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An inexact subgradient algorithm for Equilibrium Problems

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Abstract. We present an inexact subgradient projection type method for solving a nonsmooth Equilibrium Problem in a finite-dimensional space. The proposed algorithm has a low computational cost per iteration. Some numerical results are reported.

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Key words: Equilibrium problem, Projection method, Subgradient method.

1 Introduction

Let C be a nonempty closed convex subset of $\mathbb{R}^n$ and let $f\colon \mathbb{R}^n \times \mathbb{R}^n \longrightarrow (-\infty, +\infty]$ be a bifunction such that $f(x, x) = 0$ for all $x \in C$ and $C \times C$ is contained in the domain of f. We consider the following *Equilibrium Problem*:

$$EP(f, C) \quad \begin{cases} \text{Find } \ x^* \in C \ \text{ such that} \\ f(x^*, y) \geq 0 \quad \forall \ y \in C. \end{cases} \tag{1}$$

The solution set of this problem (1) is denoted by $S(f, C)$.

This formulation gives a unified framework for several problems in the sense that it includes, as particular cases, optimization problems, Nash equilibria problems, complementarity problems, fixed point problems, variational inequalities and vector minimization problems (see for instance [4]).

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In this work we assume that the function $f(x,\cdot)\colon \mathbb{R}^n \longrightarrow (-\infty,+\infty]$ is convex and subdifferentiable at x, for all $x \in C$ (see [8, 13, 15, 18]). In [8] the subdifferential of this function is called diagonal subdifferential. We define by diagonal subgradients the elements of this set.

The aim of this paper is to develop and to analyze an inexact projected diagonal subgradient method using a divergent series steplength rule. The algorithm is easy to implement and it has a low computational cost since only one inexact projection is done per iteration.

Recently, many algorithms have been developed for solving problem (1) combining diagonal subgradients with projections, see for instance, [6, 7, 15, 18, 19, 21] and references therein.

The paper is organized as follows: In Section 2 we recall useful basic notions. In Section 3 we define the algorithm and study its convergence. In Section 4, we report some computational experiments. In Section 5, we give some concluding remarks.

2 Preliminaries

In this section we present some basic concepts, properties, and notations that we will use in the sequel. Let $\mathbb{R}^n$ be endowed with the Euclidean inner product $\langle\cdot,\cdot\rangle$ and the associated norm $\|\cdot\|$.

Definition 2.1. Let $\xi \geq 0$ and $x \in \mathbb{R}^n$. A point $p_x \in C$ is called a ξ-projection of x onto C, if p_x is a ξ-solution of the problem

$$\min_{y\in C}\left\{\frac{1}{2}\|x-y\|^2\right\},$$

that is

$$\frac{1}{2}\|x-p_x\|^2 \leq \frac{1}{2}\|x-P_C(x)\|^2+\xi$$

where $P_C(x)$ is the orthogonal projection of x onto C.

It is easy to show that the ξ-projection of x onto C is characterized by

$$\langle x-p_x, p_x-y\rangle \geq -\xi \quad \forall \ y \in C. \tag{2}$$

Through this paper we will consider the following enlargement of the diagonal subdifferential.

Definition 2.2. The ϵ-diagonal subdifferential $\partial_2^\epsilon f(x,x)$ of a bifunction f at $x \in C$, is given by

$$\begin{aligned} \partial_2^\epsilon f(x,x) &:= \{g \in \mathbb{R}^n : f(x,y) + \epsilon \geq f(x,x) + \langle g, y - x\rangle \ \ \forall \ \ y \in \mathbb{R}^n\} \\ &= \{g \in \mathbb{R}^n : f(x,y) + \epsilon \geq \langle g, y - x\rangle \ \ \forall \ \ y \in \mathbb{R}^n\}. \end{aligned} \tag{3}$$

Let us note that the 0-diagonal subdifferential is the diagonal subdifferential $\partial_2 f(x,x)$, studied in [8].

The following well-known property will be useful in this paper.

