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Block triangular preconditioner for static Maxwell equations*

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Abstract. In this paper, we explore the block triangular preconditioning techniques applied to the iterative solution of the saddle point linear systems arising from the discretized Maxwell equations. Theoretical analysis shows that all the eigenvalues of the preconditioned matrix are strongly clustered. Numerical experiments are given to demonstrate the efficiency of the presented preconditioner.

Mathematical subject classification: 65F10.

Key words: Maxwell equations, preconditioner, Krylov subspace method, saddle point system.

1 Introduction

We consider the block triangular preconditioner for linear systems arising from the finite element discretization of the following static Maxwell equations: find u and p such that

$$\begin{aligned} &\nabla \times \nabla \times u + \nabla p = f && \text{in } \Omega \\ &\nabla \cdot u = 0 && \text{in } \Omega \\ &u \times n = 0 && \text{on } \partial\Omega \\ &p = 0 && \text{on } \partial\Omega \end{aligned} \tag{1.1}$$

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where $\Omega \subset \mathbb{R}^2$ is a simply connected domain with connected boundary $\partial\Omega$, and n represents the outward unit normal vector on $\partial\Omega$; u is vector field, p is the Lagrange multiplier and the datum f is given generic source.

There are a large variety of schemes for solving the Maxwell equations, such as the edge finite element method [1, 2, 6], the domain decomposition method [5, 9], the algebraic multigrid method [3] and so on.

Using finite element discretization with Nédélec elements of the first kind [4, 11, 7] for the approximation of the vector field and the standard nodal elements for the multiplier, we obtain the approximate solution of (1.1) by solving the following saddle point linear systems:

$$\mathcal{A}x \equiv \begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} u \\ p \end{bmatrix} = \begin{bmatrix} g \\ 0 \end{bmatrix} \equiv b, \tag{1.2}$$

where $u \in \mathbb{R}^n$ and $p \in \mathbb{R}^m$ are finite arrays denoting the finite element approximations, $g \in \mathbb{R}^n$ is the load vector connected with the datum f. The matrix $A \in \mathbb{R}^{n\times n}$ corresponding to the discrete curl-curl operator is symmetric positive semidefinite with nullity m, $B \in \mathbb{R}^{m\times n}$ is a discrete divergence operator with rank(B)$= m$. Specifically, one can see [4, 7, 11] for details.

The form of (1.2) frequently occurs in a large number of applications, such as the (linearized) Navier-Stokes equations [21], the time-harmonic Maxwell equations [7, 8, 10], the linear programming (LP) problem and the quadratic programming (QP) problem [17, 20]. At present, there usually exist four kinds of preconditioners for the saddle point linear systems (1.2): block diagonal preconditioner [22, 23, 24, 25], block triangular preconditioner [15, 16, 26, 27, 28, 37], constraint preconditioner [29, 30, 31, 32, 33] and Hermitian and skew-Hermitian splitting (HSS) preconditioner [34]. One can [12] for a general discussion.

Recently, Rees and Greif [17] presented the following triangular preconditioner:

$$R_k = \begin{bmatrix} A + B^T W^{-1} B & kB^T \\ 0 & W \end{bmatrix}, \tag{1.3}$$

where W is a symmetric positive definite matrix and $k \neq 0$. It was shown that if A is symmetric positive semidefinite with nullity q ($q \leq m$), then the preconditioned matrix $R_k^{-1}\mathcal{A}$ has five distinct eigenvalues: 1 with algebraic

multiplicity $n-m$, $\frac{-k\pm\sqrt{k^2+4}}{2}$ with algebraic multiplicity $2q$ and

$$\frac{-(k\eta-1)\pm\sqrt{(k\eta-1)^2+4\eta(1+\eta)}}{2(1+\eta)}$$

with algebraic multiplicity $2(m-q)$ where $\eta>0$ is the generalized eigenvalues of $\eta Ax = B^TW^{-1}Bx$. Obviously, if $m=q$, the preconditioned matrix $R_k^{-1}\mathcal{A}$ has three distinct eigenvalues: 1 and $\frac{-k\pm\sqrt{k^2+4}}{2}$. This is favorable to Krylov subspace methods, which rely on the matrix-vector products and the number of distinct eigenvalues of the preconditioned matrix [13, 19]. It is well-known fact that the preconditioning technique attempts to make the spectral property better to improve the rate of convergence of Krylov subspace methods [14].

In the light of the preconditioning idea, this paper is devoted to giving the new block triangular preconditioners for the linear systems (1.2). It is shown that, in contrast to the block triangular preconditioner R_k, all the eigenvalues of the proposed new preconditioned matrices are more strongly clustered. Numerical experiments show that the new preconditioners are slightly more efficient than the preconditioner R_k.

The remainder of this paper is organized as follows. In Section 2, the new block triangular preconditioners are presented and algebraic properties are derived in detail. In Section 3, a single column nonzero (1,2) block preconditioner is presented. In Section 4, numerical experiments are presented. Finally, in Section 5 some conclusions are drawn.

2 Block triangular preconditioner

To study the block triangular preconditioners for solving (1.2) conveniently, we consider the following saddle point linear systems:

$$\mathcal{A}x \equiv \begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix}\begin{bmatrix} u \\ p \end{bmatrix} = \begin{bmatrix} g \\ 0 \end{bmatrix} \equiv b, \tag{2.1}$$

where $A\in\mathbb{R}^{n\times n}$ is assumed to be symmetric positive semidefinite with highly nullity and $B\in\mathbb{R}^{m\times n}(m\le n)$. We assume that $\mathcal{A}$ is nonsingular, from which it follows that

$$\operatorname{rank}(B)=m \text{ and } \operatorname{null}(A)\cap\operatorname{null}(B)=0. \tag{2.2}$$

Now we are concerned with the following block triangular matrix as a preconditioner:

$$\mathcal{H}_{U,W} = \begin{bmatrix} A + B^T U^{-1} B & B^T \\ 0 & W \end{bmatrix},$$

where $U, W \in \mathbb{R}^{m\times m}$ are symmetric positive definite matrices.

Proposition 2.1. *Let $\{x_i\}_{i=1}^{n-m}$ be a basis of the null space of B. Then the vectors $(x_i, 0)$ are $n - m$ linear independent eigenvectors of $\mathcal{H}_{U,W}^{-1}\mathcal{A}$ with eigenvalue* 1.

