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Some results on variational inequalities and generalized equilibrium problems with applications

XIAOLONG QIN[1], SUN YOUNG CHO[2] and SHIN MIN KANG[3]

[1]Department of Mathematics, Hangzhou Normal University, Hangzhou 310036, China
[2]Department of Mathematics, Gyeongsang National University, Jinju 660-701, Korea
[3]Department of Mathematics and the RINS, Gyeongsang National University
Jinju 660-701, Korea

E-mail: smkang@gnu.ac.kr

Abstract. An iterative algorithm is considered for variational inequalities, generalized equilibrium problems and fixed point problems. Strong convergence of the proposed iterative algorithm is obtained in the framework Hilbert spaces.

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1 Introduction and preliminaries

Let H be a real Hilbert space, whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. Let C be a nonempty closed and convex subset of H and P_C be the projection of H onto C.

Let f, S, A, T be nonlinear mappings. Recall the following definitions:

(1) $f : C \to C$ is said to be α-contractive if there exists a constant $\alpha \in (0, 1)$ such that

$$\| fx - fy \| \le \alpha \| x - y \|, \quad \forall x, y \in C.$$

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(2) $S : C \to C$ is said to be nonexpansive if

$$\|Sx - Sy\| \leq \|x - y\|, \quad \forall x, y \in C.$$

Throughout this paper, we use $F(S)$ to denote the set of fixed points of the mapping S.

(3) $A : C \to H$ is said to be monotone if

$$\langle Ax - Ay, x - y\rangle \geq 0, \quad \forall x, y \in C.$$

(4) $A : C \to H$ is said to be inverse-strongly monotone if there exists $\delta > 0$ such that

$$\langle x - y, Ax - Ay\rangle \geq \delta\|Ax - Ay\|^2, \quad \forall x, y \in C.$$

Such a mapping A is also called δ-inverse-strongly monotone. We know that if $S\colon C \to C$ is nonexpansive, then $A = I - S$ is $\frac{1}{2}$-inverse-strongly monotone; see [1, 21] for more details.

(5) A set-valued mapping $T\colon H \to 2^H$ is said to be monotone if for all $x, y \in H$, $f \in Tx$ and $g \in Ty \Longrightarrow \langle x - y, f - g\rangle \geq 0$. A monotone mapping $T\colon H \to 2^H$ is maximal if the graph of $G(T)$ of T is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T is maximal if and only if for $(x, f) \in H \times H$, $\langle x - y, f - g\rangle \geq 0$ for every $(y, g) \in G(T)$ implies that $f \in Tx$. Let A be a monotone mapping of C into H and let $N_C v$ be the normal cone to C at $v \in C$, i.e., $N_C v = \{w \in H : \langle v - u, w\rangle \geq 0,\ \forall u \in C\}$ and define

$$Tv = \begin{cases} Av + N_C v, & v \in C, \\ \emptyset, & v \notin C. \end{cases} \tag{Δ}$$

Then T is maximal monotone and $0 \in Tv$ if and only if $\langle Av, u - v\rangle \geq 0$, $\forall u \in C$; see [16] for more details.

Recall that the classical variational inequality is to find an $x \in C$ such that

$$\langle Ax, y - x\rangle \geq 0, \quad \forall y \in C. \tag{1.1}$$

In this paper, we use $VI(C, A)$ to denote the solution set of the variational inequality (1.1). For given $z \in H$ and $u \in C$, we see that

$$\langle u - z, v - u \rangle \geq 0, \quad \forall v \in C \tag{1.2}$$

holds if and only if $u = P_C z$. It is known that projection operator P_C satisfies

$$\langle P_C x - P_C y, x - y \rangle \geq \|P_C x - P_C y\|^2, \quad \forall x, y \in H.$$

One can see that the variational inequality (1.1) is equivalent to a fixed point problem. An element $u \in C$ is a solution of the variational inequality (1.1) if and only if $u \in C$ is a fixed point of the mapping $P_C(I - \lambda A)u$, where $\lambda > 0$ is a constant and I is the identity mapping. This can be seen from the following. $u \in C$ is a solution of the variational inequality (1.1), this is,

$$\langle Au, y - u \rangle \geq 0, \quad \forall y \in C,$$

which is equivalent to

$$\langle (u - \lambda Au) - u, u - y \rangle \geq 0, \quad \forall y \in C,$$

where $\lambda > 0$ is a constant. This implies from (1.2) that $u = P_C(I - \lambda A)u$, that is, u is a fixed point of the mapping $P_C(I - \lambda A)$. This alternative equivalent formulation has played a significant role in the studies of the variational inequalities and related optimization problems.

Let $A : C \to H$ be a δ-inverse-strongly monotone mapping and F be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ denotes the set of real numbers. We consider the following generalized equilibrium problem:

$$\text{Find } x \in C \text{ such that } F(x, y) + \langle Ax, y - x \rangle \geq 0, \quad \forall y \in C. \tag{1.3}$$

In this paper, the set of such an $x \in C$ is denoted by $EP(F, A)$, i.e.,

$$EP(F, A) = \{x \in C : F(x, y) + \langle Ax, y - x \rangle \geq 0, \ \forall y \in C\}.$$

Next, we give some special cases of the generalized equilibrium problem (1.3).

(I) If $A \equiv 0$, the zero mapping, then the problem (1.3) is reduced to the following equilibrium problem:

$$\text{Find } x \in C \text{ such that } F(x, y) \geq 0, \quad \forall y \in C. \tag{1.4}$$

In this paper, the set of such an $x \in C$ is denoted by $EP(F)$, i.e.,

$$EP(F) = \{x \in C : F(x, y) \geq 0, \ \forall y \in C\}.$$

(II) If $F \equiv 0$, then the problem (1.3) is reduced to the classical variational inequality (1.1).

In 2005, Iiduka and Takahashi [8] considered the classical variational inequality (1.1) and a single nonexpansive mapping. To be more precise, they obtained the following results.

Theorem IT. *Let C be a closed convex subset of a real Hilbert space H. Let A be an α-inverse-strongly monotone mapping of C into H and S be a nonexpansive mapping of C into itself such that $F(S) \cap VI(C, A) \neq \emptyset$. Suppose that $x_1 = x \in C$ and $\{x_n\}$ is given by*

$$x_{n+1} = \alpha_n x + (1 - \alpha_n) S P_C(x_n - \lambda_n A x_n), \ \forall n \geq 1, \tag{1.5}$$

where $\{\alpha_n\}$ is a sequence in $[0, 1)$ and $\{\lambda_n\}$ is a sequence in $[0, 2\alpha]$. If $\{\alpha_n\}$ and $\{\lambda_n\}$ are chosen so that $\{\lambda_n\} \subset [a, b]$ for some a, b with $0 < a < b < 2\alpha$,

$$\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty, \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty \text{ and } \sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty,$$

then $\{x_n\}$ converges strongly to $P_{F(S)\cap VI(C,A)}x$.

On the other hand, we see that the problem (1.3) is very general in the sense that it includes, as special cases, optimization problems, variational inequalities, mini-max problems, the Nash equilibrium problem in noncooperative games and others; see, for instance, [2, 5, 9]. Recently, many authors considered iterative methods for the problems (1.3) and (1.4), see [3-7, 11-15, 18, 20, 22, 24] for more details.

To study the equilibrium problems (1.3) and (1.4), we may assume that F satisfies the following conditions:

(A1) $F(x, x) = 0$ for all $x \in C$;

(A2) F is monotone, i.e., $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;

(A3) for each $x, y, z \in C$,

$$\limsup_{t \downarrow 0} F(tz + (1-t)x, y) \leq F(x, y);$$

(A4) for each $x \in C$, $y \mapsto F(x, y)$ is convex and weakly lower semi-continuous.

Put $\mathcal{F}(x, y) = F(x, y) + \langle Ax, y - x \rangle$ for each $x, y \in C$. It is not hard to see that $\mathcal{F}$ also confirms (A1)-(A4).

In 2007, Takahashi and Takahashi [20] introduced the following iterative method

$$\begin{cases} F(y_n, u) + \dfrac{1}{r_n}\langle u - y_n, y_n - x_n \rangle \geq 0, & \forall u \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Ty_n, & n \geq 1, \end{cases} \tag{1.6}$$

where f is a α-contraction, T is a nonexpansive mapping. They considered the problem of approximating a common element of the set of fixed points of a single nonexpansive mapping and the set of solutions of the equilibrium problem (1.4). Strong convergence theorems of the iterative algorithm (1.6) are established in a real Hilbert space.