Lemma 2.3. *Let $\{\nu_k\}$ and $\{\delta_k\}$ be nonnegative sequences of real numbers satisfying $\nu_{k+1} \leq \nu_k + \delta_k$ with $\sum_{k=1}^{+\infty} \delta_k < +\infty$. Then the sequence $\{\nu_k\}$ is convergent.*

The next technical result will be used in the convergence analysis.

Lemma 2.4. *Let θ, β and ξ be nonnegative real numbers satisfying $\theta^2 - \beta\theta - \xi \leq 0$, then,*

$$\beta\theta \leq \beta^2 + \xi. \tag{4}$$

Proof. Consider the quadratic function $s(\theta) = \theta^2 - \beta\theta - \xi$, then $s(\theta) \leq 0$ implies that

$$\theta \leq \frac{\beta + \sqrt{\beta^2 + 4\xi}}{2},$$

since $\theta > 0$.

Multiplying the last inequality by β and using the property $ab \leq \frac{a^2+b^2}{2}$ we obtain

$$\begin{aligned} \beta\theta &\leq 2^{-1}\left[\beta^2 + \beta\sqrt{\beta^2 + 4\xi}\right] \\ &\leq 2^{-1}\left[\beta^2 + \tfrac{\beta^2+\beta^2+4\xi}{2}\right] \\ &= 2^{-1}\left[\beta^2 + \beta^2 + 2\xi\right] \\ &= \beta^2 + \xi. \end{aligned}$$

The proof is complete. □

In the convergence analysis we will assume that the solution set of (1) is contained in the solution set of its dual problem, which is given by

$$\begin{cases} \text{Find } x^* \in C \text{ such that} \\ f(y, x^*) \leq 0 \quad \forall\ y \in C. \end{cases} \tag{5}$$

The solution set of this problem is denoted by $S_d(f, C)$.

When f is a pseudomonotone bifunction on C (if $x, y \in C$ and $f(x, y) \geq 0$, then $f(y, x) \leq 0$), it holds that $S(f, C) \subseteq S_d(f, C)$. Moreover, this inclusion is also valid for monotone bifunctions ($f(x, y) + f(y, x) \leq 0$).

Now, we are in position to define our algorithm.

3 The algorithm and its convergence analysis

Take a positive parameter ρ and real sequences $\{\rho_k\}$, $\{\beta_k\}$, $\{\epsilon_k\}$ and $\{\xi_k\}$ verifying the following conditions:

$$\rho_k > \rho, \quad \beta_k > 0, \quad \epsilon_k \geq 0, \quad \xi_k \geq 0 \ \ \forall k \in \mathbb{N}, \tag{6}$$

$$\sum \frac{\beta_k}{\rho_k} = +\infty, \ \ \sum \beta_k^2 < +\infty, \tag{7}$$

$$\sum \frac{\beta_k \epsilon_k}{\rho_k} < +\infty, \quad \sum \xi_k < +\infty. \tag{8}$$

3.1 The Inexact Projected Subgradient Method (IPSM)

Step 0: Choose $x^0 \in C$. Set $k = 0$.
Step 1: Let $x^k \in C$. Obtain $g^k \in \partial_2^{\epsilon_k} f(x^k, x^k)$. Define

$$\alpha_k = \frac{\beta_k}{\gamma_k} \quad \text{where} \quad \gamma_k = \max\{\rho_k, \|g^k\|\}. \tag{9}$$

Step 2: Compute $x^{k+1} \in C$ such that:

$$\langle \alpha_k g^k + x^{k+1} - x^k, x - x^{k+1} \rangle \geq -\xi_k \ \ \forall\ x \in C. \tag{10}$$

Notice that the point x^{k+1} is a ξ_k-projection of $(x^k - \alpha_k g^k)$ onto C. In particular, if $\xi_k = 0$, then $x^{k+1} = P_C(x^k - \alpha_k g^k)$.

We also observe that the steplength rule (9) is similar with those given in [1] and [2]. In fact, in [1] is taking $\gamma_k = \max\{\beta_k, \|g^k\|\}$ with $\sum \beta_k = +\infty$, while in [2] is considered $\rho_k = 1$ for all $k \in \mathbb{N}$.