Proof. The eigenvalue problem of $\mathcal{H}_{U,W}^{-1}\mathcal{A}$ is

$$\begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} A + B^T U^{-1} B & B^T \\ 0 & W \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Then

$$Ax + B^T y = \lambda(A + B^T U^{-1} B)x + \lambda B^T y,$$
$$Bx = \lambda W y.$$

From the nonsingularity of $\mathcal{A}$ it follows that $\lambda \neq 0$ and $x \neq 0$. Substituting $y = \lambda^{-1} W^{-1} Bx$ into the first block row, we get

$$\lambda Ax + (1 - \lambda) B^T W^{-1} Bx = \lambda^2 (A + B^T U^{-1} B)x. \tag{2.3}$$

Assume that $x = x_i \neq 0$ is a null vector of B. Then (2.3) simplifies into

$$(\lambda^2 - \lambda) Ax_i = 0.$$

Since a nonzero null vector of B cannot be a null vector of A by (2.2) and $\mathcal{A}$ is nonsingular, the following natural property is derived:

$$\langle Ax, x\rangle > 0 \ \text{ for all } \ 0 \neq x \in \ker(B).$$

It follows that $Ax_i \neq 0$ and $\lambda = 1$. Since $Bx_i = 0$, it follows that $y = 0$ and $\lambda = 1$ is an eigenvalue of $\mathcal{H}_{U,W}^{-1}\mathcal{A}$ with algebraic multiplicity (at least) $n - m$, whose associated eigenvectors are $(x_i, 0)$, $i = 1, 2, \ldots, n - m$. □

Remark 2.1. From Proposition 2.1, it is easy to get that $\mathcal{H}_{U,W}^{-1}\mathcal{A}$ has at least $n-m$ eigenvalues equal to 1 regardless of U and W. The stronger clustering of the eigenvalues can be obtained by choosing two specific matrices such as $U = W$.

To this end, we consider the following indefinite block triangular matrix as a preconditioner:

$$H_s = \begin{bmatrix} A + sB^T W^{-1} B & (1+s)B^T \\ 0 & -W \end{bmatrix},$$

where $W \in \mathbb{R}^{m\times m}$ is a symmetric positive definite matrix and $s > 0$. The next lemma provides that all the eigenvalues of the preconditioned matrix $H_s^{-1}\mathcal{A}$ are strongly clustered, whose proof is similar to that of Theorem 2.4 in [36].

Lemma 2.1. *Suppose that A is symmetric positive semidefinite with nullity r ($r \le m$), B has full rank and λ is an eigenvalue of $H_s^{-1}\mathcal{A}$ with eigenvector (v, q). Then $\lambda = 1$ is an eigenvalue of $H_s^{-1}\mathcal{A}$ with multiplicity n, and $\lambda = \frac{1}{s}$ is an eigenvalue with multiplicity r. The remaining $m - r$ eigenvalues are*

$$\lambda = \frac{\mu}{s\mu + 1},$$

where μ are the nonzero generalized eigenvalues of

$$\mu A v = B^T W^{-1} B v. \tag{2.4}$$

Assume, in addition, that $\{x_i\}_{i=1}^r$ is a base of the null space of A; $\{y_i\}_{i=1}^{n-m}$ is a base of the null space of B; $\{z_i\}_{i=1}^{m-r}$ is a set of linearly independent vectors that complete $\mathrm{null}(A) \cup \mathrm{null}(B)$ *to a basis of $\mathbb{R}^n$. Then a set of linear independent eigenvectors corresponding to $\lambda = 1$ can be found: the $n-m$ vectors $(y_i, 0)$, the r vectors $(x_i, -W^{-1}Bx_i)$ and the $m - r$ vectors $(z_i, -W^{-1}Bz_i)$. The r vectors $(x_i, -sW^{-1}Bx_i)$ are eigenvectors associated with $\lambda = \frac{1}{s}$.*

Proof. Let λ be an eigenvalue of $H_s^{-1}\mathcal{A}$ with eigenvector (v, q). Then

$$\begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} v \\ q \end{bmatrix} = \lambda \begin{bmatrix} A + sB^T W^{-1} B & (1+s)B^T \\ 0 & -W \end{bmatrix} \begin{bmatrix} v \\ q \end{bmatrix},$$

which can be rewritten into

$$Av + B^T q = \lambda(A + sB^T W^{-1}B)v + (1+s)\lambda B^T q, \tag{2.5}$$

$$Bv = -\lambda W q. \tag{2.6}$$

Since $\mathcal{A}$ is nonsingular, it is not difficult to get that $\lambda \neq 0$ and $v \neq 0$. By (2.6), we get

$$q = -\lambda^{-1} W^{-1} B v.$$

Substituting it into (2.5) yields

$$(\lambda^2 - \lambda)Av = \big(-s\lambda^2 + (1+s)\lambda - 1\big)B^T W^{-1} B v. \tag{2.7}$$

If $\lambda = 1$, then (2.7) is satisfied for any arbitrary nonzero vector $v \in \mathbb{R}^n$, and hence $(v, -W^{-1}Bv)$ is an eigenvector of $H_s^{-1}\mathcal{A}$.

If $x \in \text{null}(A)$, then from (2.7) we obtain

$$(\lambda - 1)(s\lambda - 1)B^T W^{-1} B x = 0,$$

from which it follows that $\lambda = 1$ and $\lambda = \frac{1}{s}$ are eigenvalues associated with $(x, -W^{-1}Bx)$ and $(x, -sW^{-1}Bx)$, respectively.

Assume that $\lambda \neq 1$. Combining (2.4) and (2.7) yields

$$\lambda^2 - \lambda = \mu\big(-s\lambda^2 + (1+s)\lambda - 1\big).$$

It is easy to see that the rest $m - r$ eigenvalues are

$$\lambda = \frac{\mu}{s\mu + 1}. \tag{2.8}$$

A specific set of linear independent eigenvectors for $\lambda = 1$ and $\lambda = \frac{1}{s}$ can be readily found. From (2.2), it is not difficult to see that $(y_i, 0)$, $(x_i, -W^{-1}Bx_i)$ and $(z_i, -W^{-1}Bz_i)$ are eigenvectors associated with $\lambda = 1$. The r vectors $(x_i, -sW^{-1}Bx_i)$ are eigenvectors associated with $\lambda = \frac{1}{s}$. □

Remark 2.2. (2.8) gives an explicit formula in terms of the generalized eigenvalues of (2.7) and becomes tightly clustered as $\mu \to \infty$. To illustrate this,

we examine the case $s = 1$, i.e., H_1. We have $\lambda = 1$ with multiplicity $n + r$. The rest $m - r$ eigenvalues are

$$\lambda = \frac{\mu}{\mu + 1}.$$

Since λ is a strictly increasing function of μ on $(0, \infty)$, it is easy to find that the remaining eigenvalues $\lambda \to 1$ as $\mu \to \infty$. In [17], authors considered $k = -1$, i.e., R_{-1} and obtained five distinct eigenvalues: $\lambda = 1$ (with multiplicity $n - m$), $\lambda_{\pm} = \frac{1+\sqrt{5}}{2}$ (each with multiplicity q), the remaining eigenvalues are

$$\lambda_{\pm} = \frac{1 \pm \sqrt{1 + \frac{4\mu}{1+\mu}}}{2} \ (\mu > 0),$$

which lie in the intervals

$$\left(\frac{1-\sqrt{5}}{2}, 0\right) \cup \left(1, \frac{1+\sqrt{5}}{2}\right) \text{ as } \mu \to \infty.$$

