce four iterative methods which is based on the central-difference and forward-difference approximations to derivatives. It is proved that three of the four methods have cubic convergence and another method has quadratic convergence. The best property of these methods are that do not need to calculate any derivative. In order to demonstrate convergence properties of the introduced methods, several numerical examples are given.

Mathematical subject classification: 65H05; 41A25.

Key words: Newton's method, cubic convergence, quadratic convergence, nonlinear equations, iterative method.

1 Introduction

A large number of papers have been written about iterative methods for the solution of the the nonlinear equations [3, 7, 8, 9, 10, 12, 13]. In this paper, we consider the problem of finding a simple root x^* of a function $f : D \subset R \to R$ i.e., $f(x^*) = 0$ and $f'(x^*) \neq 0$. The famous Newton's method for finding x^* uses the iterative method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},$$

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starting from some initial value x_0. The Newton's method is an important and basic method where converges quadratically in some neighborhood of simple root x^*. Chun [5] obtained the iterative method with convergence cubically given by

$$x_{n+1} = x_n - \frac{f(z_{n+1}) - f(x_n)}{f'(x_n)}, \quad \text{where} \quad z_{n+1} = x_n + \frac{f(x_n)}{f'(x_n)}. \tag{1.1}$$

Also there is an modification of the Newton's method with third-order convergence as [14]

$$x_{n+1} = x_n - \frac{f(y_{n+1}) + f(x_n)}{f'(x_n)}, \quad \text{where} \quad y_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \tag{1.2}$$

It is well-known that the forward-difference approximation for $f'(x)$ at x is

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}.$$

If the derivative $f'(x_n)$ is replaced by the forward-difference approximation with $h = f(x_n)$ i.e.

$$f'(x_n) \approx \frac{f(x_n + f(x_n)) - f(x_n)}{f(x_n)},$$

the Newton's method becomes

$$x_{n+1} = x_n - \frac{(f(x_n))^2}{f(x_n + f(x_n)) - f(x_n)},$$

which is the famous Steffensen's method [11]. The Steffensen's method is based on forward-difference approximation to derivative. This method is a tough competitor of Newton's method. Both the methods are of quadratic convergence, both require two functions evaluation per iteration but in contrast to Newton's method, Steffensen's method is derivative free. Chen [4] studied a particular class of these methods which contain the Steffensen's method as a special case. In [1], a modified Steffensen's method for the numerical solution of the system of nonlinear equations is studied. Amat et al. [2] considered a class of the generalized Steffensen iterations procedures for solving nonlinear equations on Banach space without any derivative.

Sometimes the applications of the iterative methods depending on derivatives are restricted in engineering. In this paper we introduce some methods which are based on the approximations to the derivative $f'(x_n)$ in each iteration. These methods are based on the central-difference and forward-difference approximations to the derivatives, respectively. The central-difference approximation for $f'(x)$ at x is

$$f'(x) \approx \frac{f(x+f(x)) - f(x-f(x))}{2f(x)}.$$

We know that the leading errors in forward and central-difference formulae are $\mathcal{O}(h)$ and $\mathcal{O}(h^2)$, respectively. It follows that the central-difference approximation is a more efficient than the forward-difference approximation to the derivative $f'(x)$. If in (1.1) and (1.2) we replace derivatives $f'(x_n)$ by

$$\frac{f(x_n+f(x_n)) - f(x_n-f(x_n))}{2f(x_n)},$$

we obtain two free derivative methods as follows:

$$x_{n+1} = x_n - \frac{2f(x_n)\left[f(z_{n+1}) - f(x_n)\right]}{f(x_n+f(x_n)) - f(x_n-f(x_n))}, \tag{1.3}$$

where

$$z_{n+1} = x_n + \frac{2f(x_n)^2}{f(x_n+f(x_n)) - f(x_n-f(x_n))}, \tag{1.4}$$

and

$$x_{n+1} = x_n - \frac{2f(x_n)\left[f(y_{n+1}) + f(x_n)\right]}{f(x_n+f(x_n)) - f(x_n-f(x_n))}, \tag{1.5}$$

where

$$y_{n+1} = x_n - \frac{2f(x_n)^2}{f(x_n+f(x_n)) - f(x_n-f(x_n))}. \tag{1.6}$$

Now we use the forward-difference approximation. If in (1.1) and (1.2), we replace derivatives $f'(x_n)$ by

$$\frac{f(x_n+f(x_n)) - f(x_n)}{f(x_n)}, \tag{1.7}$$

we get two free derivative methods by the following

$$x_{n+1} = x_n - \frac{f(x_n)\left[f(z_{n+1}) - f(x_n)\right]}{f(x_n+f(x_n)) - f(x_n)}, \tag{1.8}$$