In the *exact version* of IPSM is considered $\epsilon_k = \xi_k = 0$ for all $k \in \mathbb{N}$ and the following stopping criteria are included: $g^k = 0$ (at step 1) and $x^k = x^{k+1}$ (at step 2).

3.2 Convergence analysis

The first result concerns the exact version of the algorithm.

Proposition 3.1. *If the exact version of Algorithm IPSM generates a finite sequence, then the last point is a solution of problem $EP(f, C)$.*

Proof. Since $\epsilon_k = 0$, we have that $g^k \in \partial_2 f(x^k, x^k)$. If the algorithm stops at step 1 we have $g^k = 0$. So, our conclusion follows from (3).

Now, assume that the algorithm finishes at step 2, that is, $x^k = x^{k+1}$. Suppose, for the sake of contradiction, that $x^k \notin S(f, C)$. Then, there exists $x \in C$ such that $f(x^k, x) < 0$. Using again (3) we get

$$0 > f(x^k, x) \geq \langle g^k, x - x^k \rangle. \tag{11}$$

On the other hand, by replacing x^{k+1} by x^k in (10) and taking in account that $\xi_k = 0$, it results

$$\langle \alpha_k g^k, x - x^k \rangle \geq 0. \tag{12}$$

Therefore, from (11) and (12) we get a contradiction because $\alpha_k > 0$. Hence, $x^k \in S(f, C)$. □

From now on, we assume that the algorithm IPSM generates an infinite sequence denoted by $\{x^k\}$.

We derive the following auxiliary property.

Lemma 3.2. *For each k, the following inequalities hold*

(i) $\alpha_k \|g^k\| \leq \beta_k$;

(ii) $\beta_k \|x^{k+1} - x^k\| \leq \beta_k^2 + \xi_k$.

Proof. (i) From (9) it follows

$$\alpha_k \|g^k\| \;=\; \frac{\beta_k \|g^k\|}{\max\{\rho_k, \|g^k\|\}} \;\leq\; \beta_k. \tag{13}$$

(ii) By taking $x = x^k$ in (10) it results

$$\begin{aligned} \|x^{k+1} - x^k\|^2 &\leq \langle \alpha_k g^k, x^k - x^{k+1}\rangle + \xi_k \\ &\leq \alpha_k \|g^k\| \|x^{k+1} - x^k\| + \xi_k \\ &\leq \beta_k \|x^{k+1} - x^k\| + \xi_k, \end{aligned} \tag{14}$$

where the Cauchy-Schwarz inequality is used in the second inequality and the last follows from (13).

Therefore, the desired result is obtained from Lemma 2.4 with $\theta = \|x^{k+1} - x^k\|$, $\beta = \beta_k$ and $\xi = \xi_k$, for each $k \in \mathbb{N}$. □

The next requirement will be used in the subsequent discussions.

A1. The solution set $S(f, C)$ is nonempty;

Notice that this is a common assumption for $EP(f, C)$ (see for example, [11, 13, 15, 18] and references therein). Regarding the existence of solutions for equilibrium problems we refer to [9, 12, 20] and references therein.

Proposition 3.3. *Assume that A1 is verified. Then, for every $x^* \in S(f, C)$ and for each k, the following assertion holds*

$$\|x^{k+1} - x^*\|^2 \leq \|x^k - x^*\|^2 + 2\alpha_k f(x^k, x^*) + \delta_k, \tag{15}$$

where $\delta_k = 2\alpha_k \epsilon_k + 2\beta_k^2 + 4\xi_k$.

Proof. By a simple algebraic manipulation we have that

$$\begin{aligned} \|x^{k+1} - x^*\|^2 &= \|x^k - x^*\|^2 - \|x^{k+1} - x^k\|^2 + 2\langle x^k - x^{k+1}, x^* - x^{k+1}\rangle \\ &\leq \|x^k - x^*\|^2 + 2\langle x^k - x^{k+1}, x^* - x^{k+1}\rangle. \end{aligned} \tag{16}$$

