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The rates in complete moment convergence for negatively associated sequences

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Abstract. Let $X_1, X_2, \ldots$ be a strictly stationary and negatively associated sequence of random variables with mean zero and positive, finite variance, set $S_n = X_1 + \cdots + X_n$, $M_n = \max_{1 \le k \le n} |S_k|$. Under appropriate moment conditions, we obtain precise rates in law of the logarithm for the moment convergence of S_n and M_n.

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Key words: the rates, complete moment convergence, negatively associated sequences.

1 Introduction and main results

A finite family of random variables, $X_1, X_2, \ldots, X_n$, is said to be negatively associated if, for every pair of disjoint subsets T_1 and T_2 of $\{1, 2, \ldots, n\}$,

$$Cov\big(f_1(X_i, i \in T_1), f_2(X_j, j \in T_2)\big) \le 0,$$

whenever f_1 and f_2 are coordinatewise increasing and the covariance exists. An infinite family is negatively associated if every finite subfamily is negatively associated. This definition was introduced by Alam and Saxena [1] and Joag-Dev and Proschan [7], and has found many applications in percolation theory, multivariate statistical analysis and reliability theory [2].

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Under appropriate conditions, lots of results have been obtained for negatively associated sequences, the central limit theorem (CLT) [13], probability inequalities [15, 17], weak convergence [19, 20], almost sure convergence [12], law of the iterated logarithm (LIL) [16] and complete convergence [8, 9], precise rates [5, 21, 22].

Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed (i.i.d.) random variables. Gut and Spătaru [6] obtained a result below.

Theorem A. *Suppose that* $EX_1 = 0$ *and* $EX_1^2 = \sigma^2 < \infty$. *Then, for* $0 \le \delta \le 1$,

$$\lim_{\epsilon \downarrow 0} \epsilon^{2\delta+2} \sum_{n=1}^{\infty} \frac{(\log n)^{\delta}}{n} P\big(|S_n| \ge \epsilon\sqrt{n \log n}\big) = \frac{\sigma^{2\delta+2} E|N|^{2\delta+2}}{\delta+1}. \tag{1.1}$$

where N is a standard normal random variable.

Liu and Lin [11] proved the following theorem for i.i.d. random variables.

Theorem B. *Suppose that*

$$EX_1 = 0,\ EX_1^2 = \sigma^2 \ \textit{and} \ EX_1^2(\log^+ |X_1|)^{\alpha} < \infty. \tag{1.2}$$

for $0 < \alpha \le 1$. *Then*

$$\lim_{\epsilon \downarrow 0} \epsilon^{2\alpha} \sum_{n=2}^{\infty} \frac{(\log n)^{\alpha-1}}{n^2} ES_n^2 I\big(|S_n| \ge \epsilon\sqrt{n \log n}\big) = \frac{\sigma^{2\alpha+2}}{\alpha} E|N|^{2\alpha+2}. \tag{1.3}$$

Conversely, if (1.3) *is true, then* (1.2) *holds.*

The purpose of the present paper is to investigate precise asymptotics in complete moment convergence, our results not only extend (1.3) to negatively associated sequences, but give a maximal analog of (1.3) and other versions. To formulate our results,we need some extra notation. Let $X_1, X_2, \ldots$ be strictly stationary and negatively associated random variables, $EX_1 = 0$, $EX_1^2 < \infty$, $\sigma^2 = EX_1^2 + 2\sum_{n=2}^{\infty} EX_1X_n > 0$, set $S_n = X_1 + \cdots + X_n$, $M_n = \max_{1 \le k \le n} |S_k|$, write log for the natural logarithm, $\log x = \log_e(x \vee e)$, $[z]$ denotes the integer part of z, C stands for a positive constant whose value may be different from line to line. Our results read as follows.

Theorem 1.1. *If* $EX_1^2\big(\log|X_1|\big)^{1-2/\delta} < \infty$ *for any* $\delta > 2$, *then we have*

$$\begin{aligned}\lim_{\epsilon\downarrow 0}\epsilon^{\delta-2}\sum_{n=2}^{\infty}\frac{1}{n^2(\log n)^{2/\delta}}ES_n^2I\big(|S_n|\geq\epsilon\sigma\sqrt{n}(\log n)^{1/\delta}\big)\\ =\frac{\delta(\sqrt{2})^{\delta}}{\sqrt{\pi}(\delta-2)}\Gamma\left(\frac{\delta+1}{2}\right),\end{aligned}\tag{1.4}$$

and

$$\begin{aligned}\lim_{\epsilon\downarrow 0}\epsilon^{\delta-2}\sum_{n=2}^{\infty}\frac{1}{n^2(\log n)^{2/\delta}}EM_n^2I\big(M_n\geq\epsilon\sigma\sqrt{n}(\log n)^{1/\delta}\big)\\ =\frac{2\delta(\sqrt{2})^{\delta}\Gamma\left(\frac{\delta+1}{2}\right)}{\sqrt{\pi}(\delta-2)}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^{2\delta+2}}.\end{aligned}\tag{1.5}$$