Obviously, the eigenvalues of our preconditioned matrix are more clustered than those stated in [17]. That is, the preconditioner H_1 is slightly better than R_{-1} from the viewpoint of eigenvalue clustering. In fact, it may lead to the ill-conditioning of H_1 as $\mu \to \infty$. Golub et al. [18] considered the minimizing of the condition number of the (1,1) block of H_1. The simplest choice is that $W^{-1} = \gamma I (\gamma > 0)$, which leads to all the eigenvalues that are not equal to 1 are

$$\lambda = \frac{\gamma\delta}{1 + \gamma\delta},$$

where δ is the positive generalized eigenvalue of $\delta Ax = B^T Bx$. Obviously, the parameter γ should be chosen to be large such that the eigenvalues are strongly clustered, but not too large such that the (2,2) block of H_1 is too near singular.

From (2.2), it is to get that the nullity of A must be m at most. Lemma 2.1 shows that the higher it is, the more strongly the eigenvalues are clustered. Combining Lemma 2.1 with (1.2), the following theorem is given:

Theorem 2.1. *Suppose that A is symmetric positive semidefinite with nullity m. Then the preconditioned matrix $H_s^{-1}\mathcal{A}$ has precisely two eigenvalues: $\lambda = 1$, of multiplicity n, and $\lambda = \frac{1}{s}$, of multiplicity m. Moreover, if $s = 1$, then*

the preconditioned matrix $H_1^{-1}\mathcal{A}$ has precisely one eigenvalue: $\lambda = 1$ with multiplicity $n + m$.

Remark 2.3. The important consequence of Theorem 2.1 is that the preconditioned matrix $H_s^{-1}\mathcal{A}$ have minimal polynomials of degree at most 2. Therefore, a Krylov subspace method like GMRES applied to a preconditioned linear systems with coefficient matrix $H_s^{-1}\mathcal{A}$ converges in 2 iterations or less, in exact arithmetic [38]. By the above discussion, the choice of the optimal parameter s of the preconditioner H_s is equal to 1. Investigating the preconditioner R_k, it is very difficult to determine the optimal parameter k.

Next, we consider the positive definite block triangular preconditioner as follows:

$$T_h = \begin{bmatrix} A + hB^T W^{-1} B & (1-h)B^T \\ 0 & W \end{bmatrix},$$

where $W \in \mathbb{R}^{m\times m}$ is a symmetric positive definite matrix and $h > 0$.

Similarly, we can get the following results.

Lemma 2.2. *Suppose that A is symmetric positive semidefinite with nullity r ($r \leq m$), B has full rank and λ is an eigenvalue of $T_h^{-1}\mathcal{A}$ with eigenvector (v, q). Then $\lambda = 1$ is an eigenvalue of $T_h^{-1}\mathcal{A}$ with multiplicity n, and $\lambda = -\frac{1}{h}$ is an eigenvalue with multiplicity r. The remaining $m - r$ eigenvalues are*

$$\lambda = -\frac{\mu}{h\mu + 1}, \tag{2.9}$$

where μ are defined by (2.4). *In addition, $\{x_i\}_{i=1}^{r}$, $\{y_i\}_{i=1}^{n-m}$ and $\{z_i\}_{i=1}^{m-r}$ are defined by Lemma* 2.1. *Then a set of linear independent eigenvectors corresponding to $\lambda = 1$ can be found: the n-m vectors $(y_i, 0)$, the r vectors $(x_i, W^{-1}Bx_i)$ and the $m - r$ vectors $(z_i, W^{-1}Bz_i)$. The r vectors $(x_i, -hW^{-1}Bx_i)$ are eigenvectors associated with $\lambda = -\frac{1}{h}$.*

Theorem 2.2. *Suppose that A is symmetric positive semidefinite with nullity m. Then the preconditioned matrix $T_h^{-1}\mathcal{A}$ has precisely two eigenvalues: $\lambda = 1$, of multiplicity n, and $\lambda = -\frac{1}{h}$, of multiplicity m. Moreover, if $h = 1$, the preconditioned matrix $T_1^{-1}\mathcal{A}$ has precisely two eigenvalues: $\lambda = 1$, of multiplicity n, and $\lambda = -1$, of multiplicity m.*

From Theorem 2.2, it is not difficult to find that the choice of the optimal parameter $h(>0)$ of the preconditioner T_h is equal to 1.

3 A single column nonzero (1,2) block preconditioner

We consider the following single column nonzero (1,2) block preconditioner:

$$T = \begin{bmatrix} A + B^T \tilde{W} B & -b_i e_i^T \\ 0 & W \end{bmatrix},$$

where b_i denotes the column i of B^T, and e_i is the i-th column of the $m \times m$ identify matrix,

$$W = \gamma I (\gamma > 0) \text{ and } \tilde{W} = \frac{1}{\gamma} I + \frac{1}{\gamma} e_i e_i^T.$$

It is not difficult to find that $A + B^T \tilde{W} B$ is nonsingular because A is symmetric positive semidefinite and $\tilde{W}$ is symmetric positive definite.

The spectral properties of $T^{-1}\mathcal{A}$ are presented in the following theorem:

Theorem 3.1. *The preconditioned matrix $T^{-1}\mathcal{A}$ has $\lambda = 1$ with multiplicity n and $\lambda = -1$ with multiplicity $m - 1$. Corresponding eigenvectors can be explicitly found in terms of the null space and column space of A.*

Proof. Let λ be any eigenvalue of $T^{-1}\mathcal{A}$, and $z = (x, y)$ be the corresponding eigenvector. Then $T^{-1}\mathcal{A}z = \lambda z$, i.e.,

$$\begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} A + B^T \tilde{W} B & -b_i e_i^T \\ 0 & W \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}. \tag{3.1}$$

