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A stabilized finite element method to pseudoplastic flow governed by the Sisko relation

MARCIO ANTÔNIO BORTOLOTI[1] and JOSÉ KARAM FILHO[2*]

[1]Departamento de Ciências Exatas, Universidade Estadual do Sudoeste da Bahia, UESB, Estrada do Bem-Querer, Km 4, 45083-900 Vitória da Conquista, BA, Brazil
[2]Coordenação de Mecânica Computacional, Laboratório Nacional de Computação Científica, LNCC, Av. Getúlio Vargas, 333, 25651-070 Petrópolis, RJ, Brazil

E-mails: marciob@lncc.br / jkfi@lncc.br

Abstract. In this work, a consistent stabilized mixed finite element formulation for incompressible pseudoplastic fluid flows governed by the Sisko constitutive equation is mathematically analysed. This formulation is constructed by adding least-squares of the governing equations and of the incompressibility constraint, with discontinuous pressure approximations, allowing the use of same order interpolations for the velocity and the pressure. Numerical results are presented to confirm the mathematical stability analysis.

Mathematical subject classification: Primary: 65M60; Secondary: 65N30.

Key words: finite element method, numerical analysis, Sisko relation, non-Newtonian flow.

1 Introduction

In modeling some kinds of fluids, the Sisko constitutive relation comes out from the Cross model [6], when the apparent viscosity lies in a range between the pseudoplastic region and the lower Newtonian plateau. A good alternative constitutive equation of the Sisko type for blood flow has been proposed by [13], for example.

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*Corresponding author

The nonlinearity of this relation together with the incompressibility constraint may generate numerical instabilities when some classical numerical methods are used. For classical methods, in case of velocity and pressure formulations, it is well known that, even for the linear case, different interpolation orders for these variables have to be used in order to satisfy the Babuška-Brezzi stability condition [2, 4].

In this work, to avoid the use of penalization methods or reduced integration and to recover the stability and accuracy of the solution of same interpolation orders in primitive variables, a consistent mixed stabilized finite element formulation is presented. It is constructed by adding the least-squares of the governing equations and the incompressibility constraint, with continuous velocity and discontinuous pressure interpolations. The present formulation is here mathematically analysed based on Scheurer's theorem, [12]. Stability conditions and error estimates are established when the Sisko relation is considered. Numerical examples are presented to confirm the stability analysis.

2 Definition of the problem

Let Ω be a bounded domain in $\mathbb{R}^n$ where the positive integer n, denotes the space dimension. We consider the stationary incompressible creep flow problem governed by $-\operatorname{div}\sigma = \mathbf{f}$ in Ω, where $\sigma : \Omega \to \mathbb{R}^n \times \mathbb{R}^n$ denotes the Cauchy stress tensor for the fluid and $\mathbf{f}$ denotes the body forces.

The governing equation, written above, is subjected to the incompressibility constraint $\operatorname{div}\mathbf{u} = 0$ in Ω, where $\mathbf{u}$ denotes the velocity field.

The Sisko model is characterized by a linear superspostion between the Newtonian and the pseudoplastic effects, presenting a dependence of the viscosity μ with the shear-strain rate tensor, defining an apparent viscosity $\mu(|\varepsilon(\mathbf{u})|)$, leading to the stress tensor of the form

$$\sigma = -p\mathbf{I} + \mu(|\varepsilon(\mathbf{u})|)\varepsilon(\mathbf{u}) \quad \text{with} \quad \mu(|\varepsilon(\mathbf{u})|) = \lambda_1 + \lambda_2\nu(|\varepsilon(\mathbf{u})|)$$

and $\nu(s) = s^{\alpha-2}$, where λ_1, λ_2 are two positive constitutive constants, $\alpha \in]1, 2[$ is the power index, p is the hydrostatic pressure, $\mathbf{I} \in \mathbb{R}^n \times \mathbb{R}^n$ is the identity tensor, $|\cdot|$ denotes the Euclidean tensor norm and

$$\varepsilon(\mathbf{u}) = \frac{1}{2}\big[\nabla\mathbf{u} + \nabla^T\mathbf{u}\big]$$

is the symmetric part of the gradient of $\mathbf{u}$.

With the above considerations, together with boundary condition of Dirichlet type, the resulting problem is: find $(\mathbf{u}, p) \in C^2(\Omega) \times C^1(\Omega)$ such that

$$\begin{cases} -\operatorname{div}(\mu(|\varepsilon(\mathbf{u})|)\varepsilon(\mathbf{u})) + \nabla p &= \mathbf{f} \text{ in } \Omega \\ \operatorname{div}\mathbf{u} &= 0 \text{ in } \Omega \\ \mathbf{u} &= \overline{\mathbf{u}} \text{ on } \partial\Omega \end{cases} \tag{1}$$

where $\partial\Omega$ denotes the boundary of Ω.

Physically, pseudoplastic flows are characterized by a viscosity decreasing continuously and smoothly with increasing of shear rate, and this behaviour occurs in a limited range of shear rate, generating the viscosity plateaus, where we can see that the apparent viscosity $\mu(s)$ is a bounded continuous function such that

$$\mu_\infty \leq \mu(s) \leq \mu_0 \tag{2}$$

for $0 < g_0 \leq s \leq g_\infty$ with g_0 and g_∞ being the shear rate finite limits and μ_0 and μ_∞ corresponding to the finite limiting Newtonian plateaus for low and high shear rate, respectively, [3].

It can be seen that from continuous classical Galerkin formulation associated with problem (1), we can obtain

$$\|\varepsilon(\mathbf{u})\|_0 \leq \frac{C}{\mu_\infty}\|\mathbf{f}\|_0 \tag{3}$$

for all $\mathbf{u} \in W_0^{1,2}(\Omega)$, where C is the constant in Poincaré's inequality.

3 Petrov-Galerkin-Like formulation

To generate the stabilized finite element method proposed here, the following definitions will be used.