Conversely, if (1.5) *is true, then* $EX_1^2(\log|X_1|)^{1-2/\delta} < \infty$. *Where* $\Gamma(\cdot)$ *is the Gamma function.*

Theorem 1.2. *If* $EX_1^2(\log|X_1|)^{\delta} < \infty$ *for any* $\delta > 0$, *then we have*

$$\begin{aligned}\lim_{\epsilon\downarrow 0}\epsilon^{2\delta}\sum_{n=2}^{\infty}\frac{(\log n)^{\delta-1}}{n^2}ES_n^2I\big(|S_n|\geq\epsilon\sigma\sqrt{n\log n}\big)\\ =\frac{2^{\delta+1}}{\sqrt{\pi}\delta}\Gamma\left(\delta+\frac{3}{2}\right),\end{aligned}\tag{1.6}$$

and

$$\begin{aligned}\lim_{\epsilon\downarrow 0}\epsilon^{2\delta}\sum_{n=2}^{\infty}\frac{(\log n)^{\delta-1}}{n^2}EM_n^2I\big(M_n\geq\epsilon\sigma\sqrt{n\log n}\big)\\ =\frac{2^{\delta+2}\Gamma\left(\delta+\frac{3}{2}\right)}{\sqrt{\pi}\delta}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^{2\delta+2}}.\end{aligned}\tag{1.7}$$

Conversely, if (1.7) *is true, then* $EX_1^2\big(\log|X_1|\big)^{\delta} < \infty$.

Without loss of generality, throughout the paper, we will suppose that $\sigma^2 = 1$.

2 Proof of Theorem 1.1

In order to verify this result, we first give three elementary but useful lemmas.

Lemma 2.1. [13]. *Let $\{X_n\colon n \geq 1\}$ be a strictly stationary and negatively associated sequence of random variables with mean zero and*

$$0 < \sigma^2 = EX_1^2 + 2\sum_{n=2}^{\infty} EX_1X_n < \infty,$$

then

$$S_n/\sigma\sqrt{n} \xrightarrow{\mathcal{D}} N(0,1) \ \ as \ \ n \to \infty. \tag{2.1}$$

Lemma 2.2. [18]. *Let $\{X_n\colon n \geq 1\}$ be strictly stationary and negatively associated sequence of random variables, $EX_1 = \mu$, $0 < Var X_1 = \sigma^2 < \infty$ and $B^2 = EX_1^2 + 2\sum_{n=2}^{\infty} EX_1X_n > 0$, set $S_m = \sum_{k=1}^{m} X_k$, write*

$$W_n(t) = \frac{1}{B\sqrt{n}}\big(S_m + (nt-m)X_{m+1} - nt\mu\big), \quad m \leq n < m+1, 0 \leq t \leq T.$$

Then

$$W_n(t) \xrightarrow{\mathcal{D}} W(t) \ \ in \ \ C[0,T], \tag{2.2}$$

where $\{W(t)\colon t \geq 0\}$ is a standard Wiener process and $C[0,T]$ is the usual C space on $[0,T]$.

Lemma 2.3. [10]. *Let $\{X_n\colon n \geq 1\}$ be a negatively associated sequence of random variables with mean zero, $EX_n^2 < \infty$, set $S_n = \sum_{k=1}^{n} X_k$, $B_n^2 = \sum_{k=1}^{n} EX_k^2$. Then, for any $a > 0$ and $b > 0$, we have*

$$P\Big(\max_{1\leq k\leq n} |S_k| \geq a\Big) \leq \\ 2P\Big(\max_{1\leq k\leq n} |X_k| \geq b\Big) + 2e\exp\left(\frac{a}{b} - \left(\frac{a}{b} + \frac{B_n^2}{b^2}\right)\log\left(1 + \frac{ab}{B_n^2}\right)\right). \tag{2.3}$$

Observe the following formula.

$$\sum_{n=2}^{\infty} \frac{1}{n^2(\log n)^{2/\delta}} ES_n^2 I\big(|S_n| \geq \epsilon\sqrt{n}(\log n)^{1/\delta}\big)$$

$$= \epsilon^2 \sum_{n=2}^{\infty} \frac{1}{n} P\big(|S_n| \geq \epsilon\sqrt{n}(\log n)^{1/\delta}\big) \tag{2.4}$$

$$+ \sum_{n=2}^{\infty} \frac{1}{n^2(\log n)^{2/\delta}} \int_{\epsilon\sqrt{n}(\log n)^{1/\delta}}^{\infty} 2xP\big(|S_n| \geq x\big)dx. \tag{2.5}$$

Similarly, one can obtain the corresponding equation for M_n. In the rest of this section, we give the following propositions.