Recently, Takahashi and Takahashi [22] further considered the generalized equilibrium problem (1.3). They obtained the following result in a real Hilbert space.

Theorem TT. *Let C be a closed convex subset of a real Hilbert space H and $F : C \times C \to \mathbb{R}$ be a bi-function satisfying* (A1)-(A4). *Let A be an α-inverse-strongly monotone mapping of C into H and S be a non-expansive mapping of C into itself such that $F(S) \cap EP(F, A) \neq \emptyset$. Let $u \in C$ and $x_1 \in C$ and let $\{z_n\} \subset C$ and $\{x_n\} \subset C$ be sequences generated by*

$$\begin{cases} F(z_n, y) + \langle Ax_n, y - z_n \rangle + \dfrac{1}{r}\langle y - z_n, z_n - x_n \rangle \geq 0, & \forall y \in C, \\ x_{n+1} = \beta_n x_n + (1-\beta_n)S[\alpha_n u + (1-\alpha_n)z_n], & \forall n \geq 1, \end{cases} \tag{1.7}$$

where $\{\alpha_n\} \subset [0, 1]$, $\{\beta_n\} \subset [0, 1]$ and $\{r_n\} \subset [0, 2\alpha]$ satisfy

$$0 < c \leq \beta_n \leq d < 1, \quad 0 < a \leq \lambda_n \leq b < 2\alpha,$$

$$\lim_{n\to\infty}(\lambda_n - \lambda_{n+1}) = 0, \ \lim_{n\to\infty}\alpha_n = 0, \ and \ \sum_{n=1}^{\infty}\alpha_n = \infty.$$

Then, $\{x_n\}$ *converges strongly to* $z = P_{F(S)\cap EP(F,A)}u$.

Very recently, Chang, Lee and Chan [5] introduced a new iterative method for solving equilibrium problem (1.4), variational inequality (1.1) and the fixed point problem of nonexpansive mappings in the framework of Hilbert spaces. More precisely, they proved the following theorem.

Theorem CLC. *Let H be a real Hilbert space, C be a nonempty closed convex subset of H and F be a bifunction satisfying the conditions* (A1)-(A4). *Let $A : C \to H$ be an α-inverse-strongly monotone mapping and $\{S_i : C \to C\}$ be a family of infinitely nonexpansive mappings with $F \cap VI(C, A) \cap EP(F) \neq \emptyset$, where $F := \cap_{i=1}^{\infty} F(S_i)$ and $f : C \to C$ be a ξ-contractive mapping. Let $\{x_n\}$, $\{y_n\}$ $\{k_n\}$ and $\{u_n\}$ be sequences defined by*

$$\begin{cases} F(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, & \forall y \in C, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n W_n k_n, & \forall n \geq 1, \\ k_n = P_C(y_n - \lambda_n A y_n), \\ y_n = P_C(u_n - \lambda_n A u_n), \end{cases} \tag{1.8}$$

where $\{W_n : C \to C\}$ is the sequence defined by (1.9), $\{\alpha_n\}$, $\{\beta_n\}$ *and* $\{\gamma_n\}$ *are sequences in* $[0, 1]$, $\{\lambda_n\}$ *is a sequence in* $[a, b] \subset (0, 2\alpha)$ *and* $\{r_n\}$ *is a sequence in* $(0, \infty)$. *If the following conditions are satisfied:*

(1) $\alpha_n + \beta_n + \gamma_n = 1$;

(2) $\lim_{n\to\infty}\alpha_n = 0$*;* $\sum_{n=1}^{\infty}\alpha_n = \infty$*;*

(3) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$;

(4) $\liminf_{n\to\infty} r_n > 0$*;* $\sum_{n=1}^{\infty}|r_{n+1} - r_n| < \infty$;

(5) $\lim_{n\to\infty}|\lambda_{n+1} - \lambda_n| = 0$*,*

then $\{x_n\}$ *and* $\{u_n\}$ *converge strongly to* $z \in F \cap VI(C, A) \cap EP(F)$.

In this paper, motivated and inspired by the research going on in this direction, we introduce a general iterative method for finding a common element of the set of solutions of generalized equilibrium problems, the set of solutions of variational inequalities, and the set of common fixed points of a family of nonexpansive mappings in the framework of Hilbert spaces. The results presented in this paper improve and extend the corresponding results of Ceng and Yao [3, 4], Chang Lee and Chan [5], Iiduka and Takahashi [8], Qin, Shang and Zhou [12], Su, Shang and Qin [18], Takahashi and Takahashi [20, 22], Yao and Yao [25] and many others.

In order to prove our main results, we need the following definitions and lemmas.

A space X is said to satisfy Opial condition [10] if for each sequence $\{x_n\}$ in X which converges weakly to point $x \in X$, we have

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|, \quad \forall y \in X, y \neq x.$$

Lemma 1.1 ([2]). *Let C be a nonempty closed convex subset of H and $F : C \times C \to \mathbb{R}$ be a bifunction satisfying* (A1)-(A4)*. Then, for any $r > 0$ and $x \in H$, there exists $z \in C$ such that*

$$F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C.$$

Lemma 1.2 ([2], [7]). *Suppose that all the conditions in Lemma* 1.1 *are satisfied. For any give $r > 0$ define a mapping $T_r : H \to C$ as follows:*

$$T_r x = \left\{ z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \ \forall y \in C \right\}, \ \forall x \in H,$$

then the following conclusions hold:

(1) *T_r is single-valued;*

(2) *T_r is firmly nonexpansive, i.e., for any $x, y \in H$,*

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle;$$

(3) *$F(T_r) = EP(F)$;*

(4) *$EP(F)$ is closed and convex.*

Lemma 1.3 ([23]). *Assume that $\{\alpha_n\}$ is a sequence of nonnegative real numbers such that*

$$\alpha_{n+1} \leq (1 - \gamma_n)\alpha_n + \delta_n,$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence such that

(1) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(2) $\limsup_{n\to\infty} \delta_n/\gamma_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} \alpha_n = 0$.

Definition 1.4 ([19]). Let $\{S_i : C \to C\}$ be a family of infinitely nonexpansive mappings and $\{\gamma_i\}$ be a nonnegative real sequence with $0 \leq \gamma_i < 1, \forall i \geq 1$. For $n \geq 1$ define a mapping $W_n : C \to C$ as follows:

$$\begin{aligned}
&U_{n,n+1} = I,\\
&U_{n,n} = \gamma_n S_n U_{n,n+1} + (1 - \gamma_n)I,\\
&U_{n,n-1} = \gamma_{n-1} S_{n-1} U_{n,n} + (1 - \gamma_{n-1})I,\\
&\vdots\\
&U_{n,k} = \gamma_k S_k U_{n,k+1} + (1 - \gamma_k)I,\\
&U_{n,k-1} = \gamma_{k-1} S_{k-1} U_{n,k} + (1 - \gamma_{k-1})I,\\
&\vdots\\
&U_{n,2} = \gamma_2 S_2 U_{n,3} + (1 - \gamma_2)I,\\
&W_n = U_{n,1} = \gamma_1 S_1 U_{n,2} + (1 - \gamma_1)I.
\end{aligned} \tag{1.9}$$

Such a mapping W_n is nonexpansive from C to C and it is called a W-mapping generated by $S_n, S_{n-1}, \ldots, S_1$ and $\gamma_n, \gamma_{n-1}, \ldots, \gamma_1$.

Lemma 1.5 ([19]). *Let C be a nonempty closed convex subset of a Hilbert space H, $\{S_i : C \to C\}$ be a family of infinitely nonexpansive mappings with $\cap_{i=1}^{\infty} F(S_i) \neq \emptyset$ and $\{\gamma_i\}$ be a real sequence such that $0 < \gamma_i \leq l < 1$, $\forall i \geq 1$. Then*

(1) *W_n is nonexpansive and $F(W_n) = \cap_{i=1}^{\infty} F(S_i)$, for each $n \geq 1$;*

(2) *for each* $x \in C$ *and for each positive integer* k, *the limit* $\lim_{n\to\infty} U_{n,k}$ *exists.*

(3) *the mapping* $W : C \to C$ *defined by*

$$Wx := \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x, \quad x \in C,$$

is a nonexpansive mapping satisfying $F(W) = \cap_{i=1}^{\infty} F(S_i)$ *and it is called the* W*-mapping generated by* $S_1, S_2, \ldots$ *and* $\gamma_1, \gamma_2, \ldots$.

