COMPUTATIONAL & APPLIED MATHEMATICS

Volume 31, N. 1, pp. 59–83, 2012

ISSN 0101-8205 / ISSN 1807-0302 (Online)
www.scielo.br/cam

The global convergence of a descent PRP conjugate gradient method

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Abstract. Recently, Yu and Guan proposed a modified PRP method (called DPRP method) which can generate sufficient descent directions for the objective function. They established the global convergence of the DPRP method based on the assumption that stepsize is bounded away from zero. In this paper, without the requirement of the positive lower bound of the stepsize, we prove that the DPRP method is globally convergent with a modified strong Wolfe line search. Moreover, we establish the global convergence of the DPRP method with a Armijo-type line search. The numerical results show that the proposed algorithms are efficient.

Mathematical subject classification: Primary: 90C30; Secondary: 65K05.

Key words: PRP method, sufficient descent property, modified strong Wolfe line search, Armijo-type line search, global convergence.

1 Introduction

In this paper, we consider the unconstrained problem

$$\min f(x), \quad x \in R^n, \tag{1.1}$$

#CAM-307/10. Received: 12/XII/10. Accepted: 17/IV/11.

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**This work is supported by NSF grant HHUY07-10 of HuaiHua University, China.

where $f : R^n \to R$ is continuously differentiable. Nonlinear conjugate gradient methods are efficient for problem (1.1). The nonlinear conjugate gradient methods generate iterates by letting

$$x_{k+1} = x_k + \alpha_k d_k, \qquad k = 0, 1, \cdots, \tag{1.2}$$

with

$$d_k = \begin{cases} -g_0, & \text{if} \quad k = 0, \\ -g_k + \beta_k d_{k-1}, & \text{if} \quad k \geq 1, \end{cases} \tag{1.3}$$

where α_k is the steplength, $g_k = g(x_k)$ denote the gradient of f at x_k, and β_k is a suitable scalar.

Well-known nonlinear conjugate method include Fletcher-Reeves (FR), Polak-Ribière-Polyak (PRP), Hestenes-Stiefel (HS), Conjugate-Descent (CD), Liu-Story (LS) and Dai-Yuan (DY) [1–7]. We are particularly interested in the PRP method in which the parameter β_k is defined by

$$\beta_k^{\text{PRP}} = \frac{g_k^T y_{k-1}}{\|g_{k-1}\|^2}.$$

Here and throughout, $\|\cdot\|$ stands the Euclidean norm of vector and $y_{k-1} = g_k - g_{k-1}$. The PRP method has been regarded as one of the most efficient conjugate gradient methods in practical computation and studied extensively.

Now let us simply introduce some results on the PRP method. The global convergence of the PRP method with exact line search has been proved by Polak and Ribière [3] under strong convexity assumption of f. However, Powell constructed an example which showed that the PRP method with exact line searches can cycle infinitely without approaching a solution point [8]. Powell also gave a suggestion to restrict $\beta_k = \max\{\beta_k^{\text{PRP}}, 0\}$ to ensure the convergence. Based on Powell's work, Gilbert and Nocedal [9] conducted an elegant analysis and showed the PRP method is globally convergent if β_k^{PRP} is restricted to be non-negative and the steplength satisfies the sufficient descent condition $g_k^T d_k \leq -c\|g_k\|^2$ in each iteration. In [10], Zhang, Zhou and Li proposed a three term PRP method. They calculate d_k by

$$d_k = \begin{cases} -g_0, & \text{if} \quad k = 0, \\ -g_k + \beta_k^{\text{PRP}} d_{k-1} - \theta_k y_{k-1}, & \text{if} \quad k \geq 1, \end{cases} \tag{1.4}$$

where $\theta_k = \frac{g_k^T d_{k-1}}{\|g_{k-1}\|^2}$. It is easy to see from (1.4) that $d_k^T g_k = -\|g_k\|^2$. Zhang, Zhou and Li proved that this three term PRP method convergent globally with a Armijo-type line search.

In [11], Yu and Guan proposed a modified PRP conjugate gradient formula:

$$\beta_k^{\mathrm{DPRP}} = \frac{g_k^T y_{k-1}}{\|g_{k-1}\|^2} - t\frac{\|y_{k-1}\|^2 g_k^T d_{k-1}}{\|g_{k-1}\|^4}, \tag{1.5}$$

where $t > 1/4$ is a constant. In this paper we simply call it DPRP method. This modified formula is similar to the well-known CG_DESCENT method which was proposed by Hager and Zhang in [13]. Yu and Guan has proved that the directions d_k generated by the DPRP method can always satisfied the following sufficient descent condition

$$g_k^T d_k \leq \left(\frac{1}{4t} - 1\right)\|g_k\|^2. \tag{1.6}$$

In order to obtain the global convergence of the algorithm, Yu and Guan [11] considered the following assumption: there exists a positive constant α^* such that $\alpha_k > \alpha^*$ holds for all indices k. In fact, under this assumption, if the sufficient descent condition $g_k^T d_k \leq -c\|g\|^2$ is satisfied, the conjugate gradient methods with Wolfe line search will be globally convergent.

In this paper, we focus on the global convergence of the DPRP method. In Section 2, we describe the algorithm. The convergence properties of the algorithm are analyzed in Section 3 and 4. In Section 5, we report some numerical results to test the proposed algorithms by using the test problems in the CUTEr [23] library.

2 Algorithm

From the Theorem 1 in [11] we can obtain the following useful lemma directly.

Lemma 2.1. *Let* $\{d_k\}$ *be generated by*

$$d_k = -g_k + \tau d_{k-1}, \quad d_0 = -g_0. \tag{2.1}$$

If $g_{k-1}^T d_{k-1} \neq 0$, *then* (1.6) *holds for all* $\tau \in [\beta_k^{\mathrm{DPRP}}, \max\{0, \beta_k^{\mathrm{DPRP}}\}]$.