Proposition 2.1. *We have*

$$\lim_{\epsilon\downarrow 0}\epsilon^{\delta}\sum_{n=2}^{\infty}\frac{1}{n}P\big(|N|\geq \epsilon(\log n)^{1/\delta}\big)=\frac{(\sqrt{2})^{\delta}}{\sqrt{\pi}}\Gamma\Big(\frac{\delta+1}{2}\Big), \tag{2.6}$$

and

$$\begin{aligned}\lim_{\epsilon\downarrow 0}\epsilon^{\delta}\sum_{n=2}^{\infty}\frac{1}{n}P\Big(\sup_{0\leq t\leq 1}|W(t)|\geq \epsilon(\log n)^{1/\delta}\Big)\\ =\frac{2(\sqrt{2})^{\delta}\Gamma\left(\frac{\delta+1}{2}\right)}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^{2\delta+2}},\end{aligned} \tag{2.7}$$

where N is a standard normal random variable and $\{W(t)\colon t\geq 0\}$ is a standard Wiener process.

Proposition 2.2. *Under the conditions of Theorem* 1.1*, we have*

$$\lim_{\epsilon\downarrow 0}\epsilon^{\delta}\sum_{n=2}^{\infty}\frac{1}{n}\Big|P\big(|S_n|\geq \epsilon\sqrt{n}(\log n)^{1/\delta}\big)-P\big(|N|\geq \epsilon(\log n)^{1/\delta}\big)\Big|=0, \tag{2.8}$$

and

$$\begin{aligned}\lim_{\epsilon\downarrow 0}\epsilon^{\delta}\sum_{n=2}^{\infty}\frac{1}{n}\Big|P\big(M_n\geq \epsilon\sqrt{n}(\log n)^{1/\delta}\big)\\ -P\Big(\sup_{0\leq t\leq 1}|W(t)|\geq \epsilon(\log n)^{1/\delta}\Big)\Big|=0.\end{aligned} \tag{2.9}$$

Remark 2.1. *The proofs of Propositions* 2.1 *and* 2.2 *are very standard, so we omit them.*

Proposition 2.3. *Under the conditions of Theorem* 1.1*, we have*

$$\begin{aligned}\lim_{\epsilon\downarrow 0}\epsilon^{\delta-2}\sum_{n=2}^{\infty}\frac{1}{n^2(\log n)^{2/\delta}}\int_{\epsilon\sqrt{n}(\log n)^{1/\delta}}^{\infty}2xP\big(|N|\geq x/\sqrt{n}\big)dx\\ =\frac{2(\sqrt{2})^{\delta}}{\sqrt{\pi}(\delta-2)}\Gamma\Big(\frac{\delta+1}{2}\Big),\end{aligned} \tag{2.10}$$

and

$$\lim_{\epsilon\downarrow 0}\epsilon^{\delta-2}\sum_{n=2}^{\infty}\frac{1}{n^2(\log n)^{2/\delta}}\int_{\epsilon\sqrt{n}(\log n)^{1/\delta}}^{\infty}2xP\left(\sup_{0\le t\le 1}|W(t)|\ge x/\sqrt{n}\right)dx$$
$$=\frac{4(\sqrt{2})^{\delta}\Gamma\left(\frac{\delta+1}{2}\right)}{\sqrt{\pi}(\delta-2)}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^{2\delta+2}}. \tag{2.11}$$

Proof. It follows that

$$\begin{aligned}
&\lim_{\epsilon\downarrow 0}\epsilon^{\delta-2}\sum_{n=2}^{\infty}\frac{1}{n^2(\log n)^{2/\delta}}\int_{\epsilon\sqrt{n}(\log n)^{1/\delta}}^{\infty}2xP(|N|\ge x/\sqrt{n})dx\\
&=\lim_{\epsilon\downarrow 0}\epsilon^{\delta-2}\int_2^{\infty}\frac{1}{y(\log y)^{2/\delta}}dy\int_{\epsilon(\log y)^{1/\delta}}^{\infty}2xP(|N|\ge x)dx\\
&=\delta\int_0^{\infty}t^{\delta-3}dt\int_t^{\infty}2xP(|N|\ge x)dx\\
&=\frac{2\delta}{\delta-2}\int_0^{\infty}x^{\delta-1}dx\int_x^{\infty}\frac{2}{\sqrt{2\pi}}\exp\left(\frac{-u^2}{2}\right)du\\
&=\frac{2(\sqrt{2})^{\delta}}{\sqrt{\pi}(\delta-2)}\Gamma\left(\frac{\delta+1}{2}\right).
\end{aligned}$$