Lemma 1.6 **([5]).** *Let* C *be a nonempty closed convex subset of a Hilbert space* H, $\{S_i : C \to C\}$ *be a family of infinitely nonexpansive mappings with* $\cap_{i=1}^{\infty} F(S_i) \neq \emptyset$ *and* $\{\gamma_i\}$ *be a real sequence such that* $0 < \gamma_i \leq l < 1$, $\forall i \geq 1$. *If* K *is any bounded subset of* C, *then*

$$\lim_{n\to\infty} \sup_{x\in K} \|Wx - W_n x\| = 0.$$

Throughout this paper, we always assume that $0 < \gamma_i \leq l < 1, \forall i \geq 1$.

Lemma 1.7 **([17]).** *Let* $\{x_n\}$ *and* $\{y_n\}$ *be bounded sequences in a Hilbert space* H *and* $\{\beta_n\}$ *be a sequence in* $[0, 1]$ *with*

$$0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1.$$

Suppose that $x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n$ *for all* $n \geq 0$ *and*

$$\limsup_{n\to\infty} \big(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|\big) \leq 0.$$

Then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

2 Main results

Theorem 2.1. *Let* C *be a nonempty closed convex subset of a Hilbert space* H *and* F *be a bifunction from* $C \times C$ *to* $\mathbb{R}$ *which satisfies* (A1)-(A4). *Let* $A_1 : C \to H$ *be a* δ_1*-inverse-strongly monotone mapping,* $A_2 : C \to H$ *be a* δ_2*-inverse-strongly monotone mapping,* $A_3 : C \to H$ *be a* δ_3*-inverse-strongly monotone mapping and* $\{S_i : C \to C\}$ *be a family of infinitely nonexpansive mappings.*

Assume that $\Omega := FP \cap EP(F, A_3) \cap VI \neq \emptyset$*, where* $FP = \cap_{i=1}^{\infty} F(S_i)$ *and* $VI = VI(C, A_1) \cap VI(C, A_2)$*. Let* $f : C \to C$ *be an* α*-contraction. Let* $x_1 \in C$ *and* $\{x_n\}$ *be a sequence generated by*

$$\begin{cases} F(u_n, y) + \langle A_3 x_n, y - u_n \rangle + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C, \\ z_n = P_C(u_n - \lambda_n A_2 u_n), \\ y_n = P_C(z_n - \eta_n A_1 z_n), \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n W_n y_n, \quad \forall n \geq 1, \end{cases} \tag{2.1}$$

where $\{W_n : C \to C\}$ *is the sequence generated in* (1.9)*,* $\{\alpha_n\}$*,* $\{\beta_n\}$ *and* $\{\gamma_n\}$ *are sequences in* $(0, 1)$ *such that* $\alpha_n + \beta_n + \gamma_n = 1$ *for each* $n \geq 1$ *and* $\{r_n\}$*,* $\{\lambda_n\}$ *and* $\{\eta_n\}$ *are positive number sequences. Assume that the above control sequences satisfy the following restrictions:*

(R1) $0 < a \leq \eta_n \leq b < 2\delta_1$, $0 < a' \leq \lambda_n \leq b' < 2\delta_2$, $0 < \bar{a} \leq r_n \leq \bar{b} < 2\delta_3$, $\forall n \geq 1$;

(R2) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(R3) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(R4) $\lim_{n\to\infty}(\lambda_n - \lambda_{n+1}) = \lim_{n\to\infty}(\eta_n - \eta_{n+1}) = \lim_{n\to\infty}(r_n - r_{n+1}) = 0$.

Then the sequence $\{x_n\}$ *converges strongly to* $z \in \Omega$*, which solves uniquely the following variational inequality:*

$$\langle (I - f)z, z - x \rangle \leq 0, \quad \forall x \in \Omega.$$

Proof. First, we show, for each $n \geq 1$, that the mappings $I - \eta_n A_1$, $I - \lambda_n A_2$ and $I - r_n A_3$ are nonexpansive. Indeed, for $\forall x, y \in C$, we obtain from the restriction (R1) that

$$\begin{aligned} &\|(I - \eta_n A_1)x - (I - \eta_n A_1)y\|^2 \\ &= \|(x - y) - \eta_n(A_1 x - A_1 y)\|^2 \\ &= \|x - y\|^2 - 2\eta_n \langle x - y, A_1 x - A_1 y \rangle + \eta_n^2 \|A_1 x - A_1 y\|^2 \\ &\leq \|x - y\|^2 - 2\eta_n \delta_1 \|A_1 x - A_1 y\|^2 + \eta_n^2 \|A_1 x - A_1 y\|^2 \\ &= \|x - y\|^2 + \eta_n(\eta_n - 2\delta_1)\|A_1 x - A_1 y\|^2 \\ &\leq \|x - y\|^2, \end{aligned}$$

which implies that the mapping $I - \eta_n A_1$ is nonexpansive, so are $I - \lambda_n A_2$ and $I - r_n A_3$ for each $n \geq 1$. Note that u_n can be re-written as $u_n = T_{r_n}(I - r_n A_3)x_n$ for each $n \geq 1$. Take $x^* \in \Omega$. Noticing that $x^* = P_C(I - \eta_n A_1)x^* = P_C(I - \lambda_n A_2)x^* = T_{r_n}(I - r_n A_3)x^*$, we have

$$\|u_n - x^*\| = \|T_{r_n}(I - r_n A_3)x_n - T_{r_n}(I - r_n A_3)x^*\| \leq \|x_n - x^*\|. \tag{2.2}$$

On the other hand, we have

$$\begin{aligned} \|z_n - x^*\| &= \|P_C(u_n - \lambda_n A_2 u_n) - P_C(x^* - \lambda_n A_2 x^*)\| \\ &\leq \|(u_n - \lambda_n A_2 u_n) - (x^* - \lambda_n A_2 x^*)\| \\ &\leq \|u_n - x^*\|. \end{aligned} \tag{2.3}$$

It follows from (2.2) and (2.3) that

$$\|z_n - x^*\| \leq \|x_n - x^*\|, \tag{2.4}$$

which yields that

$$\begin{aligned} \|y_n - x^*\| &= \|P_C(z_n - \eta_n A_1 z_n) - P_C(x^* - \eta_n A_1 x^*)\| \\ &\leq \|(z_n - \eta_n A_1 z_n) - (x^* - \eta_n A_1 x^*)\| \\ &\leq \|z_n - x^*\| \\ &\leq \|x_n - x^*\|. \end{aligned} \tag{2.5}$$

From the algorithm (2.1) and (2.5), we arrive at

$$\begin{aligned} \|x_{n+1} - x^*\| &= \|\alpha_n f(x_n) + \beta_n x_n + \gamma_n W_n y_n - x^*\| \\ &\leq \alpha_n \|f(x_n) - x^*\| + \beta_n \|x_n - x^*\| + \gamma_n \|W_n y_n - x^*\| \\ &\leq \alpha_n \|f(x_n) - f(x^*)\| + \alpha_n \|f(x^*) - x^*\| \\ &\quad + \beta_n \|x_n - x^*\| + \gamma_n \|y_n - x^*\| \\ &\leq \alpha\alpha_n \|x_n - x^*\| + \alpha_n \|f(x^*) - x^*\| \\ &\quad + \beta_n \|x_n - x^*\| + \gamma_n \|x_n - x^*\| \\ &= [1 - \alpha_n(1 - \alpha)]\|x_n - x^*\| + \alpha_n \|f(x^*) - x^*\|. \end{aligned}$$