By combining (16) and (10) with $x = x^*$ it follows

$$\begin{aligned} \|x^{k+1} - x^*\|^2 &\leq \|x^k - x^*\|^2 + 2\langle \alpha_k g^k, x^* - x^{k+1}\rangle + 2\xi_k \\ &= \|x^k - x^*\|^2 + 2\langle \alpha_k g^k, x^* - x^k\rangle \\ &\quad + 2\langle \alpha_k g^k, x^k - x^{k+1}\rangle + 2\xi_k. \end{aligned} \tag{17}$$

By applying the Cauchy-Schwarz inequality and Lemma 3.2 (i), it yields

$$\begin{aligned}\|x^{k+1} - x^*\|^2 \leq \|x^k - x^*\|^2 &+ 2\alpha_k \langle g^k, x^* - x^k \rangle \\ &+ 2\beta_k \|x^k - x^{k+1}\| + 2\xi_k.\end{aligned} \tag{18}$$

In virtue of (18) and Lemma 3.2 (ii) it results

$$\|x^{k+1} - x^*\|^2 \leq \|x^k - x^*\|^2 + 2\alpha_k \langle g^k, x^* - x^k \rangle + 2\beta_k^2 + 4\xi_k. \tag{19}$$

On the other hand, from the fact that $g^k \in \partial_2^{\epsilon_k} f(x^k, x^k)$, we have that $\langle g^k, x^* - x^k \rangle \leq f(x^k, x^*) + \epsilon_k$. Therefore, since $\alpha_k > 0$ we obtain

$$2\alpha_k \langle g^k, x^* - x^k \rangle \leq 2\alpha_k f(x^k, x^*) + 2\alpha_k \epsilon_k. \tag{20}$$

The conclusion follows from (19) and (20). □

The following requirement will be used to obtain the boundedness of the sequence $\{x^k\}$ generated by IPSM.

A2. $S(f, C) \subseteq S_d(f, C)$;

We point out that this assumption is weaker than the pseudomonotonicity condition. In fact, consider the following example.

Example 3.4. Let $EP(f, C)$ be defined by

$$C = [-1, 1] \subseteq \mathbb{R}, \quad f(x, y) = 2y|x|(y - x) + xy|y - x|, x, y \in \mathbb{R}.$$

Observe that $S(f, C) = \{0\}$ and $f(y, x^*) = f(y, 0) = 0$ for all $y \in C$. Hence, A2 holds. However, the bifunction f is not pseudomonotone on C. In fact, we have $f(-0.5, 0.5) = f(0.5, -0.5) = 0.25 > 0$.

Notice that $f(x, \cdot)$ is convex for all $x \in C$ and is diagonal subdifferentiable with $\partial_2 f(x, x) = [2|x|x - x^2, 2|x|x + x^2]$.

Furthermore, this example gives a negative answer to the conjecture given in [9], namely, if C is a nonempty, convex and closed set such that $f(x, x) = 0$, $f(x, \cdot) : C \to \mathbb{R}$ is convex and lower semi-continuous, $f(\cdot, y) : C \to \mathbb{R}$ is upper semi-continuous for all $x \in C$ and the primal and dual equilibrium problems have the same nonempty solution set, then f is pseudomonotone.

We observe that in [12] an example which disproves the conjecture using a pseudoconvex function $f(x,\cdot)$ instead of a convex function is given.

Theorem 3.5. *Assume that A1 and A2 are verified. Then,*

(i) $\{\|x^k - x^*\|^2\}$ *is convergent, for all* $x^* \in S(f, C)$*;*

(ii) $\{x^k\}$ *is bounded.*

Proof. (i) Let $x^* \in S(f, C)$ and $k \in \mathbb{N}$. By A2 we have $f(x^k, x^*) \leq 0$ which together with Proposition 3.3 implies

$$\|x^{k+1} - x^*\|^2 \leq \|x^k - x^*\|^2 + \delta_k, \tag{21}$$

where $\delta_k = 2\alpha_k\epsilon_k + 2\beta_k^2 + 4\xi_k$.

Therefore, in virtue of (7), (8) and (9) we obtain

$$\sum_{k=0}^{+\infty} \delta_k < +\infty. \tag{22}$$

Hence, from (21), (22) and Lemma 2.3 it results that $\{\|x^k - x^*\|^2\}$ is a convergent sequence.