Let $L^p(\Omega) = \{u | u \text{ is measurable}, \int_\Omega |u(x)|^p d\Omega < \infty\}$ be the class of all measurable functions u, such that, u is p-integrable in Ω and let $L_0^p(\Omega) = \{u \in L^p(\Omega), \int_\Omega u \; d\Omega = 0\}$ be the class of functions in $L^p(\Omega)$ such that u has null mean. Let $W^{m,p}(\Omega)$ be the Sobolev space $W^{m,p}(\Omega) = \{u \in L^p(\Omega) | D^\beta u \in L^p(\Omega), 0 \leq |\beta| \leq m\}$ with

$$D^\beta u = \frac{\partial^{|\beta|} u}{\partial^{\beta_1} x_1 \cdots \partial^{\beta_m} x_m}$$

where β_i is a natural integer and $|\beta| = \beta_1 + \cdots + \beta_m$. The $W_0^{m,p}(\Omega)$ is defined as the space of functions $u \in W^{m,p}(\Omega)$ such that $D^\beta u = 0$ on $\partial\Omega$, for all β with $|\beta| \leq m-1$, [1].

The norm in the space $W^{m,p}(\Omega)$ is defined as

$$\|u\|_{m,p} = \left(\sum_{|\beta| \leq m} \|D^\beta u\|_p \right)^{1/p}, \quad 1 \leq p < \infty,$$

where $\|u\|_p$ is the $L_p(\Omega)$ norm defined as

$$\|u\|_p = \left(\int_\Omega |u(x)|^p dx \right)^{1/p}, \quad x \in \Omega.$$

In this paper, we will denote $\|u\|_1 = \|u\|_{1,2}$ and $\|u\|_0 = \|u\|_2$.

We assume for simplicity $\Omega \subset \mathbb{R}^n$, a polygonal domain discretized by a classical uniform mesh of finite elements with N_e elements, such that

$$\overline{\Omega} = \bigcup_{e=1}^{N_e} \overline{\Omega}^e, \ \Omega^{e_i} \cap \Omega^{e_j} = \emptyset \ \text{ for all } \ i \neq j$$

where Ω^e denotes the interior of the eth element and $\overline{\Omega}^e$ is its closure.

Let $S_h^k(\Omega)$ be the finite element space of the Lagrangean continuous polynomials in Ω of degree k and $Q_h^l(\Omega)$ the finite element space of the Lagrangean discontinuous polynomials in Ω of degree l. Thus we can define the approximation spaces $\mathbf{V_h} = (S_h^k(\Omega) \cap W_0^{1,2}(\Omega))^n$ and $\mathbf{W_h} = Q_h^l(\Omega) \cap L^2(\Omega)$ to velocity and pressure respectively, that can be generated by triangles or quadrilaterals.

Remark 3.1. For the Galerkin method, k and l must be different orders even for $k \geq 2$ and one can follow [7, 8] and [9], for example, to see the limitations for the combinations of k and l.

Remark 3.2. With the present consistent stabilized formulation, all the combinations of different orders are possible, optimal and suboptimal, but it is possible to use same orders for k and l, with complete polynomials, providing $k \geq 2$, as follows.

In this work, to obtain velocity and pressure approximations to problem (1), we define the following variational form, constructed by adding the least squares of the linear momentum and of the continuity equations to the Galekin formulation, generating the following problem with homogeneous boundary condition considered without lost of generalities:

Problem PG$_{hd}$. Given $\mathbf{f} \in W^{-1,2}(\Omega)$, the dual of $W^{1,2}(\Omega)$, find $U_h \in \mathbf{V_h} \times \mathbf{W_h}$, such that

$$\begin{cases} (A_h(U_h), V_h) + B_h(p_h, \mathbf{v_h}) = F_h(V_h) \quad \forall \quad V_h \in \mathbf{V_h} \times \mathbf{W_h} \\ B_h(q_h, \mathbf{u_h}) = 0 \quad \forall \quad q_h \in \mathbf{W_h} \end{cases}$$

where

$$\begin{aligned} (A_h(U_h), V_h) &= (\mu(|\epsilon(\mathbf{u_h})|)\varepsilon(\mathbf{u_h}), \varepsilon(\mathbf{v_h})) + \delta_2 \vartheta(\operatorname{div} \mathbf{u_h}, \operatorname{div} \mathbf{v_h}) \\ &+ \frac{\delta_1 h^2}{\vartheta}(-\Delta_\mu \mathbf{u_h} + \nabla p_h, -\Delta_\mu \mathbf{v_h} + \nabla q_h)_h, \end{aligned} \tag{4}$$

$$B_h(p_h, \mathbf{v_h}) = -(p_h, \operatorname{div} \mathbf{v_h}), \tag{5}$$

$$F_h(V_h) = \mathbf{f}(\mathbf{v_h}) + \frac{\delta_1 h^2}{\vartheta}(\mathbf{f}, -\Delta_\mu \mathbf{v_h} + \nabla q_h)_h, \tag{6}$$

with h denoting the mesh parameter, $\mu(|\varepsilon(\mathbf{u_h})|)$ the apparent viscosity,

$$(u, v) = \int_\Omega u\, v\, dx, \ \ (u, v)_h = \sum_{e=1}^{N_e} \int_{\Omega^e} u\, v\, dx,$$

δ_1 and δ_2 being positive constants denoted as stability parameters, $\Delta_\mu \mathbf{u_h} = \operatorname{div}(\mu(|\varepsilon(\mathbf{u_h})|)\varepsilon(\mathbf{u_h}))$, $U_h = \{\mathbf{u}_h, p_h\}$, $V_h = \{\mathbf{v}_h, q_h\}$ and ϑ being a dimensional parameter. We can note that when $\delta_1 = \delta_2 = 0$, **Problem PG$_{hd}$** reduces to the Galerkin formulation which, for interpolations of same order, is unstable exhibiting spurious pressure modes or presenting the locking of the velocity field, [9]. The nonlinear Problem PG$_{hd}$ preserves the good properties of the linear analogous of [11] in the sense that it accommodates, for $k \geq 2$ equal-order interpolations for velocity and pressure as will be shown in the following analysis and confirmed later by the obtained numerical results.

This formulation is consistent, being easy to verify that the exact solution of problem defined in (1), $U = \{\mathbf{u}, p\}$, satisfies the PG$_{hd}$ problem.