By simple inductions, we obtain that

$$\|x_n - x^*\| \leq \max\left\{\|x_1 - x^*\|, \frac{\|f(x^*) - x^*\|}{1-\alpha}\right\},$$

which gives that the sequence $\{x_n\}$ is bounded, so are $\{y_n\}$, $\{z_n\}$ and $\{u_n\}$. Without loss of generality, we can assume that there exists a bounded set $K \subset C$ such that

$$x_n, y_n, z_n, u_n \in K, \quad \forall n \geq 1. \tag{2.6}$$

Notice that $u_{n+1} = T_{r_{n+1}}(I - r_{n+1}A_3)x_{n+1}$ and $u_n = T_{r_n}(I - r_n A_3)x_n$, we see from Lemma 1.2 that

$$F(u_{n+1}, y) + \frac{1}{r_{n+1}}\langle y - u_{n+1}, u_{n+1} - (I - r_{n+1}A_3)x_{n+1}\rangle \geq 0, \quad \forall y \in C, \tag{2.7}$$

and

$$F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - (I - r_n A_3)x_n\rangle \geq 0, \quad \forall y \in C. \tag{2.8}$$

Let $y = u_n$ in (2.7) and $y = u_{n+1}$ in (2.8). By adding up these two inequalities and using the assumption (R2), we obtain that

$$\left\langle u_{n+1} - u_n, \frac{u_n - (I - r_n A_3)x_n}{r_n} - \frac{u_{n+1} - (I - r_{n+1}A_3)x_{n+1}}{r_{n+1}}\right\rangle \geq 0.$$

Hence, we have

$$\Big\langle u_{n+1} - u_n, u_n - u_{n+1} + u_{n+1} - (I - r_n A_3)x_n$$

$$- \frac{r_n}{r_{n+1}}(u_{n+1} - (I - r_{n+1}A_3)x_{n+1})\Big\rangle \geq 0.$$

This implies that

$$\begin{aligned}
\|u_{n+1} - u_n\|^2 &\leq \Big\langle u_{n+1} - u_n, (I - r_{n+1}A_3)x_{n+1} - (I - r_n A_3)x_n \\
&\quad + \left(1 - \frac{r_n}{r_{n+1}}\right)(u_{n+1} - (I - r_{n+1}A_3)x_{n+1})\Big\rangle \\
&\leq \|u_{n+1} - u_n\|\Big(\|(I - r_{n+1}A_3)x_{n+1} - (I - r_n A_3)x_n\| \\
&\quad + |1 - \frac{r_n}{r_{n+1}}|\|u_{n+1} - (I - r_{n+1}A_3)x_{n+1})\|\Big).
\end{aligned}$$

It follows that

$$
\begin{aligned}
&\|u_{n+1} - u_n\| \\
&\leq \|(I - r_{n+1}A_3)x_{n+1} - (I - r_nA_3)x_n\| \\
&\quad + \frac{|r_{n+1} - r_n|}{r_{n+1}}\|u_{n+1} - (I - r_{n+1}A_3)x_{n+1}\| \\
&= \|(I - r_{n+1}A_3)x_{n+1} - (I - r_{n+1}A_3)x_n \\
&\quad + (I - r_{n+1}A_3)x_n - (I - r_nA_3)x_n\| \\
&\quad + \frac{|r_{n+1} - r_n|}{r_{n+1}}\|u_{n+1} - (I - r_{n+1}A_3)x_{n+1}\| \\
&\leq \|x_{n+1} - x_n\| + |r_{n+1} - r_n|M_1,
\end{aligned} \tag{2.9}
$$

where M_1 is an appropriate constant such that

$$
M_1 = \sup_{n\geq 1}\left\{\|A_3x_n\| + \frac{\|u_{n+1} - (I - r_{n+1}A_3)x_{n+1}\|}{\bar{a}}\right\}.
$$

From the nonexpansivity of P_C, we also have

$$
\begin{aligned}
&\|z_{n+1} - z_n\| \\
&= \|P_C(u_{n+1} - \lambda_{n+1}A_2u_{n+1}) - P_C(u_n - \lambda_nA_2u_n)\| \\
&\leq \|u_{n+1} - \lambda_{n+1}A_2u_{n+1} - (u_n - \lambda_nA_2u_n)\| \\
&= \|(I - \lambda_{n+1}A_2)u_{n+1} - (I - \lambda_{n+1}A_2)u_n + (\lambda_n - \lambda_{n+1})A_2u_n\| \\
&\leq \|u_{n+1} - u_n\| + |\lambda_n - \lambda_{n+1}|\|A_2u_n\|.
\end{aligned} \tag{2.10}
$$

Substituting (2.9) into (2.10), we arrive at

$$
\|z_{n+1} - z_n\| \leq \|x_{n+1} - x_n\| + |r_{n+1} - r_n|M_1 + |\lambda_n - \lambda_{n+1}|\|A_2u_n\|. \tag{2.11}
$$

In a similar way, we can obtain that

$$
\|y_{n+1} - y_n\| \leq \|z_{n+1} - z_n\| + |\eta_n - \eta_{n+1}|\|A_1z_n\|. \tag{2.12}
$$

Combining (2.11) with (2.12), we see that

$$
\|y_{n+1}-y_n\| \leq \|x_{n+1}-x_n\|+(|r_{n+1}-r_n|+|\lambda_n-\lambda_{n+1}|+|\eta_n-\eta_{n+1}|)M_2, \tag{2.13}
$$

where M_2 is an appropriate constant such that

$$M_2 = \max\left\{\sup_{n\geq 1}\{\|A_1 z_n\|\}, \sup_{n\geq 1}\{\|A_2 u_n\|\}, M_1\right\}.$$

Letting

$$x_{n+1} = (1-\beta_n)v_n + \beta_n x_n, \quad \forall n \geq 1, \tag{2.14}$$

we see that

$$\begin{aligned} v_{n+1} - v_n &= \frac{\alpha_{n+1} f(x_{n+1}) + \gamma_{n+1} W_{n+1} y_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n f(x_n) + \gamma_n W_n y_n}{1-\beta_n} \\ &= \frac{\alpha_{n+1} f(x_{n+1})}{1-\beta_{n+1}} + \frac{(1-\alpha_{n+1}-\beta_{n+1}) W_{n+1} y_{n+1}}{1-\beta_{n+1}} \\ &\quad - \left(\frac{\alpha_n f(x_n)}{1-\beta_n} + \frac{(1-\alpha_n-\beta_n) W_n y_n}{1-\beta_n}\right) \\ &= \frac{\alpha_{n+1}}{1-\beta_{n+1}}\big(f(x_{n+1}) - W_{n+1} y_{n+1}\big) - \frac{\alpha_n}{1-\beta_n}\big(f(x_n) - W_n y_n\big) \\ &\quad + W_{n+1} y_{n+1} - W_n y_n. \end{aligned}$$

It follows that

$$\begin{aligned} \|v_{n+1} - v_n\| \leq{} & \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|f(x_{n+1}) - W_{n+1} y_{n+1}\| \\ & + \frac{\alpha_n}{1-\beta_n}\|f(x_n) - W_n y_n\| + \|W_{n+1} y_{n+1} - W_n y_n\|. \end{aligned} \tag{2.15}$$

On the other hand, we have

$$\begin{aligned} & \|W_{n+1} y_{n+1} - W_n y_n\| \\ & = \|W_{n+1} y_{n+1} - W y_{n+1} + W y_{n+1} - W y_n + W y_n - W_n y_n\| \\ & \leq \|W_{n+1} y_{n+1} - W y_{n+1}\| + \|W y_{n+1} - W y_n\| + \|W y_n - W_n y_n\| \\ & \leq \sup_{x\in K}\big\{\|W_{n+1} x - W x\| + \|W x - W_n x\|\big\} + \|y_{n+1} - y_n\|, \end{aligned} \tag{2.16}$$

where K is the bounded subset of C defined by (2.6). Substituting (2.13) into (2.16), we arrive at

$$\begin{aligned} \|W_{n+1} y_{n+1} - W_n y_n\| \leq{} & \sup_{x\in K}\big\{\|W_{n+1} x - W x\| + |W x - W_n x\|\big\} + \|x_{n+1} - x_n\| \\ & + \big(|r_{n+1} - r_n| + |\lambda_n - \lambda_{n+1}| + |\eta_n - \eta_{n+1}|\big) M_2, \end{aligned}$$

which combines with (2.15) yields that

$$\begin{aligned}
&\|v_{n+1} - v_n\| - \|x_{n+1} - x_n\| \\
&\le \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|f(x_{n+1}) - W_{n+1}y_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|f(x_n) - W_n y_n\| \\
&\quad + \sup_{x\in K}\big\{\|W_{n+1}x - Wx\| + \|Wx - W_n x\|\big\} \\
&\quad + \big(|r_{n+1} - r_n| + |\lambda_n - \lambda_{n+1}| + |\eta_n - \eta_{n+1}|\big)M_2.
\end{aligned}$$