Let $QR = [Y\ Z][R^T\ 0^T]^T$ be an orthogonal factorization of B^T, where $R \in \mathbb{R}^{m\times m}$ is upper triangular, $Y \in \mathbb{R}^{n\times m}$, and $Z \in \mathbb{R}^{n\times(n-m)}$ is a basis of the null space of B. Premultiplying (3.1) by the nonsingular and square matrix

$$P = \begin{bmatrix} Z^T & 0 \\ Y^T & 0 \\ 0 & I \end{bmatrix}$$

and postmultiplying by its transpose gives

$$\begin{bmatrix} Z^TAZ & Z^TAY & 0 \\ Y^TAZ & Y^TAY & R \\ 0 & R^T & 0 \end{bmatrix} \begin{bmatrix} x_z \\ x_y \\ y \end{bmatrix}$$

$$= \lambda \begin{bmatrix} Z^TAZ & Z^TAY & 0 \\ Y^TAZ & Y^TAY + \dfrac{1}{\gamma}\left(RR^T + r_i r_i^T\right) & -r_i e_i^T \\ 0 & 0 & \gamma I \end{bmatrix} \begin{bmatrix} x_z \\ x_y \\ y \end{bmatrix}.$$

By inspection, we check $\lambda = 1$, which reduces the above equation to

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & -\dfrac{1}{\gamma}\left(RR^T + r_i r_i^T\right) & R + r_i e_i^T \\ 0 & R^T & -\gamma I \end{bmatrix} \begin{bmatrix} x_z \\ x_y \\ y \end{bmatrix} = 0.$$

Immediately, there exist $n - m$ corresponding eigenvectors of the form $(x_z, x_y, y) = (u, 0, 0)$ for $(n - m)$ linearly independent vectors u. At the same time, we can find that there have m linearly independent eigenvectors, corresponding to $\lambda = 1$, which can be written $(x_z, x_y, y) = \left(0, x_y^*, \frac{1}{\gamma}x_y^*\right)$. That is, there exist n linearly independent eigenvectors corresponding to $\lambda = 1$.

It is not difficult to get that there exist $m - 1$ eigenvectors corresponding to $\lambda = -1$. Indeed, substituting $\lambda = -1$ requires finding a solution to

$$\begin{bmatrix} 2Z^TAZ & 2Z^TAY & 0 \\ 2Y^TAZ & 2Y^TAY + \dfrac{1}{\gamma}\left(RR^T + r_i r_i^T\right) & R - r_i e_i^T \\ 0 & R^T & \gamma I \end{bmatrix} \begin{bmatrix} x_z \\ x_y \\ y \end{bmatrix} = 0.$$

Vectors x_z, x_y, y can be found to solve this equation. Consider any $x^* = Zx_z^* + Yx_y^*$ in the null space of A. Then $AZx_z^* + AYx_y^* = 0$, and we are left with finding a y such that

$$\begin{bmatrix} \dfrac{1}{\gamma}\left(RR^T + r_i r_i^T\right) & R - r_i e_i^T \\ R^T & \gamma I \end{bmatrix} \begin{bmatrix} x_y^* \\ y \end{bmatrix} = 0$$

for the fixed x_y^*. Further, we get

$$\frac{1}{\gamma}\left(RR^T + r_i r_i^T\right)x_y^* + \left(R - r_i e_i^T\right)y = 0, \tag{3.2}$$

$$R^T x_y^* + \gamma y = 0. \tag{3.3}$$

By (3.3), we get $y = -\frac{1}{\gamma}R^T x_y$. Substituting it into (3.2) requires $\frac{2}{\gamma}r_i r_i^T x_y = 0$. In general, we can find exactly $m-1$ eigenvectors orthogonal to r_i. That is, there are $m-1$ eigenvectors of the form $(x_z, x_y, y) = (x_z^*, x_y^*, -\frac{1}{\gamma}R^T x_y^*)$, where x_y^* is orthogonal to r_i, corresponding to $\lambda = -1$. □

Remark 3.1. The following preconditioner was considered in [17], that is,

$$\hat{\mathcal{M}} = \begin{bmatrix} A + B^T \bar{W} B & -b_i e_i^T \\ 0 & W \end{bmatrix},$$

where $W = \gamma I$ ($\gamma > 0$) and $\bar{W} = \frac{1}{\gamma}I - \frac{1}{\gamma}e_i e_i^T$. In practice, the preconditioner $\hat{\mathcal{M}}$ can be with riskiness. In fact, if A is a symmetric positive semidefinite matrix with highly nullity, then $A + \frac{1}{\gamma}B^T(I - e_i e_i^T)B$ may become singular because $I - e_i e_i^T$ is symmetric positive semidefinite. In our numerical experiments, we find that the preconditioner $\hat{\mathcal{M}}$ for solving (1.2) leads to the deterioration of performance when $i = 1$. In this case, the preconditioner $\hat{\mathcal{M}}$ is singular.

4 Numerical experiments

In this section, two examples are given to demonstrate the performance of our preconditioning approach. In our numerical experiments, all the computations are done with MATLAB 7.0. The machine we have used is a PC-Intel(R), Core(TM)2 CPU T7200 2.0 GHz, 1024M of RAM. The initial guess is taken to be

$$x^{(0)} = 0$$

and the stopping criterion is chosen as follows:

$$\|b - \mathcal{A}x^{(k)}\|_2 \leq 10^{-6}\|b\|_2.$$

Example 1. We consider the two-dimensional static Maxwell equations (1.1) in an L-shaped domain ($[-1, 1] \times [-1, 1] - [-1, 0] \times [0, 1]$). For simplicity,

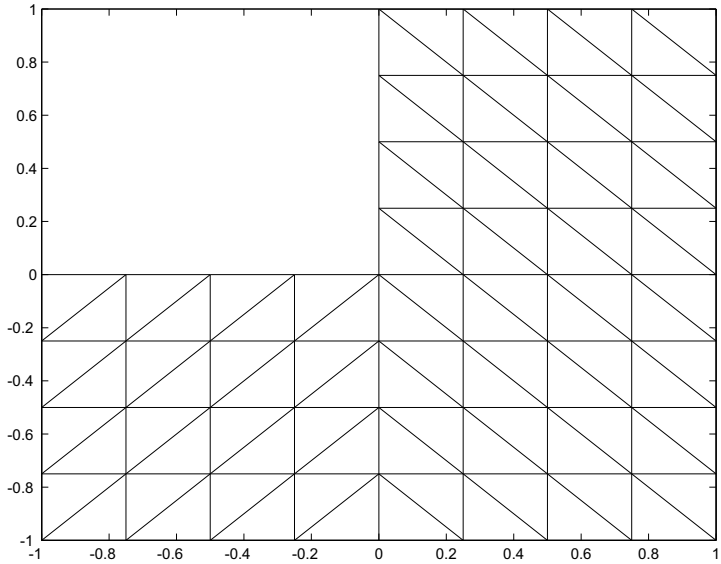


Figure 1 – 8×8 mesh dissection.