Proof. The inequality (1.6) holds for $\tau = \beta_k^{\mathrm{DPRP}}$ from the Theorem 1 in [11]. For any $\tau \in (\beta_k^{\mathrm{DPRP}}, \max\{0, \beta_k^{\mathrm{DPRP}}\}]$, it is clear that $\beta_k^{\mathrm{DPRP}} \leq \tau \leq 0$. If $g_k^T d_{k-1} \geq 0$, then (1.6) follows from (2.1). If $g_k^T d_{k-1} < 0$, then we get from (2.1) that

$$g_k^T d_k = -\|g_k\|^2 + \tau g_k^T d_{k-1} \leq -\|g_k\|^2 + \beta_k^{\mathrm{DPDP}} g_k^T d_{k-1} \leq \left(\frac{1}{4t} - 1\right) \|g_k\|^2.$$

The proof is completed. □

Lemma 2.1 shows that the directions generated by (2.1) are sufficient descent directions and this feature is independent of the line search used. Especially, if set

$$d_k = -g_k + \overline{\beta}_k^{\mathrm{DPRP}} d_{k-1}, \qquad d_0 = -g_0, \tag{2.2}$$

$$\overline{\beta}_k^{\mathrm{DPRP}} = \max\left\{\beta_k^{\mathrm{DPRP}}, \eta_k\right\}, \quad \eta_k = \frac{-1}{\|d_{k-1}\| \min\{\eta, \|g_{k-1}\|\}}, \tag{2.3}$$

where $\eta > 0$ is a constant, it follows from $\eta_k < 0$ that

$$\overline{\beta}_k^{\mathrm{DPRP}} = \max\left\{\beta_k^{\mathrm{DPRP}}, \eta_k\right\} \in \left[\beta_k^{\mathrm{DPRP}}, \max\left\{0, \beta_k^{\mathrm{DPRP}}\right\}\right].$$

So, the directions generated by (2.2) and (2.3) are descent directions due to Lemma 2.1. The update rule (2.3) to adjust the the lower bound on β_k^{DPRP} was originated in [13].

Now we present concrete algorithm as follows.

Algorithm 2.2. (The DPRP method) [11]

Step 0. Given constant $\varepsilon > 0$ and $x_0 \in R^n$. Set $k := 0$;

Step 1. Stop if $\|g_k\|_\infty \leq \varepsilon$;

Step 2. Compute d_k by (2.2);

Step 3. Determine the steplength α_k by a line search;

Step 4. Let $x_{k+1} = x_k + \alpha_k d_k$, if $\|g_{k+1}\|_\infty \leq \varepsilon$, then stop;

Step 5. Set $k := k + 1$ and go to Step 2.

It is easy to see that Algorithm 2.1 is well-defined. We analyse the convergence in the next sections. We do not specify the line search to determine the steplength α_k. It can be exact or inexact line search. In the next two sections, we devote to the convergence of the Algorithm 2.1 with modified strong Wolfe line search and Armijo-type line search.

3 Convergence of Algorithm 2.1 under modified strong Wolfe line search

In this section, we prove the global convergence of the Algorithm 2.1. We first make the following assumptions.

Assumption 3.1.

(I) *The level set*

$$\Omega = \{x \in R^n : f(x) \leq f(x_0)\}$$

is bounded.

(II) *In some neighborhood N of Ω, function f is continuously differentiable and its gradient is Lipschitz continuous, namely, there exists a constant $L > 0$ such that*

$$\|g(x) - g(y)\| \leq L\|x - y\|, \qquad \forall x, y \in N. \tag{3.1}$$

From now on, throughout this paper, we always suppose this assumption holds. It follows directly from the Assumption 3.1 that there exist two positive constants B and γ_1 such that

$$\|x\| \leq B \quad \text{and} \quad \|g(x)\| \leq \gamma_1, \quad \forall x \in \Omega. \tag{3.2}$$

In the later part of this section, we will devote to the global convergence of the Algorithm 2.1 under a modified strong Wolfe line search which determine the steplength α_k satisfying the following conditions

$$\begin{cases} f(x_k + \alpha_k d_k) \leq f(x_k) + \delta \alpha_k g_k^T d_k, \\ |g(x_k + \alpha_k d_k)^T d_k| \leq \min\left\{M, \sigma |g_k^T d_k|\right\}, \end{cases} \tag{3.3}$$

where $0 < \delta \leq \sigma < 1$ and $M > 0$ are constants. The purpose of this modification is to make $|g_k^T d_{k-1}| \leq M$ always hold with the given M. Moreover, we can set M to be large enough such that $\min\{M, \sigma|g_k^T d_k|\} = \sigma|g_k^T d_k|$ hold for almost all indices k. At first, we give the following useful lemma which was essentially proved by Zoutendijik [19] and Wolfe [20, 21].

Lemma 3.2. *Let the conditions in Assumption* 3.1 *hold,* $\{x_k\}$ *and* $\{d_k\}$ *be generated by Algorithm 2.1 with the above line search, then*

$$\sum_{k=0}^{\infty} \frac{(g_k^T d_k)^2}{\|d_k\|^2} < +\infty. \tag{3.4}$$

Proof. We get from the second inequality in (3.3)

$$\sigma g_k^T d_k \leq g_{k+1}^T d_k \leq -\sigma g_k^T d_k, \tag{3.5}$$

From the Lipschtiz condition (3.1), we have

$$(\sigma - 1) g_k^T d_k \leq (g_{k+1} - g_k)^T d_k \leq L\alpha_k \|d_k\|^2.$$

Then

$$\alpha_k \geq \frac{(\sigma - 1)}{L} \frac{g_k^T d_k}{\|d_k\|^2} = \frac{(1 - \sigma)}{L} \frac{|g_k^T d_k|}{\|d_k\|^2}. \tag{3.6}$$

Since f is bounded below and the gradient $g(x)$ is Lipschitz continuous, from the first inequality in (3.3) we have

$$-\sum_{k=0}^{\infty} \alpha_k g_k^T d_k \leq +\infty. \tag{3.7}$$

Then the Zoutendijik condition (3.4) holds from (3.6) and (3.7). □

The following lemma is similar to the Lemma 4.1 in [9] and the Lemma 3.1 in [13], which is very useful to prove that the gradients cannot be bounded away from zero.

Lemma 3.3. *Let the conditions in Assumption* 3.1 *hold,* $\{x_k\}$ *and* $\{d_k\}$ *be generated by Algorithm* 2.1 *with the modified strong Wolfe line search. If there exists a constant* $\gamma > 0$ *such that*

$$\|g_k\| \geq \gamma, \quad \forall k \geq 0, \tag{3.8}$$

then

$$d_k \neq 0,\ \forall k \geq 0, \quad \textit{and} \quad \sum_{k=0}^{\infty} \|u_k - u_{k-1}\|^2 < \infty, \tag{3.9}$$

where $u_k = d_k/\|d_k\|$.