4 Finite element analysis

The finite element analysis is developed here by considering solutions in Hilbert Spaces, as in [10]. We start the analysis by rewriting the discontinuous pressure approximation p_h as

$$p_h = p_h^* + \overline{p}_h, \tag{7}$$

with $p_h^* \in \mathbf{W_h^*}$ and $\overline{p}_h \in \overline{\mathbf{W}}_\mathbf{h}$ where $\mathbf{W_h^*} = \{p_h^* \in \mathbf{W_h} \cap L_0^2(\Omega^e), \nabla p_h^e = \nabla p_h^*\}$ is the subspace of the pressure with zero mean at the element level and $\overline{\mathbf{W}}_\mathbf{h} = \{\overline{p}_h \in \mathbf{W_h}; \nabla \overline{p}_h^e = 0, \overline{p}_h^e = \int_{\Omega^e} p_h^e d\Omega^e / \int_{\Omega^e} d\Omega^e\}$ is the subspace of the piecewise constant pressure, where p_h^e represents the constraint of p_h in element Ω^e.

The discontinuous pressure allows the satisfaction of the incompressibility constraint at element level in contrast to the continuous approximations, which satisfies the constraint only in global sense. Considering this segregation, the Problem PG$_{hd}$ can be rewritten as the following variational form, which considers the pressure variable p_h written as functions of $p_h^* \in \mathbf{W}_h^*(\Omega)$ and $\overline{p}_h \in \overline{\mathbf{W}_h}$, as was described above:

Problem PG$^*_{hd}$. Given $\mathbf{f} \in W^{-1,2}(\Omega)$, find $\{\mathbf{u_h}, p_h^*, \overline{p}_h\} \in \mathbf{V_h} \times \mathbf{W_h^*} \times \overline{\mathbf{W}}_\mathbf{h}$, such that

$$\begin{cases} (A_h^*(U_h^*), V_h^*) + B_h(\overline{p}_h, \mathbf{v_h}) = F_h^*(V_h^*) \quad \forall \quad V_h^* \in \mathbf{V_h} \times \mathbf{W_h^*} \\ B_h(\overline{q}_h, \mathbf{u_h}) = 0 \quad \forall \quad \overline{q}_h \in \overline{\mathbf{W}}_\mathbf{h} \end{cases}$$

where

$$\begin{aligned} (A_h^*(U_h^*), V_h^*) &= (\mu(|\varepsilon(\mathbf{u_h})|)\varepsilon(\mathbf{u_h}), \varepsilon(\mathbf{v_h})) + B_h(p_h^*, \mathbf{v_h}) + B_h(q_h^*, \mathbf{u_h}) \\ &+ \delta_2 \vartheta (\operatorname{div} \mathbf{u_h}, \operatorname{div} \mathbf{v_h}) + \frac{\delta_1 h^2}{\vartheta}(-\Delta_\mu \mathbf{u_h} + \nabla p_h^*, -\Delta_\mu \mathbf{v_h} + \nabla q_h^*)_h, \end{aligned} \tag{8}$$

$$B_h(p_h^*, \mathbf{v_h}) = -(p_h^*, \operatorname{div} \mathbf{v_h}), \tag{9}$$

$$B_h(\overline{p}_h, \mathbf{v_h}) = -(\overline{p}_h, \operatorname{div} \mathbf{v_h}), \tag{10}$$

$$F_h^*(V_h^*) = \mathbf{f}(\mathbf{v_h}) + \frac{\delta_1 h^2}{\vartheta}(\mathbf{f}, -\Delta_\mu \mathbf{v_h} + \nabla q_h^*)_h, \tag{11}$$

$U_h^* = \{\mathbf{u}_h, p_h^*\}$ and $V_h^* = \{\mathbf{v}_h, q_h^*\}$.

Lemma 4.1. *For $\mu(s)$ bounded, continuous and smooth real function such that $|d\mu(s)/ds| \leq M$, there exists a positive constant C, independent of h, such that $h\|\Delta_\mu \mathbf{u_h}\|_{0h} \leq C\|\varepsilon(\mathbf{u_h})\|_0$ where $\|u\|_{0h}^2 = (u, u)_h$.*

Proof. From the inverse estimate

$$h\|\operatorname{div}\varepsilon(\mathbf{u_h})\|_{0h} \leq C_h\|\varepsilon(\mathbf{u_h})\|_0, \tag{12}$$

typical of finite element methods, [5], $0 < \mu_\infty \leq \mu(s) \leq \mu_0$ and $|d\mu(s)/ds| \leq M$, we have the inverse estimate proposed, with $C = \mu_0 C_h + M$. □

Lemma 4.2. *Assuming the same considerations of Lemma 4.1, there is a positive constant C_l, independent of h, such that, $h\| - \Delta_\mu \mathbf{u_h} + \Delta_\mu \mathbf{v_h}\|_{0h} \leq C_l\|\varepsilon(\mathbf{u_h}) - \varepsilon(\mathbf{v_h})\|_0$ for all $\mathbf{u_h}$, $\mathbf{v_h} \in \mathbf{V_h}$ with $h > 0$.*

Proof. From the triangular inequality, the Lemma 4.1 and (12) we have

$$h^2\| - \Delta_\mu \mathbf{u_h} + \Delta_\mu \mathbf{v_h}\|_{0h}^2 \leq 4[(\mu_0^2 C_h^2 + M^2)\|\varepsilon(\mathbf{u_h} - \mathbf{v_h})\|_0^2$$

$$+ (1 + C_h^2)\|\varepsilon(\mathbf{v_h})\|_0^2\|\mu(\mathbf{u_h}) - \mu(\mathbf{v_h})\|_{0h}^2].$$

The mean value theorem yields

$$h^2\| - \Delta_\mu \mathbf{u_h} + \Delta_\mu \mathbf{v_h}\|_{0h}^2 \leq 4\Bigg[(\mu_0^2 C_h^2 + M^2)\|\varepsilon(\mathbf{u_h}) - \varepsilon(\mathbf{v_h})\|_0^2$$

$$+ (1 + C_h^2)\|\varepsilon(\mathbf{v_h})\|_0^2 \sup_{0\leq\theta\leq 1} \|\nabla\mu(|\varepsilon((1-\theta)\mathbf{u_h} + \theta\mathbf{v_h})|)\|_{0h}^2\|\mathbf{u_h} - \mathbf{v_h}\|_{0h}^2\Bigg].$$