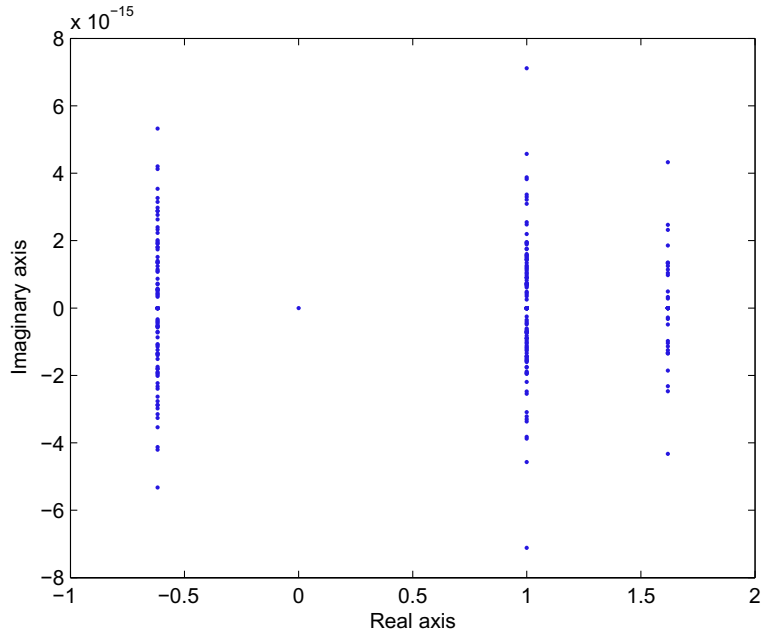
we take a finite element subdivision like Figure 1. Information on sparsity of the relevant matrices is given in Table 1. The test problem is set up so that the right hand side function is equal to 1 throughout the domain.

Mesh	n	m	$nz(A)$	$nz(B)$	order of $\mathcal{A}$
32×32	2240	961	10948	6926	3201
64×64	9088	3969	44932	29198	13057
128×128	36608	16129	182020	119822	52737
256×256	146944	65025	732676	485390	211969

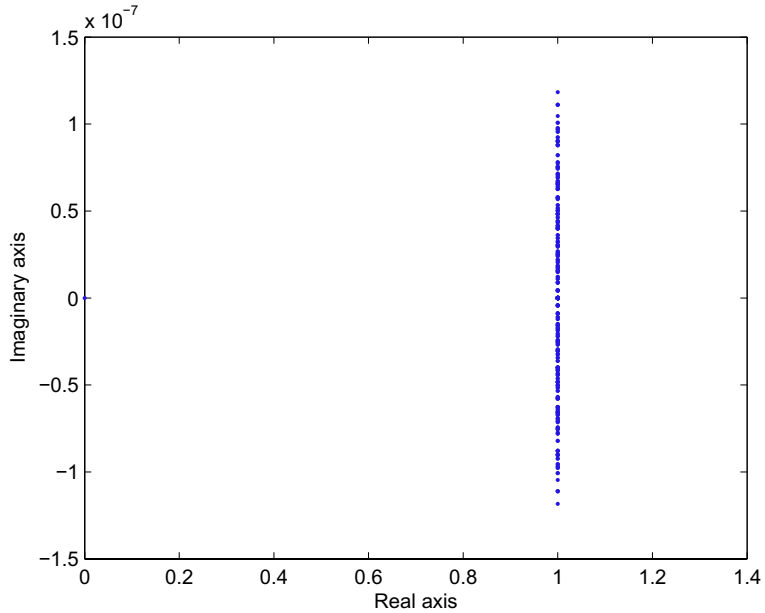
Table 1 – Values of n and m, nonzeros in A and B, order of $\mathcal{A}$.

Here we mainly test four preconditioners: R_{-1}, H_1, T_1 and T. From Remark 2.2, based on the condition number of the matrix, it ensures that the norm of the augmenting term is not too small in comparison with A [35], we set $W^{-1} = 20\frac{\|A\|_1}{\|B\|_1^2} I$. One can see [35] for details.

It is well known that the eigenvalue distribution of the preconditioned matrix gives important insight in the convergence behavior of the preconditioned Krylov subspace methods. For simplicity, we investigate the eigenvalue distribution of the preconditioned matrices $R_{-1}^{-1}\mathcal{A}$ and $H_1^{-1}\mathcal{A}$. Figure 2 plots the



(a) $k = -1$



(b) $s = 1$

Figure 2 – Eigenvalues of $R_{-1}^{-1}\mathcal{A}$ (left) and $H_1^{-1}\mathcal{A}$ (right) with 16×16 grid.

eigenvalues of the preconditioned matrices $R_{-1}^{-1}\mathcal{A}$ and $H_1^{-1}\mathcal{A}$ for 16×16 grid, where left corresponds to $R_{-1}^{-1}\mathcal{A}$ and right corresponds to $H_1^{-1}\mathcal{A}$. It is easy to see that the clustering of the eigenvalues of $H_1^{-1}\mathcal{A}$ is more stronger than that of $H_1^{-1}\mathcal{A}$ in Figure 2.

To investigating the performance of the above four preconditioners, in our numerical experiments some Krylov subspace methods with BiCGStab and GMRES(ℓ) are adopted. As is known, there is no general rule to choose the restart parameter ℓ ($\ell \ll n + m$). This is mostly a matter of experience. To illustrate the efficiency of our methods, we take $\ell = 20$. In Tables 2 and 3, we present some results to illustrate the convergence behaviors of BiCGStab and GMRES(20) preconditioned by R_{-1}, H_1, T_1 and T, respectively. Here i of T is equal to 1. Figures 3 and 4 correspond to Tables 2 and 3, which show the iteration numbers and relative residuals of preconditioned BiCGStab and GMRES(20) employed to solve the saddle point linear systems (1.2), where left in Figures 3-4 corresponds to BiCGStab and right in Figures 3-4 corresponds to GMRES(20). The purpose of these experiments is just to investigate the influence of the eigenvalue distribution on the convergence behavior of BiCGStab and GMRES(20) iterations. "IT" denotes the number of iteration. "CPU(s)" denotes the time (in seconds) required to solve a problem.

Mesh	R_{-1}		H_1		T_1		T	
	IT	CPU(s)	IT	CPU(s)	IT	CPU(s)	IT	CPU(s)
32×32	5	0.1563	3	0.0938	3	0.1406	5	0.1563
64×64	5	0.8281	3	0.4844	3	0.8281	4	0.6563
128×128	5	4.7188	3	2.7813	3	6.0156	4	3.6250
256×256	5	24.8906	2	10.0625	3	33.4688	4	19.5469

Table 2 – Iteration number and CPU(s) of BiCGStab method.