In view of the restriction (R2), (R3) and (R4), we obtain from Lemma 1.6 that

$$\limsup_{n\to\infty}\big(\|v_{n+1} - v_n\| - \|x_{n+1} - x_n\|\big) \le 0.$$

Hence, we obtain from Lemma 1.7 that

$$\lim_{n\to\infty}\|v_n - x_n\| = 0.$$

In view of (2.14), we have

$$\|x_{n+1} - x_n\| = (1-\beta_n)\|v_n - x_n\|.$$

Thanks to the restriction (R3), we see that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{2.17}$$

For any $x^* \in \Omega$, we see that

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|\alpha_n(f(x_n) - x^*) + \beta_n(x_n - x^*) + \gamma_n(W_n y_n - x^*)\|^2 \\
&\le \alpha_n\|f(x_n) - x^*\|^2 + \beta_n\|x_n - x^*\|^2 + \gamma_n\|W_n y_n - x^*\|^2 \\
&\le \alpha_n\|f(x_n) - x^*\|^2 + \beta_n\|x_n - x^*\|^2 + \gamma_n\|y_n - x^*\|^2.
\end{aligned} \tag{2.18}$$

Note that

$$\begin{aligned}
\|y_n - x^*\|^2 &= \|P_C(z_n - \eta_n A_1 z_n) - x^*\|^2 \\
&\le \|(I - \eta_n A_1)z_n - (I - \eta_n A_1)x^*\|^2 \\
&= \|(z_n - x^*) - \eta_n(A_1 z_n - A_1 x^*)\|^2 \\
&= \|z_n - x^*\|^2 - 2\eta_n\langle z_n - x^*, A_1 z_n - A_1 x^*\rangle \\
&\quad + \eta_n^2\|A_1 z_n - A_1 x^*\|^2 \\
&\le \|z_n - x^*\|^2 - 2\eta_n\delta_1\|A_1 z_n - A_1 x^*\|^2 \\
&\quad + \eta_n^2\|A_1 z_n - A_1 x^*\|^2 \\
&= \|x_n - x^*\|^2 + \eta_n(\eta_n - 2\delta_1)\|A_1 z_n - A_1 x^*\|^2.
\end{aligned} \tag{2.19}$$

Substituting (2.19) into (2.18), we arrive at

$$\|x_{n+1}-x^*\|^2 \le \alpha_n\|f(x_n)-x^*\|^2+\|x_n-x^*\|^2+\gamma_n\eta_n(\eta_n-2\delta_1)\|A_1z_n-A_1x^*\|^2.$$

This implies that

$$\begin{aligned}&\gamma_n\eta_n(2\delta_1-\eta_n)\|A_1z_n-A_1x^*\|^2\\ &\le \alpha_n\|f(x_n)-x^*\|^2+\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2\\ &\le \alpha_n\|f(x_n)-x^*\|^2+(\|x_n-x^*\|+\|x_{n+1}-x^*\|)\|x_n-x_{n+1}\|.\end{aligned}$$

By virtue of the restrictions (R1) and (R2), we obtain from (2.17) that

$$\lim_{n\to\infty}\|A_1z_n-A_1x^*\|=0. \tag{2.20}$$

Next, we show that

$$\lim_{n\to\infty}\|A_2u_n-A_2x^*\|=0. \tag{2.21}$$

Indeed, by using (2.18), we obtain that

$$\|x_{n+1}-x^*\|^2\le\alpha_n\|f(x_n)-x^*\|^2+\beta_n\|x_n-x^*\|^2+\gamma_n\|z_n-x^*\|^2. \tag{2.22}$$

On the other hand, we have

$$\begin{aligned}\|z_n-x^*\|^2&=\|P_C(u_n-\lambda_nA_2u_n)-x^*\|^2\\ &\le\|(I-\lambda_nA_2)u_n-(I-\lambda_nA_2)x^*\|^2\\ &=\|(u_n-x^*)-\lambda_n(A_2u_n-A_ux^*)\|^2\\ &=\|u_n-x^*\|^2-2\lambda_n\langle u_n-x^*,A_2u_n-A_2x^*\rangle+\lambda_n^2\|A_2u_n-A_2x^*\|^2\\ &\le\|u_n-x^*\|^2-2\lambda_n\delta_2\|A_2u_n-A_2x^*\|^2+\lambda_n^2\|A_2u_n-A_2x^*\|^2\\ &=\|u_n-x^*\|^2+\lambda_n(\lambda_n-2\delta_2)\|A_2u_n-A_2x^*\|^2.\end{aligned} \tag{2.23}$$

Substituting (2.23) into (2.22), we arrive at

$$\|x_{n+1}-x^*\|^2\le\alpha_n\|f(x_n)-x^*\|^2+\|x_n-x^*\|^2+\gamma_n\lambda_n(\lambda_n-2\delta_2)\|A_2u_n-A_2x^*\|^2.$$

This in turn gives that

$$\begin{aligned}&\gamma_n\lambda_n(2\delta_2-\lambda_n)\|A_2u_n-A_2x^*\|^2\\ &\le\alpha_n\|f(x_n)-x^*\|^2+\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2\\ &\le\alpha_n\|f(x_n)-x^*\|^2+(\|x_n-x^*\|+\|x_{n+1}-x^*\|)\|x_n-x_{n+1}\|.\end{aligned}$$

In view of the restrictions (R1) and (R2), we obtain from (2.17) that (2.21) holds.

On the other hand, we see from (2.22) that

$$\|x_{n+1} - x^*\|^2 \le \alpha_n \|f(x_n) - x^*\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n \|u_n - x^*\|^2. \quad (2.24)$$

It follows that

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\le \alpha_n \|f(x_n) - x^*\|^2 + \beta_n \|x_n - x^*\|^2 \\ &\quad + \gamma_n \|x_n - x^* - r_n(A_3x_n - A_3x^*)\|^2 \\ &\le \alpha_n \|f(x_n) - x^*\|^2 + \beta_n \|x_n - x^*\|^2 \\ &\quad + \gamma_n(\|x_n - x^*\|^2 + r_n^2 \|A_3x_n - A_3x^*\|^2 \\ &\quad - 2r_n \langle A_3x_n - A_3x^*, x_n - x^*\rangle) \\ &\le \alpha_n \|f(x_n) - x^*\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n \|x_n - x^*\|^2 \\ &\quad - r_n\gamma_n(2\delta_3 - r_n)\|A_3x_n - A_3x^*\|^2.\end{aligned}$$

This implies that

$$\begin{aligned}&r_n\gamma_n(2\delta_3 - r_n)\|A_3x_n - A_3x^*\|^2 \\ &\le \alpha_n \|f(x_n) - x^*\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\ &\le \alpha_n \|f(x_n) - x^*\|^2 + (\|x_n - x^*\| + \|x_{n+1} - x^*\|)\|x_n - x_{n+1}\|.\end{aligned}$$

In view of the restrictions (R1), (R2) and (R3), we see from (2.17) that

$$\lim_{n\to\infty} \|A_3x_n - A_3x^*\| = 0. \quad (2.25)$$

On the other hand, we see from Lemma 1.2 that

$$\begin{aligned}\|u_n - x^*\|^2 &= \|T_{r_n}(I - r_nA_3)x_n - T_{r_n}(I - r_nA_3)x^*\|^2 \\ &\le \langle (I - r_nA_3)x_n - (I - r_nA_3)x^*, u_n - x^*\rangle \\ &= \frac{1}{2}(\|(I - r_nA_3)x_n - (I - r_nA_3)x^*\|^2 + \|u_n - x^*\|^2 \\ &\quad - \|(I - r_nA_3)x_n - (I - r_nA_3)x^* - (u_n - x^*)\|^2) \\ &\le \frac{1}{2}(\|x_n - x^*\|^2 + \|u_n - x^*\|^2 - \|x_n - u_n - r_n(A_3x_n - A_3x^*)\|^2) \\ &= \frac{1}{2}(\|x_n - x^*\|^2 + \|u_n - x^*\|^2 - \|x_n - u_n\|^2 - r_n^2\|A_3x_n - A_3x^*\|^2 \\ &\quad + 2r_n\langle A_3x_n - A_3x^*, x_n - u_n\rangle).\end{aligned}$$