Since $|d\mu(s)/ds| \leq M$ for all $\mathbf{u_h} \in \mathbf{V_h}$ and from Korn's inequality, we can conclude the result, with the constant C_l given by

$$C_l = 4\left[(\mu_0^2 C_h^2 + M^2) + (1 + C_h^2)M^2 C_K^2 \sup_{\mathbf{v_h}\in\mathbf{V_h}} \|\varepsilon(\mathbf{v_h})\|_0^2\right]$$

where C_K is the constant of the Korn's inequality. The lemma is obtained as a consequence of

$$\|\varepsilon(\mathbf{u}_h)\|_0 \leq C(\lambda_1, \lambda_2, \mu_\infty, C, \delta_1, \Omega)\|\mathbf{f}\|_0 \text{ for all } \mathbf{u}_h \in \mathbf{V}_h, \tag{13}$$

that is, naturally, obtained from Problem PG_{hd} and consequently yields C_l as a finite constant, where C is the constant of the Poincaré inequality. □

Definition 4.3. *Let* $|||U_h||| = \|U_h\| + h(\|\operatorname{div} \varepsilon(\mathbf{u_h})\|_{0h} + \|\varepsilon(\mathbf{u_h})\|_{0h} + \|\nabla p_h\|_{0h})$ *be a mesh-dependent norm on the product space* $H_0^1(\Omega) \times L^2(\Omega)$*, where* h *denotes the mesh parameter and* $\|U_h\|^2 = \|\mathbf{u_h}\|_1^2 + \|p_h\|_0^2$ *is the norm defined in* $\mathbf{V_h} \times \mathbf{W_h}$.

Lemma 4.4 (Equivalence of the norms). *There exists a positive constant* κ *such that* $\|U_h\| \leq |||U_h||| \leq \kappa \|U_h\|$ *for all* $U_h \in \mathbf{V_h}$.

Proof. The inequality $\|U_h\| \leq |||U_h|||$ is immediate. In other hand, from the definition of $|||U_h|||$, from the inverse estimate (12) and from the inverse estimate for the pressure, see [5],

$$h\|\nabla p_h^*\|_{0h} \leq C_p \|p_h^*\|_0,$$

we have $|||U_h||| \leq \|U_h\| + (C_h + 1)\|\varepsilon(\mathbf{u_h})\|_0 + C_p\|p_h\|_0$. Using the classical inequality,

$$\frac{1}{\sqrt{n}}\|\operatorname{div} \mathbf{u}\|_0 \leq \|\varepsilon(\mathbf{u})\|_0 \leq \|\mathbf{u}\|_1, \tag{14}$$

we complete the proof of Lemma 4.4, with $\kappa = 1 + \max\{C_h + 1, C_p\}$. □

With the above results we can establish the following result that will be needed later to generate the estimates in Theorem 4.9.

Theorem 4.5. *There exists a positive constant* γ_c *such that*

$$|(A_h^*(U^*) - A_h^*(V_h), U_h^* - V_h^*)| \leq \gamma_c |||U^* - V_h^*||| \, \|U_h^* - V_h^*\|$$

for all $U^* \in W_0^{1,2}(\Omega) \times L^2(\Omega)$ *and* $U_h^*, V_h^* \in \mathbf{V_h} \times \mathbf{W_h^*}$.

Proof. By the consistency of the problem PG_{hd}^* and from (4) we have

$$|(A_h^*(U^*) - A^*(V_h^*), U_h^* - V_h^*)| \leq \lambda_1 |(\varepsilon(\mathbf{u} - \mathbf{v_h}), \varepsilon(\mathbf{u_h} - \mathbf{v_h}))|$$

$$+ \lambda_2 |(\nu(|\varepsilon(\mathbf{u})|)\varepsilon(\mathbf{u}) - \nu(|\varepsilon(\mathbf{v_h})|)\varepsilon(\mathbf{v_h}), \varepsilon(\mathbf{u_h} - \mathbf{v_h}))| + |B_h(p_h^* - q_h^*, \mathbf{u} - \mathbf{v_h})|$$

$$+ \delta_2 \vartheta |(\operatorname{div} \mathbf{u} - \operatorname{div} \mathbf{v_h}, \operatorname{div} \mathbf{u_h} - \operatorname{div} \mathbf{v_h})| + |B_h(p^* - q_h^*, \mathbf{u_h} - \mathbf{v_h})|$$

$$+ \frac{\delta_1 h^2}{\vartheta} \Big| \big(- \Delta_\mu \mathbf{u} + \Delta_\mu \mathbf{v_h} + \nabla(p^* - q_h^*), -\Delta_\mu \mathbf{u_h} + \Delta_\mu \mathbf{v_h} + \nabla(p_h^* - q_h^*) \big)_h \Big|$$

From the continuity of the two first terms in the right hand side above, [12], and using the inequality (14) we have

$$|(A_h^*(U^*) - A_h^*(V_h^*), U_h^* - V_h^*)| \leq [(\lambda_1 + \lambda_2 + n\vartheta\delta_2)\|\mathbf{u} - \mathbf{v_h}\|_1$$

$$+ \sqrt{n}\|p^* - q_h^*\|_0]\|\mathbf{u_h} - \mathbf{v_h}\|_1 + \frac{\delta_1 h}{\vartheta}$$

$$\times \big(\|\nu(|\varepsilon(\mathbf{u})|)\operatorname{div}\varepsilon(\mathbf{u}) - \nu(|\varepsilon(\mathbf{v_h})|)\operatorname{div}\varepsilon(\mathbf{v_h})\|_{0h}$$

$$+ \|\nabla p^* - \nabla q_h^*\|_{0h} + \|\varepsilon(\mathbf{u})\nabla\mu(\mathbf{u}) - \varepsilon(\mathbf{v_h})\nabla\mu(\mathbf{v_h})\|_{0h}\big)$$

$$\times \big(\|\mathbf{u_h} - \mathbf{v_h}\|_1 + \|p_h^* - q_h^*\|_0\big).$$

By using Lemma 4.2 and identifying the $\||\cdot|\|$ norm, we can conclude

$$|(A_h^*(U^*) - A_h^*(V_h^*), U_h^* - V_h^*)| \leq \gamma_c \||U^* - V_h^*|\| \, \|U_h^* - V_h^*\|,$$

where $\gamma_c = \max\{\lambda_1 + \lambda_2 + n\vartheta\delta_2, \sqrt{n}, \frac{\delta_1}{\vartheta}\}$. □