(ii) The conclusion follows from part (i). □

Now, we establish two different hypotheses on the data to obtain an asymptotic behavior of the sequence $\{x^k\}$.

A3. The ϵ-diagonal subdifferential is bounded on bounded subsets of C.

A3'. The sequence $\{g^k\}$ is bounded.

Let us note that, condition A3 has been considered in [14] in the setting of optimization problems. Also, a similar condition has been assumed in [10] for equilibrium problems (condition (A)). We observe that A3 and A3' hold under the conditions that there is a nonempty, open and convex set U containing C such that f is finite and continuous on $U \times U$, $f(x, x) = 0$ and $f(x,\cdot) : C \to \mathbb{R}$ is convex for all $x \in C$ ([10], Proposition 4.3). Condition A3' has been required in [7] and [15] for equilibrium problems. This condition has also been assumed in [16] and [17] for saddle point problems.

Observe that Example 3.4 satisfies both assumptions.

Theorem 3.6. *Suppose that A1 and A2 are verified. Then, under A3 or A3' it holds*

$$\lim_{k\to+\infty} \sup f(x^k, x^*) = 0 \ \ \forall \ \ x^* \in S(f, C).$$

Proof. Let $x^* \in S(f, C)$. By Proposition 3.3 and A2 it results

$$0 \le 2\alpha_k[-f(x^k, x^*)] \le \|x^k - x^*\|^2 - \|x^{k+1} - x^*\|^2 + \delta_k. \tag{23}$$

Hence,

$$\begin{array}{rcl} 0 & \le & 2\sum_{k=0}^m \alpha_k[-f(x^k, x^*)] \\ & \le & \|x^0 - x^*\|^2 - \|x^{m+1} - x^*\|^2 + \sum_{k=0}^m \delta_k \\ & \le & \|x^0 - x^*\|^2 + \sum_{k=0}^m \delta_k. \end{array} \tag{24}$$

As $m \to +\infty$ we have

$$0 \le 2\sum_{k=0}^{+\infty} \alpha_k[-f(x^k, x^*)] \le \|x^0 - x^*\|^2 + \sum_{k=0}^{+\infty} \delta_k, \tag{25}$$

which together with (22) yields

$$0 \ \le \ \sum_{k=0}^{+\infty} \alpha_k[-f(x^k, x^*)] < +\infty. \tag{26}$$

On the other hand, by A3' or A3 we have that $\{\|g^k\|\}$ is bounded. In fact, by Theorem 3.5 we get that $\{x^k\}$ is bounded. Therefore, the assertion follows from A3. In consequence, using (6) and (9) we conclude that there exists $L \ \ge \ \rho$ such that $\left\|g^k\right\| \le L$ for all $k \in \mathbb{N}$. Therefore

$$\frac{\gamma_k}{\rho_k} = \max\{1, \rho_k^{-1}\|g^k\|\} \le \frac{L}{\rho} \qquad \forall\, k \in \mathbb{N}.$$

Therefore

$$\alpha_k = \frac{\beta_k}{\gamma_k} \ge \frac{\rho}{L}\frac{\beta_k}{\rho_k} \qquad \forall\, k \in \mathbb{N}. \tag{27}$$

Consequently, by (26) and (27) we have

$$\sum_{k=0}^{+\infty} \frac{\beta_k}{\rho_k}[-f(x^k, x^*)] < +\infty. \tag{28}$$

Then, the conclusion follows from (28) and (7). □

In order to obtain the convergence of the whole sequence we introduce two additional assumptions.

A4. Let $x^* \in S(f, C)$ and $\overline{x} \in C$. If $f(\overline{x}, x^*) = f(x^*, \overline{x}) = 0$ then $\overline{x} \in S(f, C)$;

A5. $f(\cdot, y)$ is upper semicontinuous for all $y \in C$.

Assumption A4 holds, for example, when the problem $EP(f, C)$ corresponds to an optimization problem, or when it is a reformulation of the variational inequality problem with a paramonotone operator. Moreover, the requirement A4 can be considered as an extension of the cut property given in [5] from variational inequality problems to equilibrium problems. Assumption A4 can be recovered if we assume A2 and the following condition holds

$$f(z, x) \leq f(z, y) + f(y, x) \ \ \forall \ x, y, z \in C,$$

which is considered, for instance, in [3].