where

$$z_{n+1} = x_n + \frac{f(x_n)^2}{f(x_n + f(x_n)) - f(x_n)}, \tag{1.9}$$

and

$$x_{n+1} = x_n - \frac{f(x_n)\left[f(y_{n+1}) + f(x_n)\right]}{f(x_n + f(x_n)) - f(x_n)}, \tag{1.10}$$

where

$$y_{n+1} = x_n - \frac{f(x_n)^2}{f(x_n + f(x_n)) - f(x_n)}. \tag{1.11}$$

In the next section, we derive the convergence results of the iterative methods given by (1.3)-(1.6) and (1.8)-(1.11).

2 Main results

In this section we give the main results of this paper. We will give here the mathematical proof for the order of convergence of the methods given by (1.3)-(1.6) and (1.8)-(1.11).

Theorem 2.1. *Let $x^* \in D$ be a simple zero of sufficiently differentiable function $f : D \to R$ for an open interval D. If x_0 is sufficiently close to x^*, then the method defined by* (1.3) *and* (1.4) *has cubic convergence, and it satisfies the error equation*

$$e_{n+1} = \frac{3f''(x^*)^2 + 2f'(x^*)(3 + f'(x^*)^2)f^{(3)}(x^*)}{6f'(x^*)^2} e_n^3 + \mathcal{O}(e_n^4),$$

where $e_n = x_n - x^*$.

Proof. Let e_n and e^*_{n+1} be the errors in x_n and z_{n+1} respectively, i.e.

$$x_n = e_n + x^* \quad \text{and} \quad z_{n+1} = e^*_{n+1} + x^*. \tag{2.1}$$

By using Taylor' theorem, we can get

$$f(x_n) = f(x^* + e_n) = f'(x^*)e_n + \frac{f''(x^*)}{2!}e_n^2 + \frac{f^{(3)}(x^*)}{3!}e_n^3 + \mathcal{O}(e_n^4), \tag{2.2}$$

$$f(x_n)^2 = f(x^* + e_n)^2 = f'(x^*)^2 e_n^2 + f'(x^*)f''(x^*)e_n^3 + \mathcal{O}(e_n^4), \tag{2.3}$$

$$
\begin{aligned}
f(x_n + f(x_n)) &= f(x^* + e_n + f(x^* + e_n)) \\
&= (1 + f'(x^*))f'(x^*)e_n + \frac{f''(x^*)\left[f'(x^*) + (1 + f'(x^*))^2\right]}{2!}e_n^2 \\
&+ \frac{\left[f'(x^*)f^{(3)}(x^*) + 3(1 + f'(x^*))f''(x^*)^2 + (1 + f'(x^*))^3 f^{(3)}(x^*)\right]}{3!}e_n^3 \\
&+ \mathcal{O}(e_n^4),
\end{aligned}
\tag{2.4}
$$

and

$$
\begin{aligned}
f(x_n - f(x_n)) &= f(x^* + e_n - f(x^* + e_n)) \\
&= -(1 - f'(x^*))f'(x^*)e_n + \frac{f''(x^*)\left[-f'(x^*) + (1 - f'(x^*))^2\right]}{2!}e_n^2 \\
&+ \frac{\left[-f'(x^*)f^{(3)}(x^*) + 3(-1 + f'(x^*))f''(x^*)^2 - (-1 + f'(x^*))^3 f^{(3)}(x^*)\right]}{3!}e_n^3 \\
&+ \mathcal{O}(e_n^4).
\end{aligned}
\tag{2.5}
$$

It follows that

$$
\begin{aligned}
f(x_n + f(x_n)) - f(x_n - f(x_n)) &= 2f'(x^*)^2 e_n + 3f'(x^*)f''(x^*)e_n^2 \\
&+ \left(f''(x^*)^2 + \frac{1}{3}f'(x^*)(4 + f'(x^*)^2)f^{(3)}(x^*)\right)e_n^3 + \mathcal{O}(e_n^4).
\end{aligned}
\tag{2.6}
$$

Now by substituting (2.3) and (2.6) into (1.4), we have

$$
\begin{aligned}
e_{n+1}^* &= 2e_n - \frac{f''(x^*)}{2f'(x^*)}e_n^2 \\
&+ \frac{3f''(x^*)^2 - f'(x^*)(2 + f'(x^*)^2)f^{(3)}(x^*)}{6f'(x^*)^2}e_n^3 + \mathcal{O}(e_n^4).
\end{aligned}
\tag{2.7}
$$