Using the following result of Billingsley [3].

$$\begin{aligned}
P\left(\sup_{0\le s\le 1}|W(s)|\ge x\right) &= 1-\sum_{k=-\infty}^{\infty}(-1)^kP\Big((2k-1)x\le N\le(2k+1)x\Big)\\
&= 2\sum_{k=0}^{\infty}(-1)^kP\big(|N|\ge(2k+1)x\big),
\end{aligned}$$

one can obtain (2.11). □

Proposition 2.4. *Under the conditions of Theorem* 1.1*, we have*

$$\lim_{\epsilon\downarrow 0}\epsilon^{\delta-2}\sum_{n=2}^{\infty}\frac{1}{n^2(\log n)^{2/\delta}}\Bigg|\int_{\epsilon\sqrt{n}(\log n)^{1/\delta}}^{\infty}2xP\big(|S_n|\ge x\big)dx$$
$$-\int_{\epsilon\sqrt{n}(\log n)^{1/\delta}}^{\infty}2xP\big(|N|\ge x/\sqrt{n}\big)dx\Bigg|=0, \tag{2.12}$$

and

$$\lim_{\epsilon\downarrow 0}\epsilon^{\delta-2}\sum_{n=2}^{\infty}\frac{1}{n^2(\log n)^{2/\delta}}\Big|\int_{\epsilon\sqrt{n}(\log n)^{1/\delta}}^{\infty}2xP(M_n\ge x)dx \\ -\int_{\epsilon\sqrt{n}(\log n)^{1/\delta}}^{\infty}2xP\left(\sup_{0\le t\le 1}|W(t)|\ge x/\sqrt{n}\right)dx\Big|=0. \tag{2.13}$$

Proof. We only prove (2.12), let $H(\epsilon)=[\exp(M/\epsilon^{\delta})]$, $M>4$, $0<\epsilon<1/4$, $\delta>2$, denote $\Delta_n=\sup_x|P(|S_n|\ge\sqrt{n}x)-P(|N|\ge x)|$, we have

$$\begin{aligned}
&\sum_{n=2}^{\infty}\frac{1}{n^2(\log n)^{2/\delta}}\Big|\int_{\epsilon\sqrt{n}(\log n)^{1/\delta}}^{\infty}2xP\big(|S_n|\ge x\big)dx \\
&\qquad -\int_{\epsilon\sqrt{n}(\log n)^{1/\delta}}^{\infty}2xP\big(|N|\ge x/\sqrt{n}\big)dx\Big| \\
&= \sum_{n\le H(\epsilon)}\frac{1}{n^2(\log n)^{2/\delta}}\Big|\int_{\epsilon\sqrt{n}(\log n)^{1/\delta}}^{\infty}2xP\big(|S_n|\ge x\big)dx \\
&\qquad -\int_{\epsilon\sqrt{n}(\log n)^{1/\delta}}^{\infty}2xP\big(|N|\ge x/\sqrt{n}\big)dx\Big| \\
&\qquad +\sum_{n>H(\epsilon)}\frac{1}{n^2(\log n)^{2/\delta}}\Big|\int_{\epsilon\sqrt{n}(\log n)^{1/\delta}}^{\infty}2xP\big(|S_n|\ge x\big)dx \\
&\qquad -\int_{\epsilon\sqrt{n}(\log n)^{1/\delta}}^{\infty}2xP\big(|N|\ge x/\sqrt{n}\big)dx\Big| \\
&=: \Sigma_1+\Sigma_2.
\end{aligned}$$

We first estimate Σ_1, it follows that

$$\begin{aligned}
\Sigma_1 \;\le\; & \sum_{n\le H(\epsilon)}\frac{1}{n}\bigg(\int_{(\log n)^{-1/\delta}\Delta_n^{-1/4}}^{\infty}2(x+\epsilon)P\big(|N|\ge(x+\epsilon)(\log n)^{1/\delta}\big)dx \\
&+\int_{0}^{(\log n)^{-1/\delta}\Delta_n^{-1/4}}2(x+\epsilon)\Big|P\big(|S_n|\ge(x+\epsilon)\sqrt{n}(\log n)^{1/\delta}\big) \\
&-P\big(|N|\ge(x+\epsilon)(\log n)^{1/\delta}\big)\Big|dx
\end{aligned}$$

$$+\int_{(\log n)^{-1/\delta}\Delta_n^{-1/4}}^{\infty} 2(x+\epsilon)P\big(|S_n|\geq (x+\epsilon)\sqrt{n}(\log n)^{1/\delta}\big)dx\Big)$$
$$=: \sum_{n\leq H(\epsilon)} \frac{1}{n}(\Sigma_3+\Sigma_4+\Sigma_5).$$