Theorem 4.6. *Let $K_h = \{\mathbf{v_h} \in \mathbf{V_h}, B_h(\overline{q}_h, \mathbf{v_h}) = 0$, for all $\overline{q}_h \in \overline{\mathbf{W}}_\mathbf{h}\}$. Then, there exists a positive constant γ_e such that $(A_h^*(U_h^*) - A_h^*(V_h^*), U_h^* - V_h^*) \geq \gamma_e\|U_h^* - V_h^*\|^2$ for all $U_h^*, V_h^* \in K_h \times W_h^*$.*

Proof. From (8) we obtain

$$(A_h^*(U_h^*) - A_h^*(V_h^*), U_h^* - V_h^*) \geq \lambda_1\|\varepsilon(\mathbf{u_h} - \mathbf{v_h})\|_0^2$$

$$+ \delta_2\vartheta\|\operatorname{div}\mathbf{u_h} - \operatorname{div}\mathbf{v_h}\|_{0h}^2 + 2B_h(p_h^* - q_h^*, \mathbf{u_h} - \mathbf{v_h})$$

$$+ \frac{\delta_1 h^2}{\vartheta}\| - \Delta_\mu\mathbf{u_h} + \Delta_\mu\mathbf{v_h} + \nabla(p_h^* - q_h^*)\|_{0h}^2$$

$$+ \lambda_2\big(\nu(|\varepsilon(\mathbf{u_h})|)\varepsilon(\mathbf{u_h}) - \nu(|\varepsilon(\mathbf{v_h})|)\varepsilon(\mathbf{v_h}), \varepsilon(\mathbf{u_h} - \mathbf{v_h})\big).$$

From the ellipticity of the last term in the right hand side above, presented in [12], and using the Young inequality with $\xi = \frac{1}{\delta_2\vartheta}$ we have

$$(A_h^*(U_h^*) - A_h^*(V_h^*), U_h^* - V_h^*) \geq (\gamma\lambda_2 + \lambda_1)\|\varepsilon(\mathbf{u_h}) - \varepsilon(\mathbf{v_h})\|_0^2$$

$$+ \frac{\delta_1 h^2}{\vartheta}\|\Delta_\mu\mathbf{u_h} + \Delta_\mu\mathbf{v_h} + \nabla(p_h^* - q_h^*)\|_{0h}^2 - \frac{1}{\delta_2\vartheta}\|p_h^* - q_h^*\|_0^2.$$

Applying again the Young inequality it yields

$$(A_h^*(U_h^*) - A_h^*(V_h^*), U_h^* - V_h^*) \geq (\gamma\lambda_2 + \lambda_1)\|\varepsilon(\mathbf{u_h}) - \varepsilon(\mathbf{v_h})\|_0^2$$

$$-\frac{1}{\delta_2\vartheta}\|p_h^* - q_h^*\|_0^2 + \frac{\delta_1 h^2}{\vartheta}(1 - \frac{1}{\eta})\| - \Delta_\mu \mathbf{u_h} + \Delta_\mu \mathbf{v_h}\|_{0h}^2$$

$$+\frac{\delta_1 h^2}{\vartheta}(1-\eta)\|\nabla p_h^* - \nabla q_h^*\|_{0h}^2,$$

with $\eta > 0$. From Lemma 4.2 and considering r_1 and r_2 as two positive constants, such that $\frac{1}{r_1} + \frac{1}{r_2} = 1$, we can rewrite the previous inequality as

$$(A_h^*(U_h^*) - A_h^*(V_h^*), U_h^* - V_h^*) \geq \frac{\gamma\lambda_2 + \lambda_1}{r_1}\|\varepsilon(\mathbf{u_h}) - \varepsilon(\mathbf{v_h})\|_0^2$$

$$+h^2\left(\frac{\gamma\lambda_2 + \lambda_1}{r_2 C_l} + \frac{\delta_1}{\vartheta}\left(1 - \frac{1}{\eta}\right)\right)\| - \Delta_\mu \mathbf{u_h} + \Delta_\mu \mathbf{v_h}\|_{0h}^2$$

$$+\frac{\delta_1 h^2}{\vartheta}(1-\eta)\|\nabla p_h^* - q_h^*\|_{0h}^2 - \frac{1}{\delta_2\vartheta}\|p_h^* - q_h^*\|_0^2.$$

Choosing $\eta = \frac{\delta_1 r_2 C_l}{(\gamma\lambda_2+\lambda_1)\vartheta + \delta_1 r_2 C_l)}$, we obtain

$$(A_h^*(U_h^*) - A_h^*(V_h^*), U_h^* - V_h^*) \geq \frac{\gamma\lambda_2 + \lambda_1}{r_1}\|\varepsilon(\mathbf{u_h}) - \varepsilon(\mathbf{v_h})\|_0^2$$

$$-\frac{1}{\delta_2\vartheta}\|p_h^* - q_h^*\|_0^2 + \frac{\delta_1(\gamma\lambda_2 + \lambda_1)h^2}{\vartheta(\gamma\lambda_2 + \lambda_1) + \delta_1 r_2 C_l)}\|\nabla(p_h^* - q_h^*)\|_{0h}^2.$$

By using the inequality

$$h^2\|\nabla q_h^*\|_{0h}^2 \geq \|q_h^*\|_0^2, \tag{15}$$

as in [10], and the Korn's inequality, we have

$$(A_h^*(U_h^*) - A_h^*(V_h^*), U_h^* - V_h^*) \geq \gamma_e(\|\mathbf{u_h} - \mathbf{v_h}\|_1^2 + \|p_h^* - q_h^*\|_0^2)$$

with

$$\gamma_e = \min\left\{\frac{(\gamma\lambda_2 + \lambda_1)C}{r_1}, \frac{\delta_1(\gamma\lambda_2 + \lambda_1)}{\vartheta(\gamma\lambda_2 + \lambda_1) + \delta_1 r_2 C_l} - \frac{1}{\delta_2\vartheta}\right\},$$

since

$$\frac{\delta_1(\gamma\lambda_2 + \lambda_1)}{\vartheta(\gamma\lambda_2 + \lambda_1) + \delta_1 r_2 C_l} - \frac{1}{\delta_2\vartheta} > 0. \tag{16}$$

□

The inequality (16) gives a sufficient condition, providing a useful relation to be used to choose the stabilizing parameters.