We also note that Assumption A5 is a common requirement for $EP(f, C)$ (see, for example, [11, 18] and references therein).

Example 3.7. We consider the equilibrium problem defined by $C = (-\infty, 0]$ and $f(x, y) = x^2(|y| - |x|)$. Let us observe that A1-A5 hold. In fact, $x^* = 0$ is the unique solution of $EP(f, C)$, $f(y, x^*) = -|y|y^2 \leq 0$ for all $y \in C$, $f(x, y)$ is continuous and $f(\overline{x}, x^*) = f(x^*, \overline{x}) = 0$ implies that $\overline{x} = 0$, that is, $\overline{x} \in S(f, C)$.

Theorem 3.8. *Assume that A1, A2, A3 or A3', A4 and A5 are satisfied. Then, the whole sequence $\{x^k\}$ converges to a solution of $EP(f, C)$.*

Proof. Let $x^* \in S(f, C)$. By Theorem 3.6, there exists a subsequence $\{x^{k_j}\}$ of $\{x^k\}$ such that

$$\limsup_{k \to +\infty} f(x^k, x^*) = \lim_{j \to +\infty} f(x^{k_j}, x^*). \tag{29}$$

In view of Theorem 3.5, we have that $\{x^{k_j}\}$ is bounded. So, there is $\overline{x} \in C$ and a subsequence of $\{x^{k_j}\}$, without lost of generality, namely $\{x^{k_j}\}$, such that

$$\lim_{j \to +\infty} x^{k_j} = \overline{x}. \tag{30}$$

Combining assumption A5 together with Theorem 3.6 it follows

$$\begin{aligned} f(\overline{x}, x^*) &\geq \limsup_{j\to+\infty} f(x^{k_j}, x^*) \\ &= \lim_{j\to+\infty} f(x^{k_j}, x^*) \\ &= \limsup_{k\to+\infty} f(x^k, x^*) \\ &= 0. \end{aligned} \tag{31}$$

From assumption A2 we have $f(\overline{x}, x^*) \leq 0$, so, it results

$$f(\overline{x}, x^*) = 0. \tag{32}$$

Therefore, A4 implies that $\overline{x} \in S(f, C)$. Using again Theorem 3.5 we obtain that the sequence $\{\|x^k - \overline{x}\|^2\}$ is convergent, which together with (30) it yields

$$\lim_{k\to+\infty} x^k = \overline{x}, \qquad \overline{x} \in S(f, C). \qquad \square$$

Notice that Theorem 3.8 remains valid if we replace assumptions A2, A4 and A5 by the τ-strongly pseudomonotone condition on f with respect to $x^* \in S(f, C)$, that is,

$$f(x^*, y) \geq 0 \quad \Rightarrow \quad f(y, x^*) \leq -\tau\|x^* - y\|^2 \quad \forall\, y \in C.$$

This condition is weaker than the strong monotonicity of f which has been assumed in [15] for solving equilibrium problems.

4 Numerical results

In this section we illustrate the algorithm IPSM. Some comparisons are also reported. In the two first examples we compare the iterates of IPSM with such one obtained by the Relaxation Algorithm given in [6], where a constrained optimization problem and a line search have been solved at each iteration. Example 4.3 shows the computational time of IPSM versus our implementation of the Extragradient method given in [21], where a constrained optimization problem, a line search and a projection, have been performed at each iteration. Also, we present a nonsmooth example verifying our assumptions. We take $\xi_k = \epsilon_k = 0$, for all $k \in \mathbb{N}$, in order to compare the performance of the algorithms.

The algorithm has been coded in MATLAB 7.8 on a 2GB RAM Pentium Dual Core.