The estimate of Σ_3 is easy. By Toeplitz's lemma, one can complete the proof of term Σ_4. As to Σ_5, taking $a=\epsilon\sqrt{n}(\log n)^{1/\delta}$, $b=a/m$ in Lemma 2.3, it turns out that

$$\sum_{n\leq H(\epsilon)} \frac{\Sigma_5}{n} \leq C\sum_{n\leq H(\epsilon)} \frac{1}{n}\int_{(\log n)^{-1/\delta}\Delta_n^{-1/4}}^{\infty} \frac{(y+\epsilon)^{-2m+1}n^{-m}(\log n)^{-2m/\delta}}{\big(m\sum_{k=1}^{n} EX_k^2\big)^{-m}}dy$$
$$+C\sum_{n\leq H(\epsilon)} \int_{(\log n)^{-1/\delta}\Delta_n^{-1/4}}^{\infty} (y+\epsilon)P\big(|X_1|\geq (y+\epsilon)\sqrt{n}(\log n)^{1/\delta}/m\big)dy$$
$$=: \Sigma_6+\Sigma_7.$$

An easy calculation leads to

$$\begin{aligned}
\Sigma_6 &\leq C\sum_{n\leq H(\epsilon)} \frac{1}{n(\log n)^{2m/\delta}} \int_{(\log n)^{-1/\delta}\Delta_n^{-1/4}}^{\infty} (y+\epsilon)^{-2m+1}dy \\
&\leq C\frac{(\log H(\epsilon))^{1-2/\delta}}{(\log H(\epsilon))^{1-2/\delta}} \sum_{n\leq H(\epsilon)} \frac{1}{n(\log n)^{2/\delta}}\Delta_n^{(m-1)/2} \\
&= C\frac{\epsilon^{2-\delta}M^{1-2/\delta}}{(\log H(\epsilon))^{1-2/\delta}} \sum_{n\leq H(\epsilon)} \frac{1}{n(\log n)^{2/\delta}}\Delta_n^{(m-1)/2}, \qquad (2.14)
\end{aligned}$$

by Toeplitz's lemma, we have $\lim_{\epsilon\downarrow 0}\epsilon^{\delta-2}\Sigma_6=0$. Turn to Σ_7, it follows that

$$\begin{aligned}
\Sigma_7 &\leq C\sum_{n\leq H(\epsilon)} E\int_{(\log n)^{-1/\delta}\Delta_n^{-1/4}}^{\infty} (y+\epsilon)I\big(m(y+\epsilon)\leq |X_1|/\sqrt{n}(\log n)^{1/\delta}\big)dy \\
&\leq C\sum_{n\leq H(\epsilon)} \frac{1}{n(\log n)^{2/\delta}} EX_1^2 I\big(m|X_1|\geq \sqrt{n}\big) \\
&= C\frac{(\log H(\epsilon))^{-2/\delta+1}}{(\log H(\epsilon))^{-2/\delta+1}} \sum_{n\leq H(\epsilon)} \frac{1}{n(\log n)^{2/\delta}} EX_1^2 I\big(m|X_1|\geq \sqrt{n}\big) \\
&\leq C\frac{\epsilon^{2-\delta}M^{1-2/\delta}}{(\log H(\epsilon))^{-2/\delta+1}} \sum_{n\leq H(\epsilon)} \frac{1}{n(\log n)^{2/\delta}} EX_1^2 I\big(m|X_1|\geq \sqrt{n}\big), \qquad (2.15)
\end{aligned}$$

applying Toeplitz's lemma again, we have $\lim_{\epsilon\downarrow 0}\epsilon^{\delta-2}\Sigma_7 = 0$. Now let us consider Σ_2, notice that

$$\begin{aligned}
\Sigma_2 \quad &\le \quad C\sum_{n>H(\epsilon)}\frac{1}{n^2(\log n)^{2/\delta}}\int_{\epsilon\sqrt{n}(\log n)^{1/\delta}}^{\infty} 2xP\big(|N|\ge x/\sqrt{n}\big)dx \\
&\quad + C\sum_{n>H(\epsilon)}\frac{1}{n^2(\log n)^{2/\delta}}\int_{\epsilon\sqrt{n}(\log n)^{1/\delta}}^{\infty} 2xP\big(|S_n|\ge x\big)dx \\
&=: \quad \Sigma_8+\Sigma_9.
\end{aligned}$$