This in turn implies that

$$\begin{aligned}\|u_n - x^*\|^2 &\le \|x_n - x^*\|^2 - \|x_n - u_n\|^2 - r_n^2\|A_3x_n - A_3x^*\|^2\\ &\quad + 2r_n\langle A_3x_n - A_3x^*, x_n - u_n\rangle\\ &\le \|x_n - x^*\|^2 - \|x_n - u_n\|^2\\ &\quad + 2r_n\|A_3x_n - A_3x^*\|\|x_n - u_n\|.\end{aligned} \tag{2.26}$$

Combining (2.24) with (2.26), we arrive at

$$\begin{aligned}\|x_{n+1} - x^*\|^2 \le \alpha_n\|f(x_n) - x^*\|^2 + \|x_n - x^*\|^2 - \gamma_n\|x_n - u_n\|^2\\ + 2r_n\|A_3x_n - A_3x^*\|\|x_n - u_n\|.\end{aligned}$$

It follows that

$$\begin{aligned}\gamma_n\|x_n - u_n\|^2 \le \alpha_n\|f(x_n) - x^*\|^2 + (\|x_n - x^*\| + \|x_{n+1} - x^*\|)\|x_n - x_{n+1}\|\\ + 2r_n\|A_3x_n - A_3x^*\|\|x_n - u_n\|.\end{aligned}$$

Thanks to the restrictions (R2) and (R3), we see from (2.17) and (2.25) that

$$\lim_{n\to\infty}\|x_n - u_n\| = 0. \tag{2.27}$$

In view of the firm nonexpansivity of P_C, we see that

$$\begin{aligned}\|z_n - x^*\|^2 &= \|P_C(I - \lambda_nA_2)u_n - P_C(I - \lambda_nA_2)x^*\|^2\\ &\le \langle (I - \lambda_nA_2)u_n - (I - \lambda_nA_2)x^*, z_n - x^*\rangle\\ &= \frac{1}{2}\Big(\|(I - \lambda_nA_2)u_n - (I - \lambda_nA_2)x^*\|^2 + \|z_n - x^*\|^2\\ &\quad - \|(I - \lambda_nA_2)u_n - (I - \lambda_nA_2)x^* - (z_n - x^*)\|^2\Big)\\ &\le \frac{1}{2}\Big(\|u_n - x^*\|^2 + \|z_n - x^*\|^2 - \|u_n - z_n - \lambda_n(A_2u_n - A_2x^*)\|^2\Big)\\ &= \frac{1}{2}\Big(\|u_n - x^*\|^2 + \|z_n - x^*\|^2 - \|u_n - z_n\|^2\\ &\quad + 2\lambda_n\langle u_n - z_n, A_2u_n - A_2x^*\rangle - \lambda_n^2\|A_2u_n - A_2x^*\|^2\Big),\end{aligned}$$

which implies that

$$\begin{aligned}\|z_n - x^*\|^2 &\le \|u_n - x^*\|^2 - \|u_n - z_n\|^2\\ &\quad + 2\lambda_n\langle u_n - z_n, A_2u_n - A_2x^*\rangle - \lambda_n^2\|A_2u_n - A_2x^*\|^2\\ &\le \|x_n - x^*\|^2 - \|u_n - z_n\|^2 + 2\lambda_n\|u_n - z_n\|\|A_2u_n - A_2x^*\|.\end{aligned} \tag{2.28}$$

Substituting (2.28) into (2.22), we arrive at

$$\|x_{n+1} - x^*\|^2 \le \alpha_n \|f(x_n) - x^*\|^2 + \|x_n - x^*\|^2 - \gamma_n \|u_n - z_n\|^2$$
$$+ 2\lambda_n \|u_n - z_n\| \|A_2 u_n - A_2 x^*\|,$$

from which it follows that

$$\begin{aligned}\gamma_n \|u_n - z_n\|^2 &\le \alpha_n \|f(x_n) - x^*\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\ &\quad + 2\lambda_n \|u_n - z_n\| \|A_2 u_n - A_2 x^*\| \\ &\le \alpha_n \|f(x_n) - x^*\|^2 + (\|x_n - x^*\| + \|x_{n+1} - x^*\|)\|x_n - x_{n+1}\| \\ &\quad + 2\lambda_n \|u_n - z_n\| \|A_2 u_n - A_2 x^*\|.\end{aligned}$$

In view of the restrictions (R2) and (R3), we obtain from (2.17) and (2.21) that

$$\lim_{n\to\infty} \|u_n - z_n\| = 0. \tag{2.29}$$

In a similar way, we can obtain that

$$\lim_{n\to\infty} \|y_n - z_n\| = 0. \tag{2.30}$$

Note that

$$\begin{aligned}\|W_n y_n - x_n\| &\le \|x_n - x_{n+1}\| + \|x_{n+1} - W_n y_n\| \\ &\le \|x_n - x_{n+1}\| + \alpha_n \|f(x_n) - W_n y_n\| + \beta_n \|x_n - W_n y_n\|.\end{aligned}$$

It follows that

$$(1 - \beta_n)\|W_n y_n - x_n\| \le \|x_n - x_{n+1}\| + \alpha_n \|f(x_n) - W_n y_n\|.$$

In view of the restrictions (R2) and (R3), we obtain from (2.17) that

$$\lim_{n\to\infty} \|W_n y_n - x_n\| = 0. \tag{2.31}$$

Notice that

$$\|W_n y_n - y_n\| \le \|y_n - z_n\| + \|z_n - u_n\| + \|u_n - x_n\| + \|x_n - W_n y_n\|.$$

From (2.27), (2.29), (2.30) and (2.31), we arrive at

$$\lim_{n\to\infty} \|W_n y_n - y_n\| = 0. \tag{2.32}$$

Next, we prove that

$$\limsup_{n\to\infty}\langle (f-I)z, x_n - z\rangle \le 0,$$

where $z = P_\Omega f(z)$. To see this, we choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle (f-I)z, x_n - z\rangle = \lim_{i\to\infty}\langle (f-I)z, x_{n_i} - z\rangle. \tag{2.33}$$

Since $\{x_{n_i}\}$ is bounded, there exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ which converges weakly to w. Without loss of generality, we may assume that $x_{n_i} \rightharpoonup w$. On the other hand, we have

$$\|x_n - y_n\| \le \|x_n - u_n\| + \|u_n - z_n\| + \|z_n - y_n\|.$$

It follows from (2.27), (2.29) and (2.30) that

$$\lim_{n\to\infty}\|x_n - y_n\| = 0. \tag{2.34}$$

Therefore, we see that $y_{n_i} \rightharpoonup w$. First, we prove that $w \in VI(C, A_1)$. For the purpose, let T be the maximal monotone mapping defined by:

$$Tx = \begin{cases} A_1 x + N_C x, & x \in C, \\ \emptyset, & x \notin C. \end{cases}$$

For any given $(x, y) \in G(T)$, hence $y - A_1 x \in N_C$. Since $y_n \in C$, by the definition of N_C, we have

$$\langle x - y_n, y - A_1 x\rangle \ge 0. \tag{2.35}$$

Notice that

$$y_n = P_C(I - \eta_n A_1)z_n.$$

It follows that

$$\langle x - y_n, y_n - (I - \eta_n A_1)z_n\rangle \ge 0$$

and hence

$$\left\langle x - y_n, \frac{y_n - z_n}{\eta_n} + A_1 z_n\right\rangle \ge 0.$$

From the monotonicity of A_1, we see that

$$
\begin{aligned}
\langle x - y_{n_i}, y\rangle &\geq \langle x - y_{n_i}, A_1 x\rangle \\
&\geq \langle x - y_{n_i}, A_1 x\rangle - \left\langle x - y_{n_i}, \frac{y_{n_i} - z_{n_i}}{\eta_{n_i}} + A_1 z_{n_i}\right\rangle \\
&= \langle x - y_{n_i}, A_1 x - A_1 y_{n_i}\rangle + \langle x - y_{n_i}, A_1 y_{n_i} - A_1 z_{n_i}\rangle \\
&\quad - \left\langle x - y_{n_i}, \frac{y_{n_i} - z_{n_i}}{\eta_{n_i}}\right\rangle \\
&\geq \langle x - y_{n_i}, A_1 y_{n_i} - A_1 z_{n_i}\rangle - \left\langle x - y_{n_i}, \frac{y_{n_i} - z_{n_i}}{\eta_{n_i}}\right\rangle.
\end{aligned}
$$