Again using Taylor's theorem we can write

$$
\begin{aligned}
f(z_{n+1}) &= f(x^* + e_{n+1}^*) \\
&= f'(x^*)\left\{2e_n - \frac{f''(x^*)}{2f'(x^*)}e_n^2 + \frac{3f''(x^*)^2 - f'(x^*)(2 + f'(x^*)^2)f^{(3)}(x^*)}{6f'(x^*)^2}e_n^3\right\} \\
&+ \frac{f''(x^*)}{2!}\left\{4e_n^2 - \frac{2f''(x^*)}{f'(x^*)}e_n^3\right\} + \mathcal{O}(e_n^4).
\end{aligned}
\tag{2.8}
$$

Thus we get

$$
\begin{aligned}
f(z_{n+1}) &= 2f'(x^*)e_n + \frac{3}{2}f''(x^*)e_n^2 \\
+ &\left\{-\frac{f''(x^*)^2}{f'(x^*)} + \frac{3f''(x^*)^2 - f'(x^*)(2+f'(x^*)^2)f^{(3)}(x^*)}{6f'(x^*)}\right\} e_n^3 + \mathcal{O}(e_n^4).
\end{aligned}
\tag{2.9}
$$

Now it is not difficult to obtain

$$
2f(x_n)\left[f(z_{n+1}) - f(x_n)\right] = 2f'(x^*)^2 e_n^2 + 3f'(x^*)f''(x^*)e_n^3 + \mathcal{O}(e_n^4).
$$

Now by using all the previous expressions, we obtain

$$
\begin{aligned}
&\frac{2f(x_n)\left[f(z_{n+1}) - f(x_n)\right]}{f(x_n + f(x_n)) - f(x_n - f(x_n))} \\
&= e_n - \left\{\frac{3f''(x^*)^2 + 2f'(x^*)(3+f'(x^*)^2)f^{(3)}(x^*)}{6f'(x^*)^2}\right\} e_n^3 + \mathcal{O}(e_n^4).
\end{aligned}
\tag{2.10}
$$

Therefore we get the error equation

$$
e_{n+1} = \left\{\frac{3f''(x^*)^2 + 2f'(x^*)(3+f'(x^*)^2)f^{(3)}(x^*)}{6f'(x^*)^2}\right\} e_n^3 + \mathcal{O}(e_n^4). \tag{2.11}
$$

The proof is finished. □

Theorem 2.2. *Under the assumptions of Theorem* 2.1*, the method given by* (1.5) *and* (1.6) *has cubic convergence, and it verifies the error equation*

$$
e_{n+1} = \frac{f''(x^*)^2}{2f'(x^*)^2} e_n^3 + \mathcal{O}(e_n^4).
$$

Proof. Let $e_{n+1}^* = y_{n+1} - x^*$, by using the obtained equations in the proof of Theorem 2.1, we get

$$
\begin{aligned}
e_{n+1}^* &= \frac{f''(x^*)}{2f'(x^*)} e_n^2 \\
&+ \frac{-3f''(x^*)^2 + f'(x^*)(2+f'(x^*)^2)f^{(3)}(x^*)}{6f'(x^*)^2} e_n^3 + \mathcal{O}(e_n^4),
\end{aligned}
\tag{2.12}
$$

$$
\begin{aligned}
f(y_{n+1}) = f(x^* + e_{n+1}^*) = \frac{1}{2} f''(x^*) e_n^2 \\
+ \frac{-3f''(x^*)^2 + f'(x^*)(2 + f'(x^*)^2) f^{(3)}(x^*)}{6f'(x^*)} e_n^3 + \mathcal{O}(e_n^4).
\end{aligned}
\tag{2.13}
$$

It is not difficult to get

$$
2f(x_n)\left[f(y_{n+1}) + f(x_n)\right] = 2f'(x^*)^2 e_n^2 + 3f'(x^*)f''(x^*)e_n^3 + \mathcal{O}(e_n^4).
$$

By some calculations, we can show that

$$
\frac{2f(x_n)\left[f(y_{n+1}) + f(x_n)\right]}{f(x_n + f(x_n)) - f(x_n - f(x_n))} = e_n - \frac{f''(x^*)^2}{2f'(x^*)^2} e_n^3 + \mathcal{O}(e_n^4). \tag{2.14}
$$

Hence we obtain the error equation

$$
e_{n+1} = \frac{f''(x^*)^2}{2f'(x^*)^2} e_n^3 + \mathcal{O}(e_n^4).
$$

The proof is completed. □

In the next theorems, we show that if the derivatives of the methods (1.1) and (1.2) are replaced by the forward-difference approximations, only the rate of convergence (1.8) and (1.9) is decreased and the obtained method has quadratic convergence. But the method defined by (1.10), (1.11) has cubic convergence.