Proof. Since $\|g_k\| > \gamma > 0$, it follows from (1.6) that $d_k \neq 0$ for each k. We get from (3.4) and (1.6)

$$\sum_{k=0}^{\infty} \frac{\gamma^4}{\|d_k\|^2} \leq \sum_{k=0}^{\infty} \frac{\|g_k\|^4}{\|d_k\|^2} \leq \left(\frac{4t}{4t-1}\right)^2 \sum_{k=0}^{\infty} \frac{(g_k^T d_k)^2}{\|d_k\|^2} < \infty, \tag{3.10}$$

where $t > \frac{1}{4}$ is a constant. Here, we define:

$$\beta_k^+ = \max\{\overline{\beta}_k^{\mathrm{DPRP}}, 0\}, \qquad \beta_k^- = \min\{\overline{\beta}_k^{\mathrm{DPRP}}, 0\}, \tag{3.11}$$

$$r_k = \frac{-g_k + \beta_k^- d_{k-1}}{\|d_k\|}, \qquad \delta_k = \beta_k^+ \frac{d_{k-1}}{\|d_k\|}. \tag{3.12}$$

By (2.2), (3.11) and (3.12), we have

$$u_k = \frac{d_k}{\|d_k\|} = \frac{-g_k + (\beta_k^+ + \beta_k^-) d_{k-1}}{\|d_k\|} = r_k + \delta_k u_{k-1}.$$

Since the u_k are unit vectors,

$$\|r_k\| = \|u_k - \delta_k u_{k-1}\| = \|\delta_k u_k - u_{k-1}\|.$$

Since $\delta_k > 0$, we have

$$\begin{aligned} \|u_k - u_{k-1}\| &\leq \|(1+\delta_k)(u_k - u_{k-1})\| \\ &\leq \|u_k - \delta_k u_{k-1}\| + \|\delta_k u_k - u_{k-1}\| \\ &= 2\|r_k\|. \end{aligned} \tag{3.13}$$

We get from the definition of $\overline{\beta}_k^{\mathrm{DPRP}}$

$$\begin{aligned} \|r_k\|\|d_k\| &= \|-g_k + \beta_k^- d_{k-1}\| \\ &\leq \|g_k\| - \min\left\{\overline{\beta}_k^{\mathrm{DPRP}}, 0\right\}\|d_{k-1}\| \\ &\leq \|g_k\| - \eta_k \|d_{k-1}\| \\ &\leq \|g_k\| + \frac{1}{\|d_{k-1}\| \min\{\eta, \gamma\}} \|d_{k-1}\| \\ &\leq \gamma_1 + \frac{1}{\min\{\eta, \gamma\}}. \end{aligned} \tag{3.14}$$

Setting $c = \gamma_1 + \frac{1}{\min\{\eta,\gamma\}}$ and combing (3.13) with (3.14), we have

$$\|u_k - u_{k-1}\|^2 \le (2\|r_k\|)^2 \le \frac{4c^2}{\|d_k\|^2}. \tag{3.15}$$

Summing (3.15) over k and combing with the relation (3.10), we have

$$\sum_{k=0}^{\infty} \|u_k - u_{k-1}\|^2 \le \sum_{k=0}^{\infty} \frac{4c^2}{\|d_k\|^2} < \infty.$$

The proof is completed. □

Now we state a property for $\overline{\beta}_k^{\text{DPRP}}$ in (2.2), which is called Property(*) proposed by Gilbert and Nocedal in [9].

Property(*). *Suppose that Assumption* 3.1 *and inequality* (3.8) *hold. We say that the method has Property(*) if there exist constant* $b > 1$ *and* $\lambda > 0$ *such that*

$$\begin{cases} |\beta_k| \le b, \\ \|s_{k-1}\| \le \lambda \Rightarrow |\beta_k| \le \frac{1}{2b}, \end{cases} \tag{3.16}$$

where $s_{k-1} = x_k - x_{k-1} = \alpha_{k-1} d_{k-1}$.

Lemma 3.4. *If the modified strong Wolfe line search is used in Algorithm* 2.1*, then the scalar* $\overline{\beta}_k^{\text{DPRP}}$ *satisfies the Property(*).*

Proof. By the definition of $\overline{\beta}_k^{\text{DPRP}}$ in (2.2), we have

$$|\overline{\beta}_k^{\text{DPRP}}| \le |\beta_k^{\text{DPRP}}|, \quad \forall k \ge 0.$$

Under the Assumption 3.1, if (3.8) holds, we get from (3.3) and (1.6)

$$\begin{aligned} |\overline{\beta}_k^{\text{DPRP}}| &\le |\beta_k^{\text{DPRP}}| \\ &\le \frac{|g_k^T y_{k-1}|}{\|g_{k-1}\|^2} + t\frac{\|y_{k-1}\|^2 |g_k^T d_{k-1}|}{\|g_{k-1}\|^4} \\ &\le \frac{\|g_k\|\|y_{k-1}\|}{\|g_{k-1}\|^2} + t\frac{\|y_{k-1}\|^2 M}{\|g_{k-1}\|^4} \\ &\le \frac{2\gamma_1^2}{\gamma^2} + t\frac{4\gamma_1^2 M}{\gamma^4}. \end{aligned} \tag{3.17}$$

If $s_{k-1} < \lambda$, then

$$\begin{aligned} |\overline{\beta}_k^{\mathrm{DPRP}}| &\le |\beta_k^{\mathrm{DPRP}}| \\ &\le \frac{\|g_k\|\|y_{k-1}\|}{\|g_{k-1}\|^2} + t\frac{\|y_{k-1}\|^2 M}{\|g_{k-1}\|^4} \\ &\le \left(\frac{\gamma_1}{\gamma^2} + t\frac{2\gamma_1 M}{\gamma^4}\right)\|y_{k-1}\| \\ &\le \left(\frac{\gamma_1}{\gamma^2} + t\frac{2\gamma_1 M}{\gamma^4}\right) L\,\|s_{k-1}\|. \end{aligned} \tag{3.18}$$

(3.17) together with (3.18) implies the conditions in Property(*) hold by setting

$$b = \frac{2\gamma_1^2}{\gamma^2} + t\frac{4\gamma_1^2 M}{\gamma^4} \qquad \text{and} \qquad \lambda = \frac{\gamma_1}{Lb^2}. \tag{3.19}$$

The proof is completed. □

Let N^* denote the set of positive integer set. For $\lambda > 0$ and a positive integer $\triangle$, we define the index set:

$$K_{k,\triangle}^{\lambda} := \left\{ i \in N^* : k \le i \le k + \triangle - 1,\ \|s_{i-1}\| > \lambda \right\}. \tag{3.20}$$

Let $|K_{k,\triangle}^{\lambda}|$ denote the number of elements in $K_{k,\triangle}^{\lambda}$. We have the following Lemma which was essentially proved by Gilbert and Nocedal in [9].