Since $y_{n_i} \rightharpoonup w$ and A_1 is Lipschitz continuous, we obtain from (2.30) that $\langle x - w, y\rangle \geq 0$. Notice that T is maximal monotone, hence $0 \in Tw$. This shows that $w \in VI(C, A_1)$. It follows from (2.27) and (2.29), we also have

$$\lim_{n\to\infty} \|x_n - z_n\| = 0.$$

Therefore, we obtain $z_{n_i} \rightharpoonup w$. Similarly, we can prove $w \in VI(C, A_2)$. That is, $w \in VI = VI(C, A_2) \cap VI(C, A_1)$.

Next, we show that $w \in FP = \cap_{i=1}^{\infty} F(S_i)$. Suppose the contrary, $w \notin FP$, i.e., $Ww \neq w$. Since $y_{n_i} \rightharpoonup w$, we see from Opial condition that

$$
\begin{aligned}
\liminf_{i\to\infty} \|y_{n_i} - w\| &< \liminf_{i\to\infty} \|y_{n_i} - Ww\| \\
&\leq \liminf_{i\to\infty} \left\{\|y_{n_i} - Wy_{n_i}\| + \|Wy_{n_i} - Ww\|\right\} \\
&\leq \liminf_{i\to\infty} \left\{\|y_{n_i} - Wy_{n_i}\| + \|y_{n_i} - w\|\right\}.
\end{aligned}
\tag{2.36}
$$

On the other hand, we have

$$
\begin{aligned}
\|Wy_n - y_n\| &\leq \|Wy_n - W_n y_n\| + \|W_n y_n - y_n\| \\
&\leq \sup_{x\in K} \|Wx - W_n x\| + \|W_n y_n - y_n\|.
\end{aligned}
$$

From Lemma 1.6, we obtain from (2.32) that $\lim_{n\to\infty} \|Wy_n - y_n\| = 0$, which combines with (2.36) yields that that

$$\liminf_{i\to\infty} \|y_{n_i} - w\| < \liminf_{i\to\infty} \|y_{n_i} - w\|.$$

This derives a contradiction. Thus, we have $w \in FP$.

Next, we show that $w \in EP(F, A_3)$. It follows from (2.27) that $u_n \rightharpoonup w$. Since $u_n = T_{r_n}(I - rA_3)x_n$, for any $y \in C$, we have

$$F(u_n, y) + \langle A_3 x_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0.$$

From the assumption (A2), we see that

$$\langle A_3 x_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq F(y, u_n), \quad \forall y \in C.$$

Replacing n by n_i, we arrive at

$$\langle A_3 x_{n_i}, y - u_{n_i}\rangle + \left\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\right\rangle \geq F(y, u_{n_i}), \quad \forall y \in C. \tag{2.37}$$

Putting $y_t = ty + (1-t)w$ for any $t \in (0, 1]$ and $y \in C$, we see that $y_t \in C$. It follows from (2.37) that

$$\begin{aligned}\langle y_t - u_{n_i}, A_3 y_t\rangle &\geq \langle y_t - u_{n_i}, A_3 y_t\rangle - \langle A_3 x_{n_i}, y_t - u_{n_i}\rangle \\ &\quad - \left\langle y_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\right\rangle + F(y_t, u_{n_i}) \\ &= \langle y_t - u_{n_i}, A_3 y_t - A_3 u_{n_i}\rangle + \langle y_t - u_{n_i}, A_3 u_{n_i} - A_3 x_{n_i}\rangle \\ &\quad - \left\langle y_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\right\rangle + F(y_t, u_{n_i}).\end{aligned}$$

In view of the monotonicity of A_3, (2.27) and the restriction (R1), we obtain from the assumption (A4) that

$$\langle y_t - w, A_3 y_t\rangle \geq F(y_t, w). \tag{2.38}$$

From the assumptions (A1) and (A4), we see that

$$\begin{aligned}0 = F(y_t, y_t) &\leq tF(y_t, y) + (1-t)F(y_t, w) \\ &\leq tF(y_t, y) + (1-t)\langle y_t - w, A_3 y_t\rangle \\ &= tF(y_t, y) + (1-t)t\langle y - w, A_3 y_t\rangle,\end{aligned}$$

from which it follows that

$$0 \leq F(y_t, y) + (1-t)\langle y - w, A_3 y_t\rangle, \quad \forall y \in C.$$

It follows from the assumption (A3) that $w \in EP(F, A_3)$. On the other hand, we see from (2.33) that

$$\limsup_{n\to\infty}\langle (f-I)z, x_n - z\rangle = \langle (f-I)z, w-z\rangle \le 0. \tag{2.39}$$

Finally, we show that $x_n \to z$, as $n \to \infty$. Note that

$$\begin{aligned}
\|x_{n+1}-z\|^2 &= \langle \alpha_n f(x_n) + \beta_n x_n + \gamma_n W_n y_n - z, x_{n+1}-z\rangle \\
&= \alpha_n\langle f(x_n) - z, x_{n+1}-z\rangle + \beta_n\langle x_n - z, x_{n+1}-z\rangle \\
&\quad + \gamma_n\langle W_n y_n - z, x_{n+1}-z\rangle \\
&\le \alpha_n\langle f(x_n) - f(z), x_{n+1}-z\rangle + \alpha_n\langle f(z)-z, x_{n+1}-z\rangle \\
&\quad + \beta_n\|x_n - z\|\|x_{n+1}-z\| + \gamma_n\|y_n - z\|\|x_{n+1}-z\| \\
&\le \frac{\alpha}{2}\alpha_n(\|x_n - z\|^2 + \|x_{n+1}-z\|^2) + \alpha_n\langle f(z)-z, x_{n+1}-z\rangle \\
&\quad + (1-\alpha_n)\|x_n - z\|\|x_{n+1}-z\| \\
&\le \frac{\alpha}{2}\alpha_n(\|x_n - z\|^2 + \|x_{n+1}-z\|^2) + \alpha_n\langle f(z)-z, x_{n+1}-z\rangle \\
&\quad + \frac{(1-\alpha_n)}{2}(\|x_n - z\|^2 + \|x_{n+1}-z\|^2) \\
&\le \frac{1-\alpha_n(1-\alpha)}{2}\|x_n - z\|^2 + \frac{1}{2}\|x_{n+1}-z\|^2 + \alpha_n\langle f(z)-z, x_{n+1}-z\rangle,
\end{aligned}$$

which implies that

$$\|x_{n+1}-z\|^2 \le [1-\alpha_n(1-\alpha)]\|x_n - z\|^2 + 2\alpha_n\langle f(z)-z, x_{n+1}-z\rangle.$$

From the restriction (R2), we obtain from Lemma 1.3 that $\lim_{n\to\infty}\|x_n - z\| = 0$. This completes the proof. □

Corollary 2.2. *Let C be a nonempty closed convex subset of a Hilbert space H and F be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies* (A1)-(A4). *Let $A_1 : C \to H$ be a δ_1-inverse-strongly monotone mapping, $A_2 : C \to H$ be a δ_2-inverse-strongly monotone mapping and $\{S_i : C \to C\}$ be a family of infinitely nonexpansive mappings. Assume that $\Omega := FP \cap EP(F) \cap VI \neq \emptyset$, where*

$FP = \cap_{i=1}^{\infty} F(S_i)$ and $VI = VI(C, A_1) \cap VI(C, A_2)$. Let $f : C \to C$ be an α-contraction. Let $x_1 \in C$ and $\{x_n\}$ be a sequence generated by

$$\begin{cases} F(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ z_n = P_C(u_n - \lambda_n A_2 u_n), \\ y_n = P_C(z_n - \eta_n A_1 z_n), \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n W_n y_n, \quad \forall n \geq 1, \end{cases}$$

where $\{W_n : C \to C\}$ is the sequence generated in (1.9)*, $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in* $(0, 1)$ *such that $\alpha_n + \beta_n + \gamma_n = 1$ for each $n \geq 1$ and $\{r_n\}$, $\{\lambda_n\}$ and $\{\eta_n\}$ are positive number sequences. Assume that the above control sequences satisfy the following restrictions:*

(R1) $0 < a \leq \eta_n \leq b < 2\delta_1$, $0 < a' \leq \lambda_n \leq b' < 2\delta_2$, $0 < \bar{a} \leq r_n \leq \bar{b} < 2\delta_3$, $\forall n \geq 1$;

(R2) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(R3) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(R4) $\lim_{n\to\infty}(\lambda_n - \lambda_{n+1}) = \lim_{n\to\infty}(\eta_n - \eta_{n+1}) = \lim_{n\to\infty}(r_n - r_{n+1}) = 0$.