Applying Markov's inequality, one can complete the estimate of Σ_8. By Lemma 2.3, taking $m > \delta/2$, it turns out

$$\begin{aligned}
\Sigma_9 \;&\le\; C\sum_{n>H(\epsilon)}\frac{1}{n}\int_0^\infty \frac{(y+\epsilon)^{-2m+1}n^{-m}(\log n)^{-2m/\delta}}{\big(m\sum_{k=1}^n EX_k^2\big)^{-m}}dy \\
&\quad + C\sum_{n>H(\epsilon)}\int_0^\infty (y+\epsilon)P\big(|X_1|\ge (y+\epsilon)\sqrt{n}(\log n)^{1/\delta}/m\big)dy \\
&\le\; C\sum_{n>H(\epsilon)}\frac{1}{n(\log n)^{2m/\delta}}\int_0^\infty (y+\epsilon)^{-2m+1}dy \\
&\quad + CE\Big(\int_0^\infty (y+\epsilon)\sum_{n>H(\epsilon)} I(\sqrt{n}(\log n)^{1/\delta}\le m|X_1|/(y+\epsilon)dy\Big) \\
&\le\; C\epsilon^{-2m+2}\sum_{n>H(\epsilon)}\frac{1}{n(\log n)^{2m/\delta}} \\
&\quad + CE\Big(\int_0^\infty \frac{X_1^2}{(y+\epsilon)}(\log|X_1|-\log(y+\epsilon))^{-2/\delta}I\big(y+\epsilon\le m|X_1|\big)dy\Big) \\
&\le\; C\epsilon^{2-\delta}M^{-2m/\delta+1} + CEX_1^2\Big(\big(\log|X_1|\big)^{1-2/\delta} - (\log\epsilon)^{1-2/\delta}\Big), \qquad (2.16)
\end{aligned}$$

we have $\lim_{\epsilon\downarrow 0}\epsilon^{\delta-2}\Sigma_9 = 0$. The proof of (2.12) is now complete. □

Proof of Theorem 1.1. Combining Propositions 2.1 ∼ 2.4, we have

$$\lim_{\epsilon\downarrow 0}\epsilon^\delta\sum_{n=2}^\infty\frac{1}{n}P\big(|S_n|\ge\epsilon\sqrt{n}(\log n)^{1/\delta}\big) = \frac{(\sqrt{2})^\delta}{\sqrt{\pi}}\Gamma\Big(\frac{\delta+1}{2}\Big), \qquad (2.17)$$

$$\lim_{\epsilon\downarrow 0}\epsilon^{\delta-2}\sum_{n=2}^{\infty}\frac{1}{n^2(\log n)^{2/\delta}}\int_{\epsilon\sqrt{n}(\log n)^{1/\delta}}^{\infty}2xP\big(|S_n|\geq x\big)dx = \frac{2(\sqrt{2})^{\delta}}{\sqrt{\pi}(\delta-2)}\Gamma\left(\frac{\delta+1}{2}\right). \tag{2.18}$$

Then, the proof of (1.4) follows from (2.17) and (2.18), similarly, one can obtain (1.5).

For the sufficient part, using the standard method, we first show that

$$P\left(\max_{1\leq k\leq n}|X_k|\geq\epsilon\sqrt{n}(\log n)^{1/\delta}\right)\to 0 \ \text{ as } \ n\to\infty. \tag{2.19}$$

It is easy to see that

$$|X_n| = |S_n - S_{n-1}|\leq |S_n| + |S_{n-1}|, \tag{2.20}$$

furthermore, we have

$$\left(\max_{1\leq k\leq n}|X_k|\geq 2\epsilon\sqrt{n}(\log n)^{1/\delta}\right) \subset \left(\max_{1\leq k\leq n}|S_k|\geq\epsilon\sqrt{n}(\log n)^{1/\delta}\right)\cup\left(\max_{1\leq k\leq n-1}|S_k|\geq\epsilon\sqrt{n}(\log n)^{1/\delta}\right). \tag{2.21}$$

Recalling (1.5) and (2.4), which yields

$$\begin{aligned}\infty &> \sum_{n=2}^{\infty}\frac{1}{n}P\left(\max_{1\leq k\leq n}|X_k|\geq\epsilon\sqrt{n}(\log n)^{1/\delta}\right)\\ &\geq \frac{C}{2}\sum_{m=2}^{\infty}P\left(\max_{1\leq k\leq 2^m}|X_k|\geq\epsilon\sqrt{2^m}(\log 2^m)^{1/\delta}\right).\end{aligned} \tag{2.22}$$

Hence, we have

$$P\left(\max_{1\leq k\leq 2^m}|X_k|\geq\epsilon\sqrt{2^m}(\log 2^m)^{1/\delta}\right)\to 0, \tag{2.23}$$