Lemma 3.5. *Let x_k and d_k be generated by Algorithm* 2.1 *with the modified strong Wolfe line search. If* (3.8) *holds, then there exists a constant $\lambda > 0$ such that for any $\triangle \in N^*$ and any index k_0, there is an index $k > 0$ such that*

$$|K_{k,\triangle}^{\lambda}| > \frac{\triangle}{2}. \tag{3.21}$$

The proof of Lemma 3.5 is similar to that of the Lemma 4.2 in [9], so we omit here. The next theorem, which is modification of Theorem 4.3 in [9], shows that the proposed Algorithm 2.1 is globally convergent.

Theorem 3.6. *Let the conditions in Assumption* 3.1 *hold, and $\{x_k\}$ be generated by Algorithm* 2.1 *with the modified strong Wolfe line search* (3.3)*, then either $\|g_k\| = 0$ for some k or*

$$\liminf_{k\to\infty} \|g_k\| = 0. \tag{3.22}$$

Proof. We suppose for contradiction that both

$$\|g_k\| \neq 0 \quad \text{and} \quad \liminf_{k\to\infty} \|g_k\| \neq 0.$$

Denote $\gamma = \inf\{\|g_k\| : k \geq 0\}$. It is clear that

$$\|g_k\| \geq \gamma > 0, \quad \forall k \geq 0. \tag{3.23}$$

Therefore the conditions of Lemma 3.3 and 3.5 hold. Make use of Lemma 3.4, we can complete the proof in the same way as Theorem 4.3 in [9], so we omit here too. □

4 Convergence of the Algorithm 2.1 with a Armijo-type line search

In this section, we will prove the global convergence of the Algorithm 2.1 under an Armijo-type line search which determine

$$\alpha_k = \max\left\{\rho^j, j = 1, 2, \cdots\right\}$$

satisfying

$$f(x_k + \alpha_k d_k) - f(x_k) < -\delta_1 \alpha_k^2 \|d_k\|^4, \tag{4.1}$$

where $\delta_1 > 0$ and $\rho \in (0, 1)$.

We get from the definition of Algorithm 2.1 that the function value sequence $\{f(x_k)\}$ is decreasing. And what's more, if f is bounded from below, from (4.1) we have

$$\sum_{k=0}^{\infty} \alpha_k^2 \|d_k\|^4 < \infty \quad \text{or} \quad \sum_{k=0}^{\infty} \alpha_k \|d_k\|^2 < \infty. \tag{4.2}$$

Under Assumption 3.1, we have the following useful lemma.

Lemma 4.1. *Let the conditions in Assumption* 3.1 *hold,* $\{x_k\}$ *and* $\{d_k\}$ *be generated by Algorithm* 2.1 *with the above Armijo-type line search. If there exists a constant* $\gamma > 0$ *such that*

$$\|g_k\| > \gamma, \quad \forall\, k \geq 0, \tag{4.3}$$

then there exists a constant $T > 0$ *such that*

$$\|d_k\| \leq T, \quad \forall\, k \geq 0. \tag{4.4}$$

Proof. By the definition of d_k and the inequalities (3.1) and (3.2), we have

$$\begin{aligned}\|d_k\| &\leq \|g_k\| + |\beta_k^{\text{DPRP}}|\|d_{k-1}\| \\ &\leq \|g_k\| + \frac{\|g_k\|\|y_{k-1}\|\|d_{k-1}\|}{\|g_{k-1}\|^2} + t\frac{\|y_{k-1}\|^2\|g_k\|\|d_{k-1}\|}{\|g_{k-1}\|^4}\|d_{k-1}\| \\ &\leq \gamma_1 + \frac{\gamma_1 L\alpha_{k-1}\|d_{k-1}\|^2}{\gamma^2} + t\frac{2\gamma_1^2 L\alpha_{k-1}\|d_{k-1}\|^2}{\gamma^4}\|d_{k-1}\|.\end{aligned}$$

(4.2) implies that $\alpha_k\|d_k\|^2 \to 0$ as $k \to \infty$. Then for any constant $b \in (0,1)$, there exist a index k_0 such that

$$\frac{2t\gamma_1^2 L\alpha_{k-1}\|d_{k-1}\|^2}{\gamma^4} < b, \quad \forall k > k_0.$$

Then

$$\|d_k\| \leq \gamma_1 + \frac{\gamma^2 b}{2t\gamma_1} + b\|d_{k-1}\| = c + b\|d_{k-1}\|, \quad \forall k > k_0,$$

where $c = \gamma_1 + \frac{\gamma^2 b}{2t\gamma_1}$ is a constant. For any $k > k_0$ we have

$$\|d_k\| \leq c(1 + b + b^2 + \cdots + b^{k-k_0+1}) + b^{k-k_0}\|d_{k_0}\| \leq \frac{c}{1-b} + \|d_{k_0}\|.$$

We can get (4.4) by setting

$$T = \max\{\|d_1\|, \|d_2\|, \cdots, \|d_{k_0}\|, \frac{c}{1-b} + \|d_{k_0}\|\}. \qquad \square$$

Based on Lemma 4.6 we give the next global convergent theorem for Algorithm 2.1 under the above Armijo-type line search.

Theorem 4.2. *Let the conditions in Assumption* 3.1 *hold,* $\{x_k\}$ *be generated by Algorithm* 2.1 *with the above Armijo-type line search, then either* $\|g_k\| = 0$ *for some k or*

$$\liminf_{k\to\infty} \|g_k\| = 0. \tag{4.5}$$

Proof. We suppose for the sake of contradiction that $\|g_k\| \neq 0$ for all $k \geq 0$ and $\liminf_{k\to\infty}\|g_k\| \neq 0$. Denote $\gamma = \inf\{\|g_k\| : k \geq 0\}$. It is clear that

$$\|g_k\| \geq \gamma > 0, \quad \forall k \geq 0. \tag{4.6}$$

At first, we consider the case that $\liminf_{k\to\infty} \alpha_k > 0$. From (4.2) we have $\liminf_{k\to\infty} \|d_k\| = 0$. This together with (1.6) gives $\liminf_{k\to\infty} \|g_k\| = 0$, which contradicts (4.6).