Then the sequence $\{x_n\}$ converges strongly to $z \in \Omega$, which solves uniquely the following variational inequality:

$$\langle (I - f)z, z - x\rangle \leq 0, \quad \forall x \in \Omega.$$

Proof. Putting $A_3 \equiv 0$, we see that

$$\langle A_3 x - A_3 y, x - y\rangle \geq \delta \|A_3 x - A_3 y\|^2, \quad \forall x, y \in C$$

for all $\delta \in (0, \infty)$. We can conclude from Theorem 2.1 the desired conclusion easily. This completes the proof. □

Remark 2.3. If $A_1 \equiv A_2$ and $\lambda_n \equiv \eta_n$, then Corollary 2.2 is reduced to Theorem 3.1 of Chang et al. [5]. If $A_2 \equiv 0$, $f(x) \equiv e \in C$ a arbitrary fixed point and $S_i \equiv I$, the identity mapping, then Corollary 2.2 is reduced to Theorem 3.1 of Plubtieng and Punpaeng [11].

Corollary 2.4. *Let C be a nonempty closed convex subset of a Hilbert space H. Let $A_1 : C \to H$ be a δ_1-inverse-strongly monotone mapping, $A_2 : C \to H$ be a δ_2-inverse-strongly monotone mapping, $A_3 : C \to H$ be a δ_3-inverse-strongly monotone mapping and $\{S_i : C \to C\}$ be a family of infinitely nonexpansive mappings. Assume that $\Omega := FP \cap EP(F, A_3) \cap VI \neq \emptyset$, where $FP = \cap_{i=1}^{\infty} F(S_i)$ and $VI = VI(C, A_1) \cap VI(C, A_2)$. Let $f : C \to C$ be an α-contraction. Let $x_1 \in C$ and $\{x_n\}$ be a sequence generated by*

$$\begin{cases} z_n = P_C(x_n - r_n A_3 x_n), \\ z_n = P_C(u_n - \lambda_n A_2 u_n), \\ y_n = P_C(z_n - \eta_n A_1 z_n), \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n W_n y_n, \quad \forall n \geq 1, \end{cases}$$

where $\{W_n : C \to C\}$ is the sequence generated in (1.9)*, $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in* (0, 1) *such that $\alpha_n + \beta_n + \gamma_n = 1$ for each $n \geq 1$ and $\{r_n\}$, $\{\lambda_n\}$ and $\{\eta_n\}$ are positive number sequences. Assume that the above control sequences satisfy the following restrictions:*

(R1) $0 < a \leq \eta_n \leq b < 2\delta_1$, $0 < a' \leq \lambda_n \leq b' < 2\delta_2$, $0 < \bar{a} \leq r_n \leq \bar{b} < 2\delta_3$, $\forall n \geq 1$;

(R2) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(R3) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(R4) $\lim_{n\to\infty}(\lambda_n - \lambda_{n+1}) = \lim_{n\to\infty}(\eta_n - \eta_{n+1}) = \lim_{n\to\infty}(r_n - r_{n+1}) = 0$.

Then the sequence $\{x_n\}$ converges strongly to $z \in \Omega$, which solves uniquely the following variational inequality:

$$\langle (I - f)z, z - x \rangle \leq 0, \quad \forall x \in \Omega.$$

Proof. Putting $F \equiv 0$, we see that

$$\langle A_3 x_n, y - u_n \rangle + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C$$

is equivalent to

$$\langle y - u_n, x_n - r_n A_3 x_n - u_n \rangle \leq 0, \quad \forall y \in C.$$

This implies that

$$u_n = P_C(x_n - r_n A_3 x_n).$$

From the proof of Theorem 2.1, we can conclude the desired conclusion immediately. This completes the proof. □

Remark 2.5. Corollary 2.4 includes Theorem 3.1 of Yao and Yao [25] as a special case, see [25] for more details.

As some applications of our main results, we can obtain the following results.

Recall that a mapping $T: C \to C$ is said to be a k-strict pseudo-contraction if there exists a constant $k \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \ \forall x, y \in C .$$

Note that the class of k-strict pseudo-contractions strictly includes the class of nonexpansive mappings.

Put $A = I - T$, where $T: C \to C$ is a k-strict pseudo-contraction. Then A is $\frac{1-k}{2}$-inverse-strongly monotone; see [1] for more details.

Corollary 2.6. *Let C be a nonempty closed convex subset of a Hilbert space H and F be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies* (A1)-(A4). *Let $T_1: C \to C$ be a k_1-inverse-strongly monotone mapping, $T_2: C \to C$ be a k_2-inverse-strongly monotone mapping, $T_3: C \to C$ be a k_3-inverse-strongly monotone mapping and $\{S_i: C \to C\}$ be a family of infinitely nonexpansive mappings. Assume that $\Omega := FP \cap EP(F, I - T_3) \cap VI \neq \emptyset$, where $FP = \cap_{i=1}^{\infty} F(S_i)$ and $VI = F(T_1) \cap F(T_2)$. Let $f: C \to C$ be an α-contraction. Let $x_1 \in C$ and $\{x_n\}$ be a sequence generated by*

$$\begin{cases} F(u_n, y) + \langle (I - T_3)x_n, y - u_n\rangle + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ z_n = (1 - \lambda_n u_n) + \lambda_n T_2 u_n, \\ y_n = (1 - \eta_n z_n) + \eta_n T_1 z_n, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n W_n y_n, \quad \forall n \geq 1, \end{cases}$$

where $\{W_n : C \to C\}$ is the sequence generated in (1.9)*, $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in* $(0, 1)$ *such that $\alpha_n + \beta_n + \gamma_n = 1$ for each $n \geq 1$ and $\{r_n\}$, $\{\lambda_n\}$ and*

$\{\eta_n\}$ are positive number sequences. Assume that the above control sequences satisfy the following restrictions:

(R1) $0 < a \leq \eta_n \leq b < (1-k_1)$, $0 < a' \leq \lambda_n \leq b' < (1-k_2)$, $0 < \bar{a} \leq r_n \leq \bar{b} < (1-k_3)$, $\forall n \geq 1$;

(R2) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(R3) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(R4) $\lim_{n\to\infty}(\lambda_n - \lambda_{n+1}) = \lim_{n\to\infty}(\eta_n - \eta_{n+1}) = \lim_{n\to\infty}(r_n - r_{n+1}) = 0$.

Then the sequence $\{x_n\}$ converges strongly to $z \in \Omega$, which solves uniquely the following variational inequality:

$$\langle (I-f)z, z-x \rangle \leq 0, \quad \forall x \in \Omega.$$

Proof. Taking $A_j = I - T_j$, wee see that $A_j : C \to H$ is a δ_j-strict pseudo-contraction with $\delta_j = \frac{1-k_j}{2}$ and $F(T_j) = VI(C, A_j)$ for $j = 1, 2$. From Theorem 2.1, we can obtain the desired conclusion easily. This completes the proof. □

3 Conclusion

The iterative process (2.1) presented in this paper which can be employed to approximate common elements in the solution set of the generalized equilibrium problem (1.3), in the solution set of the classical variational inequality (1.1) and in the common fixed point set of a family nonexpansive mappings is general. It is of interest to improve the main results presented in this paper to the framework of real Banach spaces.

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