From Tables 2-3, it is not difficult to see that the exact preconditioners R_{-1}, H_1, T_1 and T are in relation to the CPU time, and the iteration numbers of the exact preconditioners R_{-1}, H_1, T_1 and T are insensitive to the changes in the mesh size by using BiCGStab and GMRES(20) to solve the saddle point linear systems (1.2). Although the exact preconditioners R_{-1}, H_1, T_1 and T are quite competitive in terms of convergence rate, robustness and efficiency, the preconditioner H_1 outperforms the preconditioners R_{-1}, T_1 and T from iteration

Mesh	R_{-1}		H_1		T_1		T	
	IT	CPU(s)	IT	CPU(s)	IT	CPU(s)	IT	CPU(s)
32×32	3	0.1563	2	0.1250	2	0.1719	3	0.1406
64×64	3	0.8125	2	0.6250	2	1.1094	3	0.7813
128×128	3	4.6719	2	3.7656	2	7.7031	3	4.4844
256×256	3	19.6875	2	14.6563	2	33.0625	3	19.3750

Table 3 – Iteration number and CPU(s) of GMRES(20).

Matrix name	order of $\mathcal{A}$	n	m	nnz($\mathcal{A}$)
GHSindef/k1san	67759	46954	20805	559774

Table 4 – Characteristics of the test matrix from the UF Sparse Matrix Collection.

number and CPU time. Compared with the preconditioners R_{-1}, T_1 and T, the preconditioner H_1 may be the 'best' choice. Comparing the performance of BiCGStab to the performance of GMRES(20) is not within our stated goals, but having results using more than one Krylov solver allows us to confirm the consistency of convergence behavior for most problems.

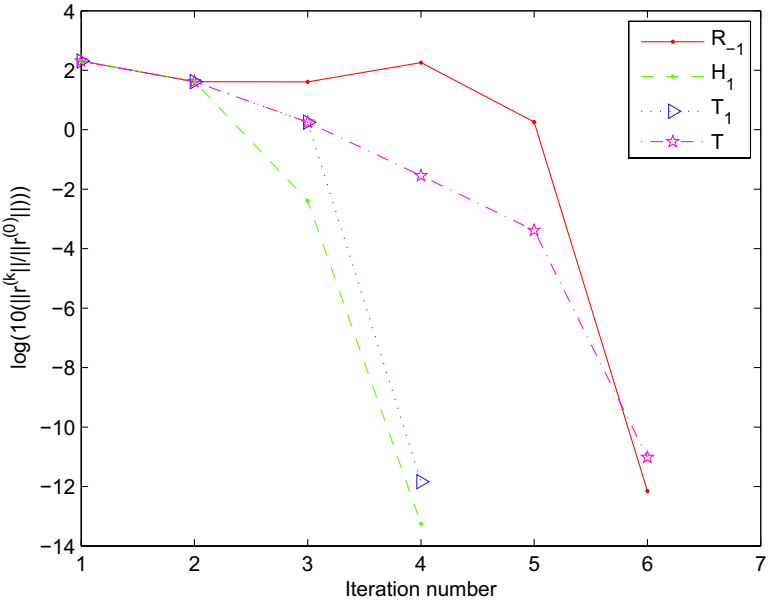
Example 2. A matrix from the UF Sparse Matrix Collection [39].

The test matrix is GHSindef/k1san, coming from UF Sparse Matrix Collection, which is an ill-conditioned matrix from Aug. system modelling the underground of Strazpod Ralskem mine by MFE. The characteristics of the test matrix are listed in Table 4. The numerical results from using the BiCGStab and GMRES(20) methods preconditioned by the above four preconditioners to solve the corresponding saddle point linear systems are given in Table 5. Figure 5 is in concord with Table 5, where left in Figure 5 corresponds to BiCGStab and right in Figure 5 corresponds to GMRES(20).

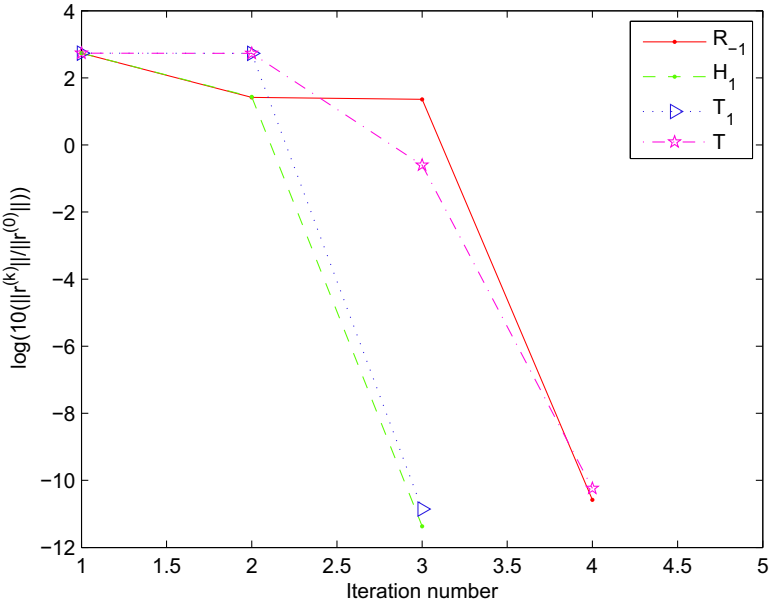
From Table 5, it is easy to see that the preconditioners R_{-1}, H_1, T_1 and T are really efficient when BiCGStab and GMRES(20) methods are used to solve

	R_{-1}		H_1		T_1		T	
	IT	CPU(s)	IT	CPU(s)	IT	CPU(s)	IT	CPU(s)
BiCGStab	53	96.3594	17	31.5625	19	38.1563	21	50.6406
GMRES(20)	31	61.3906	13	26.2656	13	27.8906	14	39

Table 5 – Iteration number and CPU(s) of BiCGStab and GMRES(20).