Example 4.1. Consider the River Basin Pollution Problem given in [6] which consists of three players with payoff functions:

$$\phi_j(x) = u_j x_j^2 + 0.01 x_j(x_1 + x_2 + x_3) - v_j x_j, \quad j = 1, 2, 3$$

where $u = (0.01, 0.05, 0.01)$ and $v = (2.90, 2.88, 2.85)$, and the constraints are given by

$$\begin{cases} 3.25x_1 + 1.25x_2 + 4.125x_3 & \leq & 100 \\ 2.291x_1 + 1.5625x_2 + 2.8125x_3 & \leq & 100. \end{cases}$$

We take $\gamma_k = \max\{3, \|g^k\|\}$, $\beta_k = \frac{168}{k}$ for all $k \in \mathbb{N}$.

Table 1 gives the results obtained by IPSM algorithm and by the Relaxation Algorithm (RA) used in [6].

	RA			IPSM		
Iter.(k)	x_1^k	x_2^k	x_3^k	x_1^k	x_2^k	x_3^k
0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
1	19.3258	17.1746	3.8115	17.4819	42.9394	−2.5431
2	20.7043	16.1053	3.0495	26.3436	−22.0781	10.1772
3	21.0366	16.0367	2.8084	21.0333	16.8576	2.5623
4	21.1181	16.0295	2.7464	21.2024	16.6129	2.8023
5	21.1382	16.0282	2.7310	21.1349	16.1052	2.7103
6	21.1431	16.0279	2.7272	21.1452	16.0284	2.7255
7	21.1444	16.0278	2.7262	21.1452	16.0279	2.7257

Table 1 – Example 4.1: Iterations of RA [6] and IPSM, where $x^* = (21.149, 16.028, 2, 722)$.

Table 1 shows that both algorithms give similar approximations to x^* at iteration 7, involving different computational effort. In fact, an optimization problem and an inexact line search are considered at each iteration of RA.

Example 4.2. Consider the Cournot oligopoly problem with shared constraints and nonlinear cost functions as described in [6]. The bifunction is defined by

$$\begin{aligned} f(x, y) &= \textstyle\sum_i^5 [\theta_i(y_{-i}, x_i) - \theta_i(x_{-i}, x_i)], \\ \theta_j(x) &= f_j(x_j) - 5000^{\frac{1}{\eta}} x_j (x_1 + \cdots + x_5)^{\frac{-1}{\eta}} \\ f_j(x_j) &= c_j x_j + \frac{\beta_j}{\beta_j + 1} K_j^{(-\frac{1}{\beta_j})} x_j^{(\frac{\beta_j+1}{\beta_j})} \end{aligned}$$

where, $\eta = 1.1$, $c = (10, 8, \dots, 2)$, $K = (5, 5, \dots, 5)$, $\beta = (1.2, 1.1, \dots, 0.8)$ and $C = \mathbb{R}^n_+$.

For this problem, we consider $\beta_k = \frac{30}{k}$, $\rho_k = 1$ for all $k \in \mathbb{N}$.

In Table 2, we show the first three components of each iterate for sake of comparison of IPSM with the relaxation algorithm RA given in [6].

	RA			IPSM		
Iter.(k)	x_1^k	x_2^k	x_3^k	x_1^k	x_2^k	x_3^k
0	10.0000	10.0000	10.0000	10.0000	10.0000	10.0000
1	55.0181	56.1830	55.7512	22.2998	23.8567	23.4060
2	27.2067	33.0054	36.1694	27.9168	29.1315	30.2456
3	42.6043	46.7481	47.9981	31.5732	33.4380	35.0173
4	33.5762	38.8631	41.1775	34.3174	36.8889	38.8577
5	38.8777	43.5262	45.1860	36.5254	40.0134	42.2881
10	36.7970	41.6992	43.6040	36.8336	41.7204	43.6016
20	36.9318	41.8175	43.7060	36.9325	41.8181	43.7065

Table 2 – Example 4.2: Iterations of RA and IPSM, where $x^* = (36.912, 41.842, 43.705, 42.665, 39.182)$.

Again, despite the algorithms RA and IPSM obtain similar results at iteration 20, the computational effort is different.