On the other hand, if $\liminf_{k\to\infty} \alpha_k = 0$, then there exists a infinite index set K such that

$$\liminf_{k\in K, k\to\infty} \alpha_k = 0 \tag{4.7}$$

The Step 3 of the Algorithm 2.1 implies that $\rho^{-1}\alpha_k$ does not satisfy (4.1). Namely

$$f(x_k + \rho^{-1}\alpha_k d_k) - f(x_k) > -\delta_1 \rho^{-2}\alpha_k^2 \|d_k\|^4. \tag{4.8}$$

By the Lipschitz condition (3.1) and the mean value theorem, there is a $\xi_k \in [0, 1]$, such that

$$\begin{aligned} & f(x_k + \rho^{-1}\alpha_k d_k) - f(x_k) \\ = \; & \rho^{-1}\alpha_k g(x_k + \xi_k \rho^{-1}\alpha_k d_k)^T d_k \\ = \; & \rho^{-1}\alpha_k g_k^T d_k + \rho^{-1}\alpha_k (g(x_k + \xi_k \rho^{-1}\alpha_k d_k) - g_k)^T d_k \\ \le \; & \rho^{-1}\alpha_k g_k^T d_k + L\rho^{-2}\alpha_k^2 \|d_k\|^2. \end{aligned}$$

This together with (4.8), (4.4) and (1.6) gives

$$\begin{aligned} \frac{4t-1}{4t}\|g_k\|^2 \le -g_k^T d_k &\le \alpha_k \rho^{-1}(\delta_1 \|d_k\|^4 + L\|d_k\|^2) \\ &\le \alpha_k \rho^{-1}(\delta_1 T^4 + L T^2). \end{aligned} \tag{4.9}$$

(4.7) and (4.9) imply that $\liminf_{k\in K, k\to\infty} \|g_k\| = 0$. This also yields contradiction, then the proof is completed. □

5 Numerical results

In this section, we do some numerical experiments to test Algorithm 2.1 with the modified strong Wolfe line search, and compare the performance with some existing conjugate gradient methods including the PRP+ method developed by Gilbert and Nocedal [9], the CG_DESCENT method proposed by Hager and Zhang [13], and the MPRP method proposed by Zhang et al., in [10].

The PRP+ code was obtained from Nocedal's web page at
`http://www.ece.northwestern.edu.nocedalsoftware.html`
and the CG_DESCENT code from Hager's web page at
`http://www.math.ufl.edu/~hager/papers/CG`. All codes were written in Fortran77 and ran on a PC with 2.8 GHZ CPU processor and 2GB RAM memory and Linux (Fedora 10 with GCC 4.3.2) operation system. We stop the iteration if $\|g_k\|_\infty \leq 10^{-6}$ or the total number of iterations is larger than 10000.

We tested *all* the unconstrained problems in CUTEr library [23] with their *default* dimensions. By the end of 2010, there were 164 unconstrained optimization problems in the CUTEr library. Moreover, five of them failed to install on our system for the limit storage space.

In order to get relatively better t value in Algorithm 2.1, we tested the problems with $t = 0.3, 0.5, 0.7, 0.9, 1.1, 1.3, 1.5$. We obtained from the data that Algorithm 2.1 with $t = 1.3$ performed slightly better than others. So in this section, we only compare the numerical results for Algorithm 2.1 with $t = 1.3$ with CG_DESCENT, MPRP and PRP+ method. The problems and their dimensions are listed in Table 1. Table 2 list all the numerical results, which include the total number of iterations (Iter), the total number of function evaluations (Nf), the total number of gradient evaluations (Ng), the CPU time (Time) in seconds, respectively. In Table 2, "$-$" means the method failed and "NaN" means that the cost function generated a "NaN" for the function value.

N	Problem	Dim	N	Problem	Dim	N	Problem	Dim
1	3PK	30	54	EIGENALS	110	107	OSBORNEA	5
2	AKIVA	2	55	EIGENBLS	110	108	OSBORNEB	11
3	ALLINITU	4	56	EIGENCLS	462	109	OSCIPATH	15
4	ARGLINA	200	57	ENGVAL1	5000	110	PALMER1C	8
5	ARGLINB	200	58	ENGVAL2	3	111	PALMER1D	7
6	ARGLINC	200	59	ERRINROS	50	112	PALMER2C	8
7	ARWHEAD	5000	60	EXPFIT	2	113	PALMER3C	8
8	BARD	3	61	EXTROSNB	1000	114	PALMER4C	8
9	BDQRTIC	5000	62	FLETCBV2	5000	115	PALMER5C	6
10	BEALE	2	63	FLETCBV3	100	116	PALMER6C	8
11	BIGGS6	6	64	FLETCHBV	100	117	PALMER7C	8
12	BOX	1000	65	FLETCHCR	1000	118	PALMER8C	8
13	BOX3	3	66	FMINSRF2	5625	119	PENALTY1	1000
14	BRKMCC	2	67	FMINSURF	5625	120	PENALTY2	200
15	BROWNAL	200	68	FREUROTH	5000	121	PENALTY3	100

Table 1 (continuation) – The problems and their dimensions.