So (2.19) follows from (2.22) and (2.23). We next show that

$$P\left(\max_{1\leq k\leq n}|X_k|\geq\epsilon\sqrt{n}(\log n)^{1/\delta}\right)\geq C\sum_{k=1}^{n}P\big(|X_k|\geq\epsilon\sqrt{n}(\log n)^{1/\delta}\big). \tag{2.24}$$

By the result of Joag-Dev et al. [7], it turns out that

$$
\begin{aligned}
&P\left(\max_{1\le k\le n}|X_k|\ge \epsilon\sqrt{n}(\log n)^{1/\delta}\right)\\
&\quad\ge\ 1-\prod_{k=1}^{n}\Big(1-P\big(|X_k|\ge \epsilon\sqrt{n}(\log n)^{1/\delta}\big)\Big)\\
&\quad\ge\ 1-\exp\Big(-\sum_{k=1}^{n}P\big(|X_k|\ge \epsilon\sqrt{n}(\log n)^{1/\delta}\big)\Big),
\end{aligned}
\tag{2.25}
$$

by (2.19), for sufficiently large n, we have

$$
\sum_{k=1}^{n}P\big(|X_k|\ge \epsilon\sqrt{n}(\log n)^{1/\delta}\big)\to 0 \ \text{ for any } \ \epsilon>0.
\tag{2.26}
$$

By (2.26) and using elementary inequality, we have

$$
\begin{aligned}
&P\left(\max_{1\le k\le n}|X_k|\ge \epsilon\sqrt{n}(\log n)^{1/\delta}\right)\\
&\ge\ \sum_{k=1}^{n}P\big(|X_k|\ge \epsilon\sqrt{n}(\log n)^{1/\delta}\big)\Big(1-\sum_{k=1}^{n}P\big(|X_k|\ge \epsilon\sqrt{n}(\log n)^{1/\delta}\big)\Big)\\
&\ge\ C\sum_{k=1}^{n}P\big(|X_k|\ge \epsilon\sqrt{n}(\log n)^{1/\delta}\big).
\end{aligned}
\tag{2.27}
$$

Finally, the sufficient part follows from (2.27) together with

$$
\begin{aligned}
\infty\ &>\ \sum_{n=2}^{\infty}\frac{1}{n}\int_0^{\infty}2(x+1)P\big(M_n\ge (x+1)\sqrt{n}(\log n)^{1/\delta}\big)dx\\
&\ge\ C\sum_{n=N}^{\infty}\int_0^{\infty}(x+1)P\big(|X_1|\ge 2(x+1)\sqrt{n}(\log n)^{1/\delta}\big)dx\\
&\ge\ CE\left(\int_0^{\infty}(y+\epsilon)\sum_{n>H(\epsilon)}I(\sqrt{n}(\log n)^{1/\delta}\le m|X_1|/(y+\epsilon)dy\right)\\
&\ge\ CE\left(\int_0^{\infty}\frac{X_1^2}{(y+\epsilon)}(\log|X_1|-\log(y+\epsilon))^{-2/\delta}I(y+\epsilon\le m|X_1|)dy\right)\\
&\ge\ CEX_1^2\big(\log|X_1|\big)^{1-2/\delta}.
\end{aligned}
$$

3 Proof of Theorem 1.2

According to the proof of Theorem 1.1, we only give the main ideas of the proofs of (1.6) and (1.7). Observe (2.4) and (2.5), it is natural to give the following Propositions.

Proposition 3.1. *We have*

$$\lim_{\epsilon\downarrow 0}\epsilon^{2\delta+2}\sum_{n=2}^{\infty}\frac{(\log n)^{\delta}}{n}P\big(|S_n|\geq\epsilon\sqrt{n\log n}\big)=\frac{2^{\delta+1}}{\sqrt{\pi}(\delta+1)}\Gamma\left(\delta+\frac{3}{2}\right),\quad(3.1)$$

and

$$\begin{aligned}&\lim_{\epsilon\downarrow 0}\epsilon^{2\delta+2}\sum_{n=2}^{\infty}\frac{(\log n)^{\delta}}{n}P\big(M_n\geq\epsilon\sqrt{n\log n}\big)\\&=\frac{2^{\delta+2}\Gamma\left(\delta+\frac{3}{2}\right)}{\sqrt{\pi}(\delta+1)}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^{2\delta+2}}.\end{aligned}\quad(3.2)$$

Proof. By a careful calculation, it follows that

$$\lim_{\epsilon\downarrow 0}\epsilon^{2\delta+2}\sum_{n=2}^{\infty}\frac{(\log n)^{\delta}}{n}P\big(|N|\geq\epsilon\sqrt{\log n}\big)=\frac{2^{\delta+1}}{\sqrt{\pi}(\delta+1)}\Gamma\left(\delta+\frac{3}{2}\right).\quad(3.3)$$

Then, along the same lines as those of the proof of Proposition 2.2, we have

$$\lim_{\epsilon\downarrow 0}\epsilon^{2\delta+2}\sum_{n=2}^{\infty}\frac{(\log n)^{\delta}}{n}\Big|P\big(|S_n|\geq\epsilon\sqrt{n\log n}\big)-P\big(|N|\geq\epsilon\sqrt{\log n}\big)\Big|=0.\quad(3.4)$$

With the help of Billingsley's result, one can complete the proof of (3.2).