Theorem 2.3. *Under the assumptions of Theorem* 2.1*, the method defined by* (1.8) *and* (1.9) *has quadratic convergence, and it satisfies the error equation*

$$
e_{n+1} = f''(x^*) e_n^2 + \mathcal{O}(e_n^3). \tag{2.15}
$$

Proof. Let $e_n = x_n - x^*$ and $e_{n+1}^* = z_{n+1} - x^*$, by substituting (2.3) and (2.4) into (1.9) and some calculations, we obtain

$$
e_{n+1}^* = 2e_n - \frac{(1 + f'(x^*)) f''(x^*)}{2f'(x^*)} e_n^2 + \mathcal{O}(e_n^3). \tag{2.16}
$$

By using Taylor's theorem, we have

$$
\begin{aligned}
f(z_{n+1}) = f(x^* + e_{n+1}^*) = 2f'(x^*) e_n \\
+ \left\{ 2f''(x^*) - \frac{1}{2}(1 + f'(x^*)) f''(x^*) \right\} e_n^2 + \mathcal{O}(e_n^3).
\end{aligned}
\tag{2.17}
$$

Also we obtain

$$f(x_n)\left[f(y_{n+1})+f(x_n)\right]=f'(x^*)e_n^2+\mathcal{O}(e_n^3).$$

Now by some calculations, it is not difficult to obtain

$$e_{n+1}=f''(x^*)e_n^2+\mathcal{O}(e_n^3). \tag{2.18}$$

The proof is finished. □

Theorem 2.4. *Under the assumptions of Theorem* 2.1*, the method given by* (1.10) *and* (1.11) *has cubic convergence, and it verifies the error equation*

$$e_{n+1}=\frac{(2+3f'(x^*)+f'(x^*)^2)f''(x^*)^2}{4f'(x^*)^2}e_n^3+\mathcal{O}(e_n^4). \tag{2.19}$$

Proof. Let $e_n=x_n-x^*$ and $e_{n+1}^*=z_{n+1}-x^*$, by substituting (2.3) and (2.4) into (1.11) and some calculations, we have

$$\begin{aligned} e_{n+1}^* &= \frac{(1+f'(x^*))f''(x^*)}{2f'(x^*)}e_n^2\\ &+\frac{-3(2+2f'(x^*)+2f'(x^*)^2)f''(x^*)^2+2f'(x^*)(2+3f'(x^*)+f'(x^*)^2)f^{(3)}(x^*)}{12f'(x^*)^2}e_n^3\\ &+\mathcal{O}(e_n^4),\end{aligned} \tag{2.20}$$

$$\begin{aligned} f(y_{n+1}) &= \frac{(1+f'(x^*))f''(x^*)}{2}e_n^2\\ &+\frac{-3(2+2f'(x^*)+2f'(x^*)^2)f''(x^*)^2+2f'(x^*)(2+3f'(x^*)+f'(x^*)^2)f^{(3)}(x^*)}{12f'(x^*)}e_n^3\\ &+\mathcal{O}(e_n^4),\end{aligned} \tag{2.21}$$

Hence we obtain

$$f(x_n)\left[f(y_{n+1})+f(x_n)\right]=f'(x^*)^2+\frac{1}{2}f'(x^*)(3+f'(x^*))f''(x^*)e_n^3+\mathcal{O}(e_n^4).$$

It follows from the above equations that

$$e_{n+1}=\frac{(2+3f'(x^*)+f'(x^*)^2)f''(x^*)^2}{4f'(x^*)^2}e_n^3+\mathcal{O}(e_n^4). \tag{2.22}$$