N	Problem	Dim	N	Problem	Dim	N	Problem	Dim
16	BROWNBS	2	69	GENHUMPS	5000	122	PFIT1LS	3
17	BROWNDEN	4	70	GENROSE	500	123	PFIT2LS	3
18	BROYDN7D	5000	71	GROWTHLS	3	124	PFIT3LS	3
19	BRYBND	5000	72	GULF	3	125	PFIT4LS	3
20	CHAINWOO	4000	73	HAIRY	2	126	POWELLSG	5000
21	CHNROSNB	50	74	HATFLDD	3	127	POWER	10000
22	CLIFF	2	75	HATFLDE	3	128	QUARTC	5000
23	COSINE	10000	76	HATFLDFL	3	129	ROSENBR	2
24	CRAGGLVY	5000	77	HEART6LS	6	130	S308	2
25	CUBE	2	78	HEART8LS	8	131	SBRYBND	100
26	CURLY10	1000	79	HELIX	3	132	SCHMVETT	5000
27	CURLY20	1000	80	HIELOW	3	133	SCOSINE	100
28	CURLY30	1000	81	HILBERTA	2	134	SCURLY10	100
29	DECONVU	61	82	HILBERTB	10	135	SCURLY20	100
30	DENSCHNA	2	83	HIMMELBB	2	136	SCURLY30	100
31	DENSCHNB	2	84	HIMMELBF	4	137	SENSORS	100
32	DENSCHNC	2	85	HIMMELBG	2	138	SINEVAL	2
33	DENSCHND	3	86	HIMMELBH	2	139	SINQUAD	5000
34	DENSCHNE	3	87	HUMPS	2	140	SISSER	2
35	DENSCHNF	2	88	HYDC20LS	99	141	SNAIL	2
36	DIXMAANA	3000	89	INDEF	5000	142	SPARSINE	1000
37	DIXMAANB	3000	90	JENSMP	2	143	SPARSQUR	10000
38	DIXMAANC	3000	91	KOWOSB	4	144	SPMSRTLS	4999
39	DIXMAAND	3000	92	LIARWHD	5000	145	SROSENBR	5000
40	DIXMAANE	3000	93	LOGHAIRY	2	146	TESTQUAD	5000
41	DIXMAANF	3000	94	MANCINO	100	147	TOINTGOR	50
42	DIXMAANG	3000	95	MARATOSB	2	148	TOINTGSS	5000
43	DIXMAANH	3000	96	MEXHAT	2	149	TOINTPSP	50
44	DIXMAANI	3000	97	MEYER3	3	150	TOINTQOR	50
45	DIXMAANJ	3000	98	MODBEALE	2000	151	TQUARTIC	5000
46	DIXMAANK	15	99	MOREBV	5000	152	TRIDIA	5000
47	DIXMAANL	3000	100	MSQRTALS	1024	153	VARDIM	200
48	DIXON3DQ	10000	101	MSQRTBLS	1024	154	VAREIGVL	50
49	DJTL	2	102	NONCVXU2	5000	155	VIBRBEAM	8
50	DQDRTIC	5000	103	NONCVXUN	100	156	WATSON	12
51	DQRTIC	5000	104	NONDIA	5000	157	WOODS	4000
52	EDENSCH	2000	105	NONDQUAR	5000	158	YFITU	3
53	EG2	1000	106	NONMSQRT	9	159	ZANGWIL2	2

We used the profiles by Dolan and Moré [24] to compare the performance of those methods. We discarded the data of the problems for which different methods converged to different local minimizers. We consider that two methods converged to the same solution if the optimal costs f_1 and f_2 satisfy the following conditions

$$\frac{|f_1 - f_2|}{1 + |f_i|} \leq 10^{-6}, \quad i = 1, 2.$$

N	CG_DESCENT	Algorithm 2.1	MPRP	PRP+
	Iter/Nf/Ng/Time	Iter/Nf/Ng/Time	Iter/Nf/Ng/Time	Iter/Nf(Ng)/Time
1	–/–/–/–	–/–/–/–	–/–/–/–	–/–/–
2	17/40/28/0.002	13/30/22/0.001	11/24/15/0.002	8/26/0.001
3	14/33/19/0	15/33/21/0.001	11/26/15/0	10/28/0
4	1/3/2/0.006	1/3/2/0.005	1/3/2/0.006	1/5/0.008
5	8/16/19/0.028	7/14/16/0.025	8/15/18/0.026	–/–/–
6	8/15/18/0.026	9/19/23/0.032	4/10/9/0.016	–/–/–
7	10/23/16/0.033	12/26/18/0.041	9/24/19/0.045	–/–/–
8	27/59/38/0.001	26/56/34/0.002	23/50/31/0	22/58/0.001
9	3763/6992/8726/18.3	3240/5933/7665/15.9	1950/4049/4468/10.1	–/–/–
10	22/45/25/0.001	22/47/29/0	21/43/26/0	11/30/0
11	123/260/165/0.004	202/436/285/0.008	142/315/204/0.005	167/373/0.006
12	16/40/31/0.017	15/38/29/0.015	36/60/77/0.038	6/17/0.007
13	18/37/22/0.001	17/35/20/0	20/41/26/0.002	20/51/0
14	6/13/8/0.001	6/13/8/0.001	6/13/8/0	6/13/0
15	4/9/6/0.007	6/18/14/0.011	15/31/21/0.018	7/47/0.026
16	–/–/–/–	17/178/173/0	15/181/176/0	–/–/–
17	36/73/57/0.001	32/62/48/0	31/61/47/0.001	–/–/–
18	1502/2988/1524/14.1	1420/2842/1446/13.4	1435/2858/1454/13.9	6007/12369/83.2
19	36/74/39/0.141	38/82/46/0.16	32/67/37/0.137	26/66/0.16
20	312/588/394/0.792	519/958/642/1.305	410/765/579/1.194	412/872/1.319
21	272/545/273/0.006	282/566/284/0.005	236/473/237/0.005	314/636/0.008
22	34/93/61/0.001	33/89/59/0	–/–/–/–	14/101/0.001
23	12/32/28/0.119	10/25/21/0.092	11/27/25/0.111	9/28/0.107
24	116/208/150/0.641	124/245/171/0.736	124/241/175/0.792	–/–/–
25	62/166/122/0	61/186/140/0	75/198/143/0.001	15/45/0
26	9431/14475/14406/4.2	8841/13371/13854/3.9	9624/14677/14981/4.7	–/–/–
27	9757/15481/15084/7.07	9765/16590/17351/7.92	9672/15210/15052/7.3	–/–/–
28	9765/15713/15122/9.46	9703/16847/17210/10.7	9971/15941/15722/10.3	–/–/–
29	457/916/462/0.03	397/797/401/0.027	394/790/398/0.026	696/1398/0.062
30	10/21/11/0	10/21/11/0.001	10/21/11/0	7/21/0.001
31	5/11/6/0	7/15/8/0	6/13/7/0	7/19/0
32	17/39/25/0	14/33/20/0.001	16/37/22/0	7/24/0
33	170/346/207/0.001	120/251/153/0.001	162/331/203/0.001	22/65/0
34	19/62/47/0.001	13/59/48/0.001	14/44/32/0.001	13/53/0
35	10/21/11/0	10/21/11/0	10/21/11/0	10/37/0
36	9/19/10/0.012	8/17/9/0.012	7/15/8/0.011	7/20/0.014
37	9/19/10/0.012	8/17/9/0.012	8/17/9/0.011	6/23/0.017
38	10/21/11/0.013	9/19/10/0.013	9/19/10/0.014	8/26/0.019
39	12/25/13/0.017	11/23/12/0.016	11/23/12/0.015	7/25/0.018
40	225/451/226/0.269	226/453/227/0.268	227/455/228/0.288	228/462/0.334

Table 2 (continuation) – The numerical results of the methods.