Theorem 4.7. *There exists a positive constant γ_B such that*

$$B_h(\overline{p} - \overline{p}_h, \mathbf{u_h} - \mathbf{v_h}) \leq \gamma_B \|\overline{p} - \overline{p}_h\|_0 \|U_h^* - V_h^*\|$$

for all $\overline{p} \in L^2(\Omega), U_h^*, V_h^* \in \mathbf{V_h} \times \mathbf{W_h^*}$.

Proof. This result comes from the application of the Hölder-Schwarz inequality and by the use of (14). □

Theorem 4.8. *For $k \geq 2$ and since $\overline{V}_h \in \mathbf{V_h} \times \overline{\mathbf{W}}_\mathbf{h}$, there exists a positive constant β_h, independent of h, such that*

$$\sup_{\mathbf{v_h} \in \mathbf{V_h}} \frac{|B_h(\overline{q}_h, \mathbf{v_h})|}{\|\mathbf{v_h}\|_1} \geq \beta_h \|\overline{q}_h\|_0 \textit{ for all } \overline{q}_h \in \overline{\mathbf{W}}_\mathbf{h}.$$

Proof. This result may be seen in [7]. □

With the above results, we can establish the following error approximation estimates.

Theorem 4.9. *There exists a positive constant ζ_h, independent of h, such that the following estimate holds* $\||U - U_h|\| \leq \zeta_h \||U - V_h|\|$.

Proof. From the definition of $(A_h^*(\cdot), \cdot)$ and the consistency of the formulation we can write

$$\begin{aligned}\big(A_h^*(U_h^*) - A_h^*(V_h^*), U_h^* - V_h^*\big) &= \big(A_h^*(U^*) - A_h^*(V_h^*), U_h^* - V_h^*\big) \\ &\quad + \big(A_h^*(U_h^*) - A_h^*(U^*), U_h^* - V_h^*\big)\end{aligned} \tag{17}$$

with

$$\big(A_h^*(U_h^*) - A_h^*(U^*), U_h^* - V_h^*\big) = B_h\big(\overline{p} - \overline{q}_h, \mathbf{u_h} - \mathbf{v_h}\big). \tag{18}$$

Replacing (18) in (17) we have

$$\begin{aligned}\big(A_h^*(U_h^*) - A_h^*(V_h^*), U_h^* - V_h^*\big) &= \big(A_h^*(U^*) - A_h^*(V_h^*), U_h^* - V_h^*\big) \\ &\quad + B_h\big(\overline{p} - \overline{q}_h, U_h^* - V_h^*\big).\end{aligned}$$

From Theorem 4.5, Theorem 4.6, Theorem 4.7 and considering the norm equivalence between $|||\cdot|||$ and $\|\cdot\|$ established in Lemma 4.4, we have

$$|||U^* - U_h^*||| \leq \left(1 + \frac{\gamma_c}{\gamma_e}\right) |||U^* - V_h^*||| + \frac{\gamma_B}{\gamma_e}\|\overline{p} - \overline{q}_h\|_0. \qquad (19)$$

In order to obtain an estimate to $\|\overline{p} - \overline{p}_h\|_0$, we note that from PG$_{hd}$ problem, we have

$$B_h\big(\overline{p}_h, \mathbf{u_h} - \mathbf{v_h}\big) = \big(A_h^*(U^*) - A_h^*(U_h^*), U_h^* - V_h^*\big) + B_h\big(\overline{p}, \mathbf{u_h} - \mathbf{v_h}\big).$$

Since $\mathbf{v_h} \in K_h$, then

$$B_h\big(\overline{p}_h - \overline{q}_h, \mathbf{u_h} - \mathbf{v_h}\big) = \big(A_h^*(U^*) - A_h^*(U_h^*), U_h^* - V_h^*\big)$$
$$+ B_h\big(\overline{p} - \overline{q}_h, \mathbf{u_h} - \mathbf{v_h}\big).$$

Using Theorem 4.5 and Theorem 4.7 we have

$$\sup_{\mathbf{w_h} \in \mathbf{V_h^*}} \frac{B_h(\overline{p}_h - \overline{q}_h, \mathbf{w_h})}{\|\mathbf{w_h}\|_1 + \|p_h^* - q_h^*\|_0} \leq |||U^* - V_h^*||| + \|\overline{p} - \overline{q}_h\|_0.$$

By the use of Theorem 4.8, we can see that

$$\|\overline{p} - \overline{p}_h\|_0 \leq \frac{1}{\beta_h}\|U^* - V_h^*\|_h + \left(1 + \frac{1}{\beta_h}\right)\|\overline{p} - \overline{q}_h\|_0. \qquad (20)$$

Combining (19) and (20) we have

$$|||U^* - U_h^*||| \leq \zeta_h |||U^* - V_h^*|||,$$

where

$$\zeta_h = 1 + \frac{1}{\beta_h} + \frac{1}{\gamma_e}\max\big\{\gamma_c, \gamma_B\big\},$$

since $\|p_h\|_0^2 = \|p_h^*\|_0^2 + \|\overline{p}_h\|_0^2$. □

From Theorem 4.9, applying inverse estimates and the very classical interpolation results presented in, for example, Chapter 3 of [5], we obtain the following error estimate

$$\|U - U_h\| \leq \zeta_h(2 + C_h)c_1 h^k |\mathbf{u}|_{k+1} + \zeta_h(1 + C_p)c_2 h^{l+1}|p|_{l+1} \qquad (21)$$

with $c_1, c_2 \in \mathbb{R}$, $\mathbf{u_h} \in S_h^k(\Omega)$ and $p_h \in Q_h^l(\Omega)$ and $|\cdot|_{m+1}$ the semi-norm of the $W^{m,2}(\Omega)$.