Proposition 3.2. *Under the conditions of Theorem* 1.2*, we have*

$$\begin{aligned}&\lim_{\epsilon\downarrow 0}\epsilon^{2\delta}\sum_{n=2}^{\infty}\frac{(\log n)^{\delta-1}}{n^2}\int_{\epsilon\sqrt{n\log n}}^{\infty}2xP\big(|S_n|\geq x\big)dx\\&=\frac{2^{\delta+1}}{\sqrt{\pi}\delta(\delta+1)}\Gamma\left(\delta+\frac{3}{2}\right),\end{aligned}\quad(3.5)$$

and

$$\begin{aligned}\lim_{\epsilon\downarrow 0}\epsilon^{2\delta}\sum_{n=2}^{\infty}\frac{(\log n)^{\delta-1}}{n^2}\int_{\epsilon\sqrt{n\log n}}^{\infty}2xP\big(M_n\geq x\big)dx\\ =\frac{2^{\delta+2}\Gamma\left(\delta+\frac{3}{2}\right)}{\sqrt{\pi}\delta(\delta+1)}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^{2\delta+2}}.\end{aligned}\tag{3.6}$$

Proof. Recalling the proof of Proposition 2.3, it is easy to get

$$\begin{aligned}\lim_{\epsilon\downarrow 0}\epsilon^{2\delta}\sum_{n=2}^{\infty}\frac{(\log n)^{\delta-1}}{n^2}\int_{\epsilon\sqrt{n\log n}}^{\infty}2xP\big(|N|\geq x/\sqrt{n}\big)dx\\ =\frac{2^{\delta+1}}{\sqrt{\pi}\delta(\delta+1)}\Gamma\left(\delta+\frac{3}{2}\right).\end{aligned}$$

The following proof is similar to that of Proposition 2.4, so we have

$$\begin{aligned}\lim_{\epsilon\downarrow 0}\epsilon^{2\delta}\sum_{n=2}^{\infty}\frac{(\log n)^{\delta-1}}{n^2}\bigg|\int_{\epsilon\sqrt{n\log n}}^{\infty}2xP\big(|S_n|\geq x\big)dx\\ -\int_{\epsilon\sqrt{n\log n}}^{\infty}2xP\big(|N|\geq x/\sqrt{n}\big)dx\bigg|=0.\end{aligned}\tag{3.7}$$

The moment condition $EX_1^2(\log|X_1|)^{\delta}<\infty$ is used as follows, note the corresponding part of Σ_9, we have

$$\begin{aligned}&C\sum_{n>H(\epsilon)}\frac{(\log n)^{\delta-1}}{n}\int_{\epsilon\sqrt{n\log n}}^{\infty}2xP\big(|X_1|\geq x\big)dx\\
&\leq\ C\sum_{n>H(\epsilon)}(\log n)^{\delta}\int_0^{\infty}(y+\epsilon)P\big(|X_1|\geq(y+\epsilon)\sqrt{n\log n}/m\big)dy\\
&\leq\ CE\left(\int_0^{\infty}(y+\epsilon)\sum_{n>H(\epsilon)}(\log n)^{\delta}I(\sqrt{n\log n}\leq m|X_1|/(y+\epsilon)dy\right)\\
&\leq\ CE\left(\int_0^{\infty}\frac{X_1^2}{(y+\epsilon)}(\log|X_1|-\log(y+\epsilon))^{\delta-1}I(y+\epsilon\leq m|X_1|)dy\right)\\
&\leq\ CEX_1^2\big((\log|X_1|)^{\delta}-(\log\epsilon)^{\delta}\big),\end{aligned}\tag{3.8}$$

we then complete the proof of (3.5). Finally, observe that

$$P\left(\sup_{0\le t\le 1}|W(t)|\ge x\right)\le 2P(|N|\ge x),$$

along the same proof lines of (3.5), one can complete the proof of (3.6).

Proof of Theorem 1.2. By virtue of Propositions 3.1 and 3.2, one can obtain (1.6) and (1.7). With the help of (2.27), the sufficient part is obvious.

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