Example 4.3. Consider two equilibrium problems given in [21], where

$$C = \left\{ x \in \mathbb{R}^5 : \sum_{i=1}^{5} x_1 \geq -1, \quad -5 \leq x_i \leq 5, \ i = 1, \dots, 5 \right\}$$

and the bifunction is of the form

$$f(x, y) = \langle Px + Qy + q, y - x \rangle. \tag{33}$$

The matrices P, Q and the vector q are defined by

$$P_1 = \begin{bmatrix} 3.1 & 2 & 0 & 0 & 0 \\ 2 & 3.6 & 0 & 0 & 0 \\ 0 & 0 & 3.5 & 2 & 0 \\ 0 & 0 & 2 & 3.3 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 3.1 & 2 & 0 & 0 & 0 \\ 2 & 3.6 & 0 & 0 & 0 \\ 0 & 0 & 3.5 & 2 & 0 \\ 0 & 0 & 2 & 3.3 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix},$$

$$Q = \begin{bmatrix} 1.6 & 1 & 0 & 0 & 0 \\ 1 & 1.6 & 0 & 0 & 0 \\ 0 & 0 & 1.5 & 1 & 0 \\ 0 & 0 & 1 & 1.5 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix} \quad \text{and} \quad q = \begin{bmatrix} 1 \\ -2 \\ -1 \\ 2 \\ -1 \end{bmatrix},$$

where the i-th problem considers $P = P_i, \quad i = 1, 2$.

For the first problem, we take $\beta_k = \frac{7}{2k}$ and $\rho_k = 3$ for all $k \in \mathbb{N}$. Like in [21], we use $tol = 10^{-3}$ and $x_0 = (1, 3, 1, 1, 2)$.

In Table 3, we compare IPSM with two Extragradient Algorithms (EA) given in [21].

Scheme	Iter.(k)	Ls. step	cpu(s)
EA-a	8	7	0.0562
EA-b	25	–	0.1148
IPSM	10	–	0.0006

Table 3 – Example 4.3: Problem 1.

In the second problem, we use $P = P_2$, $\beta_k = \frac{10}{3k}$ and $\rho_k = 3$ for all $k \in \mathbb{N}$. Like in [21], we take $tol = 10^{-3}$ and $x_0 = (1, 3, 1, 1, 2)$.

In Table 4, we compare IPSM with algorithm EA given in [21].

Scheme	Iter.(k)	Ls. step	cpu(s)
EA-a	10	9	0.0605
IPSM	10	–	0.0006

Table 4 – Example 4.3: Problem 2.

Tables 3 and 4 show a good performance of algorithm IPSM.

Example 4.4. Consider the nonsmooth equilibrium problem defined by the bifunction $f(x, y) = |y_1| - |x_1| + y_2^2 - x_2^2$ and the constraint set $C = \{x \in \mathbb{R}^2_+ : x_1 + x_2 = 1\}$. The optimal point is $x^* = (\frac{1}{2}, \frac{1}{2})$ and the partial subdifferential of the equilibrium bifunction f is given by

$$\partial_2 f(x, x) = \begin{cases} (1, 2x_2) & \text{if } x_1 > 0, \\ ([-1, 1], 2x_2) & \text{if } x_1 = 0, \\ (-1, 2x_2) & \text{if } x_1 < 0. \end{cases}$$

We use $\gamma_k = \max\{1, \|g^k\|\}$ and $\|x^k - x^*\| \leq 10^{-4}$ as stop criteria. In Table 5, we show our results by considering different initial points.

x_1^0	x_2^0	β_k	Iter.(k)	cpu(s)
0.0000	1.0000	$1/k$	1	0.0057
0.1111	0.8889	$9/k$	8	0.0563
0.3333	0.6667	$9/k$	8	0.0560
0.6667	0.3333	$4/k$	5	0.0359
0.8889	0.1111	$8/k$	7	0.0478
1.0000	0.0000	$1/k$	1	0.0061

Table 5 – Example 4.4.

5 Concluding remarks

In this paper we have presented a subgradient-type method, denoted by IPSM, for solving equilibrium problems and established its convergence under mild assumptions.

Numerical results were reported for test problems given in the literature of computational methods for solving nonsmooth equilibrium problems. The comparison with other two schemes has shown a satisfactory behavior of the algorithm IPSM in terms of the computational time.

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