N	CG_DESCENT	Algorithm 2.1	MPRP	PRP+
	Iter/Nf/Ng/Time	Iter/Nf/Ng/Time	Iter/Nf/Ng/Time	Iter/Nf(Ng)/Time
41	174/349/175/0.21	164/329/165/0.196	169/339/170/0.22	167/342/0.245
42	170/341/171/0.206	163/327/164/0.191	167/335/168/0.218	159/327/0.243
43	167/335/168/0.197	163/327/164/0.196	169/339/170/0.211	257/523/0.389
44	2552/5105/2553/3.00	3094/6189/3095/3.63	2632/5265/2633/3.35	2399/4804/3.457
45	297/595/298/0.356	313/627/314/0.372	313/627/314/0.396	360/728/0.542
46	63/127/64/0.001	47/95/49/0.001	55/111/57/0	48/102/0
47	231/463/232/0.272	234/469/235/0.277	281/563/282/0.357	283/575/0.419
48	10000/20001/10002/21	10000/20001/10002/21	10000/20001/10002/23	10000/20006/25
49	179/872/743/0.003	107/550/481/0.003	125/698/602/0.003	−/−/−
50	7/15/8/0.019	7/15/8/0.018	7/15/8/0.018	5/15/0.019
51	33/67/34/0.033	33/67/34/0.033	33/67/34/0.036	17/66/0.035
52	33/61/43/0.041	28/54/36/0.036	32/60/45/0.042	−/−/−
53	4/9/6/0.004	3/7/4/0.003	4/9/6/0.004	−/−/−
54	376/757/389/0.067	394/794/408/0.071	449/903/463/0.08	1502/3009/0.376
55	342/687/345/0.059	326/656/330/0.056	393/790/397/0.068	397/800/0.104
56	1776/3575/1802/2.75	1757/3524/1769/2.71	1634/3279/1647/2.52	1867/3746/4.23
57	26/46/37/0.07	24/45/33/0.065	25/44/34/0.07	−/−/−
58	173/361/249/0.001	100/231/163/0	88/188/128/0.001	112/335/0
59	1013/2023/1444/0.024	993/1965/1437/0.025	1306/2590/1809/0.032	−/−/−
60	17/40/27/0	19/44/31/0.001	13/37/28/0	12/35/0.001
61	6423/13272/6951/1.89	8030/16297/8318/2.28	8009/16217/8268/2.42	50/117/0.021
62	0/1/1/0.004	0/1/1/0.005	0/1/1/0.004	4101/8203/17.33
63	1854/4600/3078/0.201	−/−/−/−	9510/22981/14091/0.96	2446/8908/0.372
64	−/−/−/−	−/−/−/−	−/−/−/−	−/−/−
65	6828/14236/7479/2.88	5048/10287/5248/2.06	4306/8642/4344/1.82	4371/8767/2.2
66	363/729/366/1.043	359/725/367/1.086	305/611/306/0.916	350/707/1.443
67	492/985/493/1.485	442/886/445/1.393	410/821/411/1.304	471/949/2.027
68	65/126/95/0.248	40/80/68/0.171	54/108/81/0.217	−/−/−
69	9412/18948/9575/54.4	6600/13290/6711/38.3	6078/12900/6945/38.5	7442/15241/47.1
70	1259/2559/1309/0.26	1116/2277/1171/0.23	1089/2219/1146/0.23	1121/2269/0.28
71	−/−/−/−(NaN)	−/−/−/−(NaN)	−/−/−/−(NaN)	−/−/−
72	−/−/−/−(NaN)	−/−/−/−(NaN)	−/−/−/−(NaN)	120/322/0.053
73	59/189/147/0.002	38/129/100/0.001	20/67/56/0.001	16/62/0
74	51/125/89/0.002	57/140/104/0.002	66/158/110/0.002	100/237/0.002
75	108/236/158/0.005	68/145/95/0.002	67/148/100/0.002	25/72/0.001
76	531/1228/882/0.002	915/2091/1517/0.003	447/996/711/0.001	46/140/0
77	6850/24080/20394/0.09	9502/31328/26301/0.12	4974/16343/13645/0.07	−/−/−
78	301/719/487/0.003	322/778/530/0.004	283/694/482/0.003	1172/2442/0.01
79	48/103/59/0.001	57/121/73/0	51/107/66/0.001	157/324/0
80	67/152/128/0.442	65/157/121/0.438	57/135/112/0.395	−/−/−
81	2/5/3/0	2/5/3/0	2/5/3/0.001	2/5/0
82	4/9/5/0.001	4/9/5/0	4/9/5/0	4/10/0
83	21/43/22/0	21/43/22/0	21/43/22/0	−/−/−
84	8972/21280/17580/0.05	6599/17322/15980/0.05	3891/9501/7701/0.02	−/−/−
85	9/25/18/0.001	8/24/17/0	8/22/15/0.001	8/22/0
86	10/24/14/0	8/20/12/0.001	8/20/12/0.002	5/13/0

Table 2 (continuation) – The numerical results of the methods.