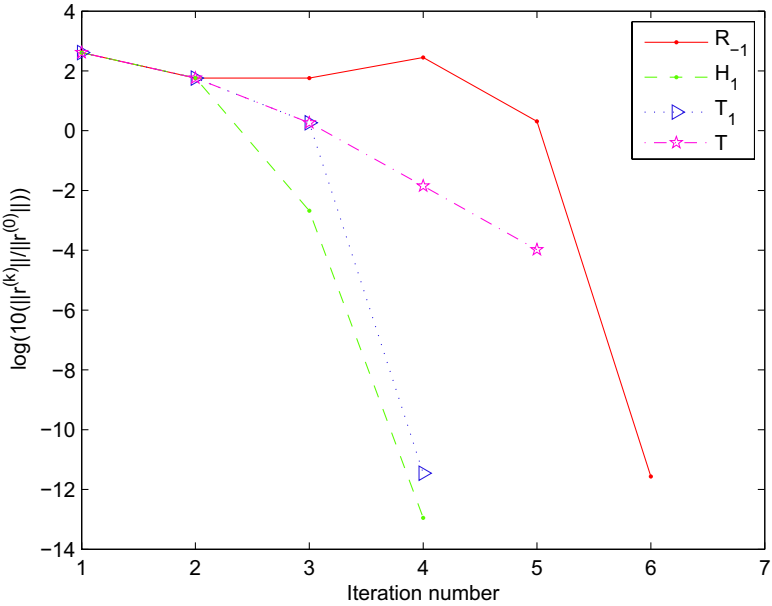


(a) 32×32 mesh

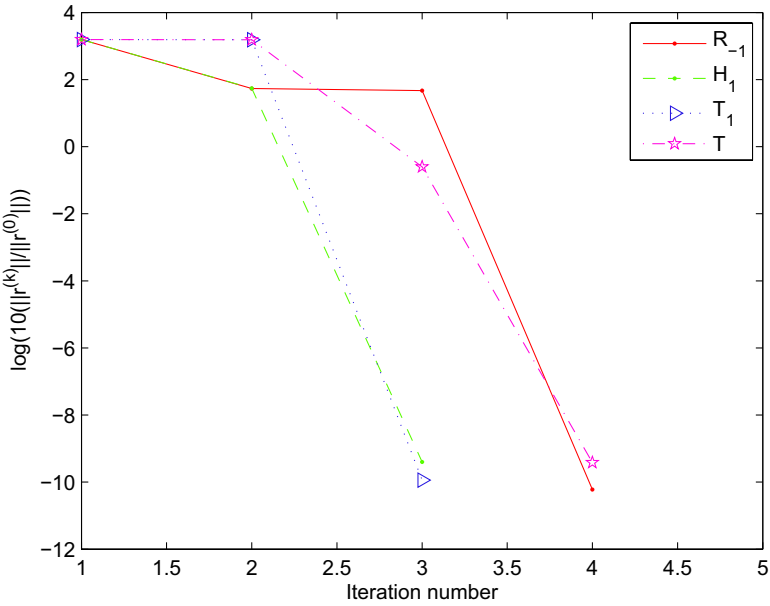


(b) 32×32 mesh

Figure 3 (to be continue) – Iteration number of BiCGStab (top) and GMRES(20) (bottom).

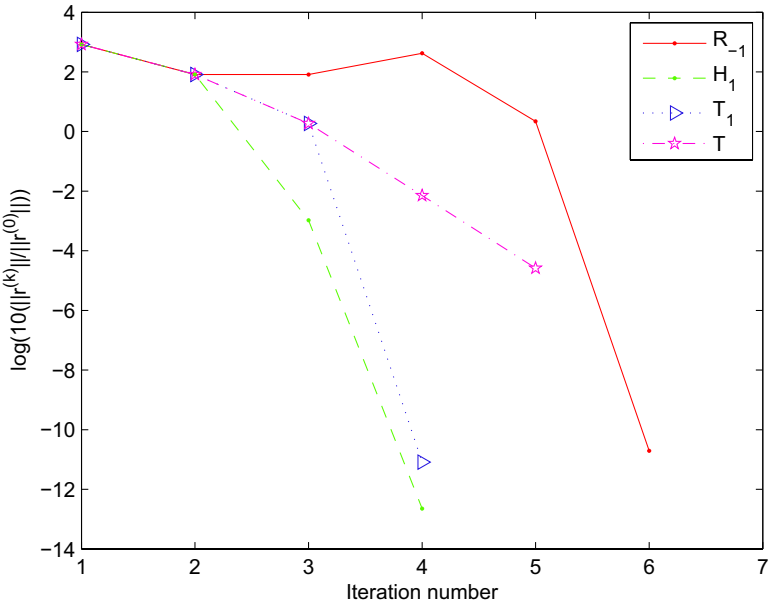


(c) 64×64 mesh

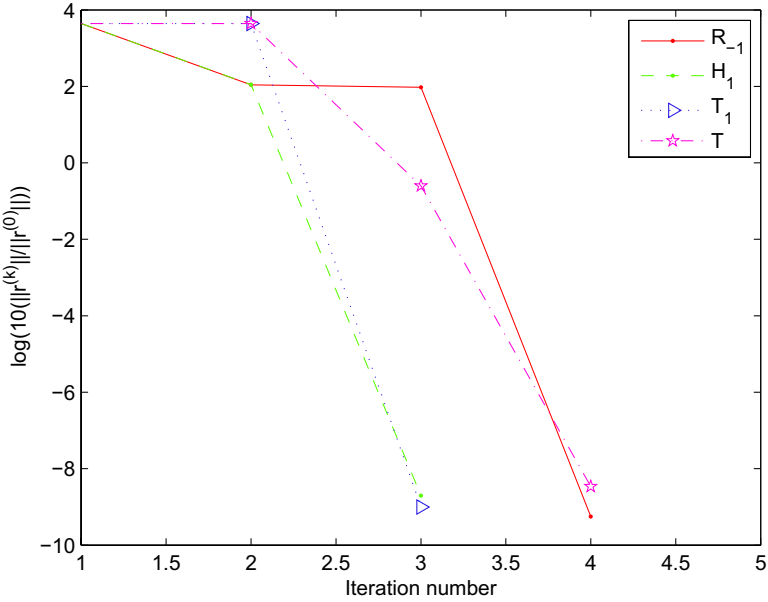


(d) 64×64 mesh

Figure 3 (to be continue) – Iteration number of BiCGStab (top) and GMRES(20) (bottom).



(e) 128×128 mesh



(f) 128×128 mesh

Figure 3 (conclusion) – Iteration number of BiCGStab (top) and GMRES(20) (bottom).