The proof is completed. □

Method	n	x_n	$\|f(x_n)\|$
Newton's method	1	-7	0.0701888458437128
	2	-10.67709617664	0.0225666081098759
	3	-13.2791673756327	0.00436601933391256
	4	-14.0536558542692	0.000239019777052984
	5	-14.1011099568664	$7.99584812360976\times10^{-7}$
Steffensen's method	1	-7	0.0701888458437128
	2	-10.708837736245	0.0222921271890313
	3	-13.2995315760352	0.00425139483858206
	4	-14.0562246958444	0.000226083226453122
	5	-14.1011274539331	$7.12043263462192\times10^{-7}$
The method given by (1.3) and (1.4)	1	-7	0.0701888458437128
	2	-13.0388144901168	0.00574574535987593
	3	-14.1011589883491	$5.54270412900237\times10^{-7}$
	4	-14.1012697727344	$2.79776202205539\times10^{-14}$
	5	-14.1012697727366	$1.68753899743024\times10^{-14}$
The method given by (1.5) and (1.6)	1	-7	0.0701888458437128
	2	-11.8591341170014	0.0133207790518093
	3	-14.0049302623335	0.000485292604529342
	4	-14.1012609462887	$4.41597016731521\times10^{-8}$
	5	-14.10126977274	0
The method given by (1.8) and (1.9)	1	-7	0.0701888458437128
	2	-13.1709615370047	0.00498098138198277
	3	-14.1019247141922	$3.27659042276274\times10^{-6}$
	4	-14.1012697730407	$1.50435219836709\times10^{-12}$
	5	-14.1012697727234	$8.28226376370367\times10^{-14}$
The method given by (1.10) and (1.11)	1	-7	0.0701888458437128
	2	-11.8867725141432	0.0131262898051761
	3	-14.009072627247	0.00046428970071144
	4	-14.101262092531	$3.84249230211964\times10^{-8}$
	5	-14.10126977274	0

Table 1 – The comparison of the introduced methods in this paper with Newton's method and Steffensen's method for $f(x) = e^x - 1.5 - \tan^{-1} x$.

Method	n	x_n	$\|g(x_n)\|$
Newton's method	1	0	1
	2	1	1.17797952259091
	3	0.724644697567095	0.221820009620489
	4	0.64465890487027	0.0134025850038723
	5	0.639177807467281	$5.74817089998847 \times 10^{-5}$
Steffensen's method	1	0	1
	2	0.459141139587212	0.38055490048224
	3	0.611228478941219	0.0662370181759029
	4	0.638317069569666	0.00202779223976352
	5	0.639153314504489	$1.89530859256992 \times 10^{-6}$
The method given by (1.3) and (1.4)	1	0	1
	2	0.360452364589399	0.548782536293577
	3	0.576508750644072	0.144654947339254
	4	0.637957420142172	0.00289826619895661
	5	0.639154087002152	$2.26174761697173 \times 10^{-8}$
The method given by (1.5) and (1.6)	1	0	1
	2	0.633973846554925	0.0125070972128752
	3	0.63915392812122	$4.0777728727015 \times 10^{-7}$
	4	0.639154096332008	$1.11022302462516 \times 10^{-16}$
	5	0.639154096332007	$1.66533453693773 \times 10^{-16}$
The method given by (1.8) and (1.9)	1	0	1
	2	0.182435774509363	0.797739575083025
	3	0.328331547459196	0.598445908724964
	4	0.454070648621023	0.389819960978116
	5	0.639143306963761	$2.61554132035546 \times 10^{-5}$
The method given by (1.10) and (1.11)	1	0	1
	2	0.633869550270125	0.0127578660747175
	3	0.639154041910595	$1.31928625801692 \times 10^{-7}$
	4	0.639154096332008	$1.11022302462516 \times 10^{-16}$
	5	0.639154096332007	$1.66533453693773 \times 10^{-16}$

Table 2 – The comparison of the introduced methods in this paper with Newton's method and Steffensen's method for $g(x) = \cos\, x - xe^x + x^2$.

3 Numerical examples

In this section, in order to compare the methods introduced in this paper with Newton's method and Steffensen's method, we present some numerical examples. For this purpose we take two examples from the literature. Consider two nonlinear equations as

$$f(x) = e^x - 1.5 - \tan^{-1} x, \tag{3.1}$$

$$g(x) = \cos x - xe^x + x^2. \tag{3.2}$$

Tables 1 and 2 demonstrate the comparison of these methods for $f(x)$ and $g(x)$, respectively. All the tests are performed using MATLAB 7 which has a machine precision of around 10^{-16} on a Pentium IV. The numerical results indicate that the proposed iterative methods may be very efficient.

4 Conclusion

The problem of locating roots of nonlinear equations (or zeros of functions) occurs frequently in scientific work. In this paper, we have introduced some techniques for solving nonlinear equations. The techniques were based on the central-difference and forward-difference approximations to derivatives. We have shown that that three of the four methods have cubic convergence and another method has quadratic convergence. The introduced methods can be used for solving nonlinear equations without computing derivatives. Meanwhile, the methods introduced in this paper can be used to more class of nonlinear equations. The numerical examples shown in this paper illustrated the the efficiency of the new methods. We used the well known software MATLAB 7 to calculate the numerical results obtained from the proposed techniques.

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