5 Numerical results

In order to obtain numerical results for the finite element method presented here to nonlinear problem, the following numerical algorithm will be used. There are many methods to solve nonlinear equations. In this case, we lag nonlinear terms in the system of equations and start with an initial guess generating a sequence of functions that is expected to converge for the solution. In this sense, our scheme is constructed by: given $\mathbf{u_h^0} \in \mathbf{V_h}$, lets find $(\mathbf{u_h^n}, p_h^n) \in \mathbf{V_h} \times \mathbf{W_h}$, $n = 1, 2, 3, \ldots$ such that

$$(\mu(|\varepsilon(\mathbf{u_h^n})|)\varepsilon(\mathbf{u_h^{n+1}}), \varepsilon(\mathbf{v_h})) + B_h(p_h^{n+1}, \mathbf{v_h}) + B_h(q_h, \mathbf{u_h^{n+1}})$$

$$+\frac{\delta_1 h^2}{\vartheta}(-\mathrm{div}\,(\mu(\mathbf{u_h^n})\varepsilon(\mathbf{u_h^{n+1}})) + \nabla p_h^{*n+1}, -\Delta_\mu \mathbf{v_h} + \nabla q_h)_h$$

$$+\delta_2\vartheta(\mathrm{div}\,\mathbf{u_h^{n+1}}, \mathrm{div}\,\mathbf{v_h}) = F_h^*(V_h^*), \forall\ V_h \in \mathbf{V_h} \times \mathbf{W_h}.$$

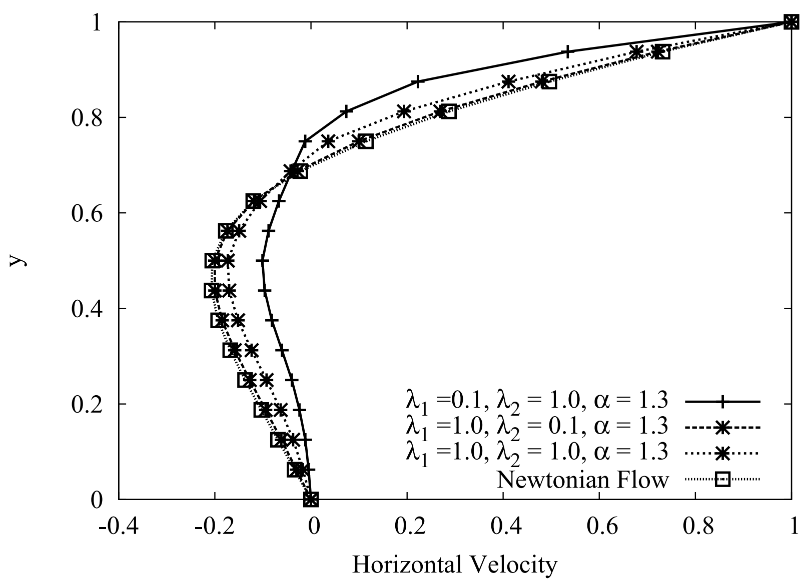
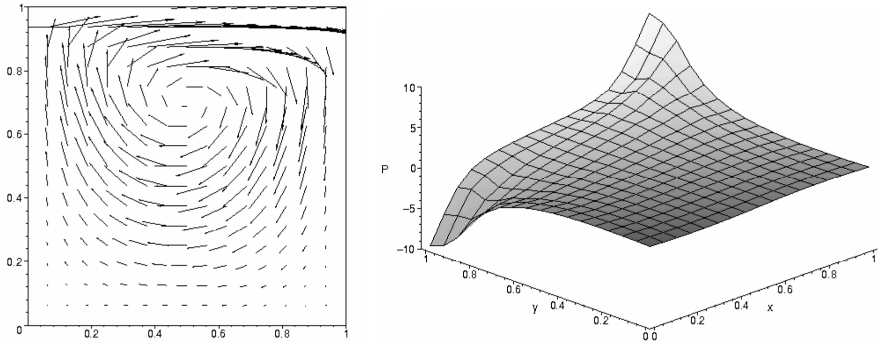


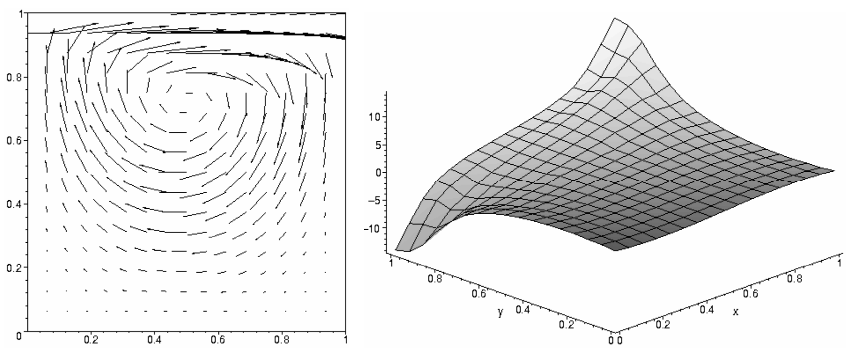
Figure 1 – Horizontal velocity at $x = 0.5$

The algorithm above, was applied to obtain numerical results for the classical driven cavity flow problem with bounbary conditions: $\mathbf{u}(\mathbf{x}) = (1, 0)$ on $\mathbf{x} \in [0, 1] \times \{1\}$ and $\mathbf{u}(\mathbf{x}) = (0, 0)$ on the other boundaries. A finite element



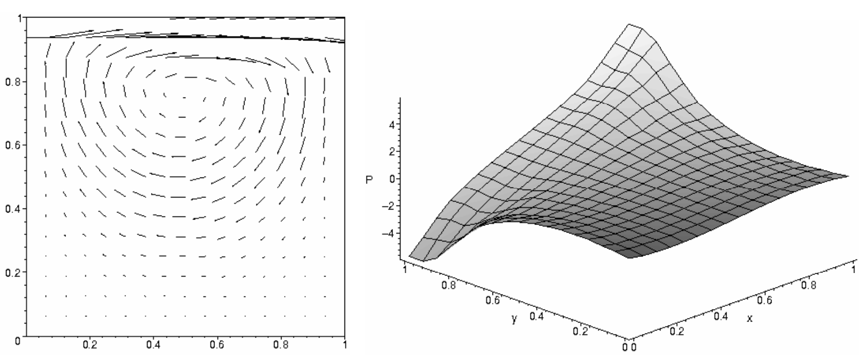
(a) Velocity field (b) Pressure field

Figure 2 – Results for $\lambda_1 = 1.0$, $\lambda_2 = 0.1$ and $\alpha = 1.3$ using $\delta_1 = 1.0$ and $\delta_2 = 10.0$.