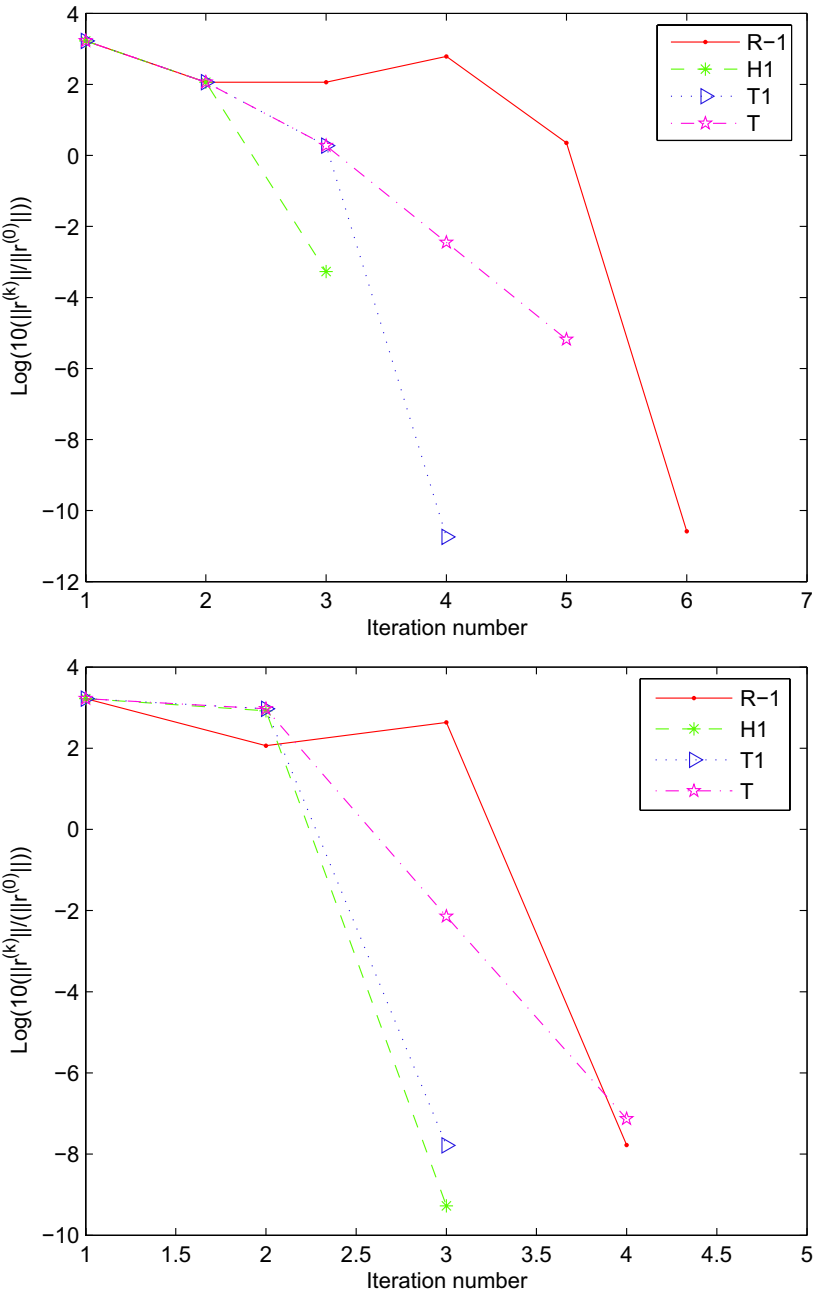


Figure 4 – Iteration number of BiCGStab (top) and GMRES(20) (bottom) with 256×256.

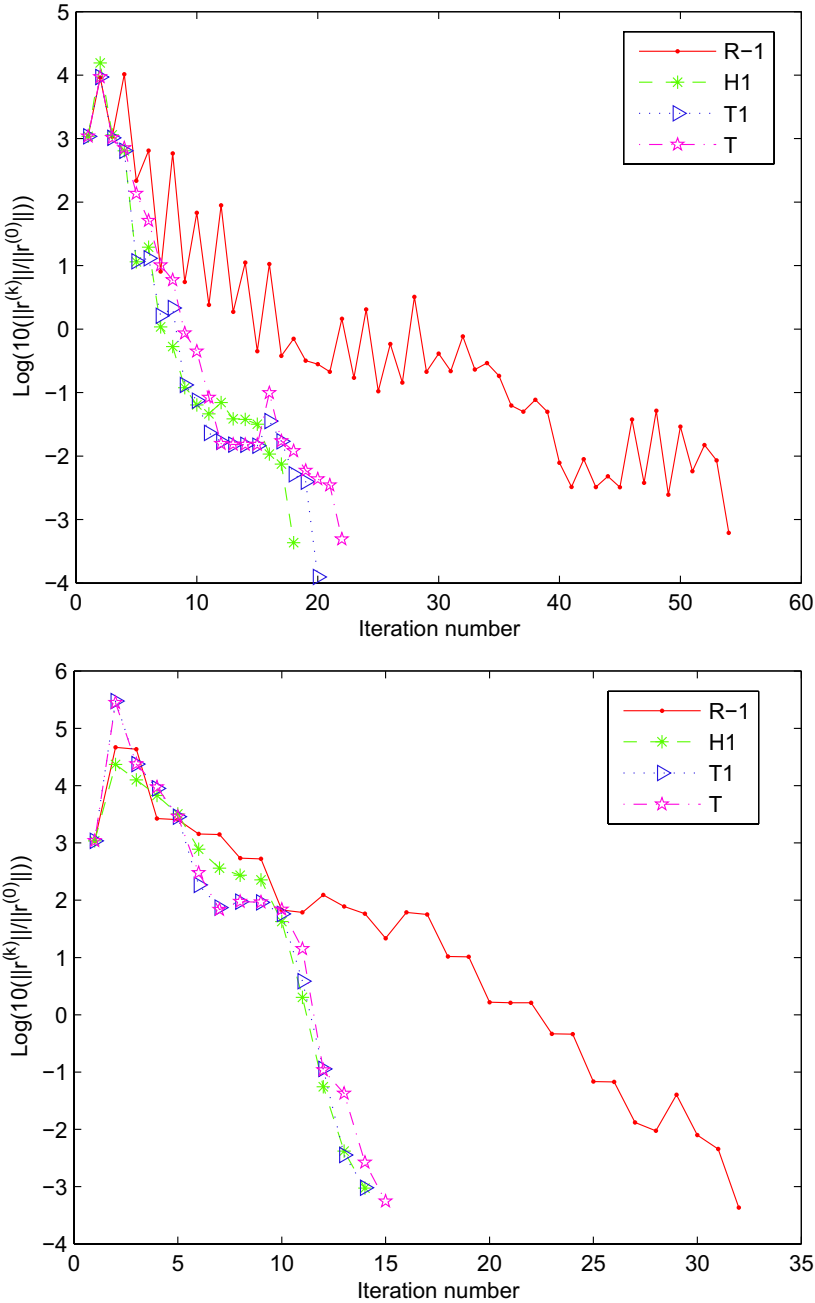


Figure 5 – Iteration number of BiCGStab (top) and GMRES(20) (bottom) method for GHSindef/k1san.

the saddle point systems with the coefficient matrix being GHSindef/k1san. It is not difficult to find that the preconditioner H_1 are superior to the preconditioners R_{-1}, T_1 and T from iteration number and CPU time under certain conditions. That is, the preconditioner H_1 is quite competitive in terms of convergence rate, robustness and efficiency.

5 Conclusion

In this paper, we have proposed three types of block triangular preconditioners for iteratively solving linear systems arising from finite element discretization of the Maxwell equations. The preconditioners have the attractive property to improve the eigenvalue clustering of the coefficient matrix. Furthermore, numerical experiments confirm the effectiveness of our preconditioners.

In fact, in Section 2, our methodology can extend the unsymmetrical case, that is, the (1,2) block and the (2,1) block of the saddle point systems are unsymmetrical.

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