(a) Velocity field (b) Pressure field

Figure 3 – Results for $\lambda_1 = 1.0$, $\lambda_2 = 1.0$ and $\alpha = 1.3$ using $\delta_1 = 1.0$ and $\delta_2 = 10.0$.



(a) Velocity field (b) Pressure field

Figure 4 – Results for $\lambda_1 = 0.1$, $\lambda_2 = 1.0$ and $\alpha = 1.3$ using $\delta_1 = 1.0$ and $\delta_2 = 10.0$.

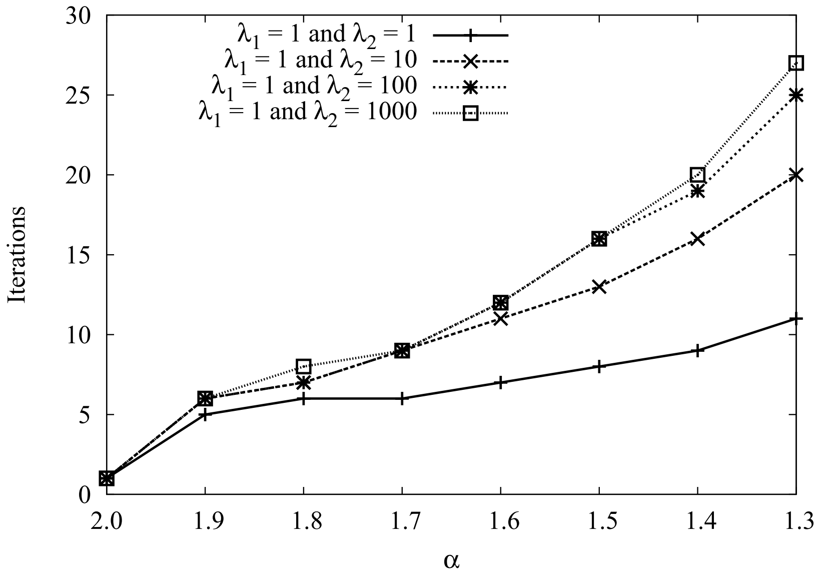


Figure 5 – Number of iterations with $Tol \leq 10^{-6}$ for 17×17 nodes with biquadratic elements in the case of various combinations between λ_1 and λ_2.

mesh of 17×17 nodes and 8×8 biquadratic quadrilateral elements has been used. The numerical results were performed using the following stabilizing parameters: $\delta_1 = 1.0$ and $\delta_2 = 10.0$. For the convergence of the algorithm, we imposed a tolerence of 10^{-6}. Numerical results are shown for some combinations of the constitutive parameters λ_1 and λ_2. In Figure 1 we can note the characteristic of the pseudoplastic behavior comparing the velocity profiles on $x = 0.5$ for the λ_i combinations presented. Velocity and pressure fields are shown in Figures 2-4 for the same λ_i combinations of those in Figure 1. We can see, from these results, how λ_1 and λ_2 control the Newtonian and the pseudoplastic contributions respectively. We note magnitude decreasing in the velocity and in the pressure fields and also the flatteness of the pressure next to the two corners (0, 1) and (1, 1) due to the pseudoplastic effect, as expected. Formulations that use the continuous pressure interpolations may present lack of accuracy in those regions, where critical boundary conditions exist. Unlike, discontinuous pressure interpolations, as is the case here, recover the accuracy at those regions, due to the satisfaction of the incompressibility constraint locally. The convergence behaviour of the algorithm used, together with the Petrov-

Galerkin-like formulation PG_{hd}, is shown in Figure 5, as a function of the α power index for four Sisko fluids. It can be seen that the greater is the non-Newtonian effect, the larger is the number of iteractions required to achieve convergence, as expected for a fixed mesh. Note that even for a higher nonlinearity (higher power index α) convergence and stability are achieved.

6 Summary and conclusions

In this work, a consistent stabilized mixed Petrov-Galerkin-like finite element formulation in primitive variables, with continuous velocity and discontinuous pressure interpolations, has been mathematically analyzed for flows governed by the nonlinear Sisko relation. Stability, convergence and error estimates have been proven for same order interpolations of the primitive variables for any combinations when $k \geq 2$.

To generate the mathematical stability conditions, it was possible to split the discontinuous pressure. Only the constant by part pressure resulted as responsible to fulfill the LBB condition. The other part, the null mean pressure part, contributed to achieve the required ellipticity in the Scheurer's theorem sense together with the stabilizing terms. For this formulation ellipticity was the key for the stability, since the constant part of the pressure fulfills in standard ways the LBB. It was possible from the ellipticity to provide a sufficient condition to choose the stabilizing parameters not only as a function of the quasi-newtonian constitutive parameter but considering both constitutive constants coming from the Sisko relation.

Numerical results have been presented for the benchmark driven cavity flow problem to confirm the mathematical analysis.

From the results, stability and convergence have been reached for several combinations of the constitutive parameters of the Sisko relation, ranging from lower to highly pseudoplastic (low α and/or high λ_2) effects, although with more interactions in the last case.

The use of discontinuous pressure interpolations ensured accuracy for the pressure field even in the regions where discontinuous boundary conditions are present, since, now, the weakened internal constraint is satisfied locally, in contrast with continuous approximations.

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