N	CG_DESCENT	Algorithm 2.1	MPRP	PRP+
	Iter/Nf/Ng/Time	Iter/Nf/Ng/Time	Iter/Nf/Ng/Time	Iter/Nf(Ng)/Time
87	100/336/264/0.002	70/256/212/0.001	45/139/109/0	20/136/0
88	–/–/–/–	–/–/–/–	–/–/–/–	–/–/–
89	–/–/–/–	–/–/–/–	–/–/–/–	–/–/–
90	16/32/23/0	15/33/24/0	21/46/34/0	–/–/–
91	59/131/86/0.001	104/222/146/0	104/219/140/0.001	488/990/0.004
92	21/48/32/0.061	30/66/42/0.082	22/47/30/0.062	16/46/0.063
93	92/362/306/0.001	98/428/361/0.001	104/369/304/0.001	53/333/0.001
94	11/23/12/0.137	11/23/12/0.137	10/21/11/0.126	11/27/0.17
95	2333/9160/7879/0.01	2586/11296/9870/0.01	2695/10016/8557/0.01	577/2164/0.01
96	28/60/38/0.001	27/56/35/0	31/64/38/0	15/59/0.001
97	–/–/–/–	–/–/–/–	–/–/–/–	–/–/–
98	779/1465/949/1.93	214/431/365/0.63	–/–/–/–	203/482/0.634
99	147/295/149/0.27	165/331/167/0.299	164/329/166/0.31	161/323/0.369
100	3393/6793/3402/16.7	3621/7252/3633/17.8	3708/7423/3717/18.29	2934/5873/20.95
101	2318/4642/2325/11.54	2078/4162/2085/10.25	2210/4427/2219/11.01	2396/4797/17.21
102	9118/17827/9529/32.7	9044/17701/9433/32	7494/13246/9238/28.9	7224/14458/29
103	168/329/179/0.012	168/324/186/0.013	150/298/154/0.01	228/466/0.019
104	8/27/22/0.034	10/22/13/0.025	10/37/32/0.049	5/26/0.034
105	5014/10053/5154/5.638	5234/10483/5636/5.995	3397/6800/3504/4.11	–/–/–
106	–/–/–/–	–/–/–/–	–/–/–/–	–/–/–
107	–/–/–/–(NaN)	–/–/–/–(NaN)	–/–/–/–(NaN)	1130/2756/0.072
108	399/803/432/0.088	675/1359/724/0.141	693/1395/745/0.143	291/598/0.067
109	12/25/13/0.001	12/25/13/0	12/25/13/0	12/25/0
110	–/–/–/–	–/–/–/–	–/–/–/–	–/–/–
111	–/–/–/–	–/–/–/–	–/–/–/–	–/–/–
112	–/–/–/–	–/–/–/–	–/–/–/–	–/–/–
113	–/–/–/–	–/–/–/–	–/–/–/–	–/–/–
114	–/–/–/–	–/–/–/–	–/–/–/–	–/–/–
115	7/15/8/0	7/15/8/0	7/15/8/0	6/15/0
116	–/–/–/–	–/–/–/–	–/–/–/–	1910/4935/0.013
117	–/–/–/–	–/–/–/–	–/–/–/–	–/–/–
118	–/–/–/–	–/–/–/–	–/–/–/–	–/–/–
119	50/121/77/0.018	61/171/120/0.026	47/124/83/0.019	42/173/0.026
120	199/234/365/0.126	189/221/348/0.123	210/245/392/0.139	–/–/–
121	83/195/137/0.374	–/–/–/–	76/184/136/0.369	–/–/–
122	–/–/–/–(NaN)	–/–/–/–(NaN)	–/–/–/–(NaN)	3559/10403/0.037
123	–/–/–/–(NaN)	–/–/–/–(NaN)	–/–/–/–(NaN)	–/–/–
124	–/–/–/–(NaN)	–/–/–/–(NaN)	–/–/–/–(NaN)	–/–/–
125	–/–/–/–(NaN)	–/–/–/–(NaN)	–/–/–/–(NaN)	–/–/–
126	162/332/187/0.178	256/517/302/0.281	113/228/122/0.13	148/346/0.207
127	369/739/370/0.56	368/737/369/0.56	1116/2233/1117/1.89	355/719/0.67
128	33/67/34/0.032	33/67/34/0.033	33/67/34/0.035	17/66/0.034
129	50/123/86/0.001	46/148/117/0	58/153/114/0.001	23/72/0
130	10/21/11/0.001	10/21/11/0	10/21/11/0	8/25/0
131	–/–/–/–	–/–/–/–	–/–/–/–	–/–/–
132	39/66/53/0.383	41/68/57/0.408	39/68/51/0.377	41/91/0.44

Table 2 (continuation) – The numerical results of the methods.

N	CG_DESCENT	Algorithm 2.1	MPRP	PRP+
	Iter/Nf/Ng/Time	Iter/Nf/Ng/Time	Iter/Nf/Ng/Time	Iter/Nf(Ng)/Time
133	–/–/–/–	–/–/–/–	–/–/–/–	–/–/–
134	–/–/–/–	–/–/–/–	–/–/–/–	–/–/–
135	–/–/–/–	–/–/–/–	–/–/–/–	–/–/–
136	–/–/–/–	–/–/–/–	–/–/–/–	–/–/–
137	25/57/44/0.507	29/67/49/0.578	31/73/57/0.661	27/66/0.495
138	138/465/380/0.001	147/480/391/0.001	165/530/431/0.001	43/137/0
139	46/111/108/0.313	66/120/147/0.407	339/531/804/2.16	–/–/–
140	13/27/14/0	13/27/14/0.001	13/27/14/0.001	4/21/0
141	25/70/54/0	25/84/67/0	30/97/78/0	60/218/0.001
142	4515/9031/4516/3.69	4348/8697/4349/3.72	5282/10565/5283/4.41	4072/8149/4.38
143	22/45/23/0.168	22/45/23/0.17	22/45/23/0.177	41/131/0.783
144	218/443/227/0.72	196/402/208/0.633	206/419/215/0.682	212/430/0.898
145	12/26/16/0.018	14/31/21/0.022	10/22/13/0.016	10/29/0.021
146	1715/3431/1716/1.31	1682/3365/1683/1.28	1664/3329/1665/1.42	1590/3183/1.45
147	122/224/154/0.006	119/217/148/0.005	120/222/150/0.005	–/–/–
148	4/9/5/0.023	4/9/5/0.024	4/9/5/0.023	1/7/0.019
149	155/327/211/0.005	139/304/215/0.004	128/267/188/0.003	–/–/–
150	32/61/41/0.001	31/58/39/0.002	32/60/40/0	29/60/0
151	21/52/38/0.049	20/65/51/0.066	18/71/61/0.071	9/32/0.031
152	782/1565/783/0.80	784/1569/785/0.83	784/1569/785/0.87	781/1565/0.96
153	28/57/29/0.002	25/53/28/0.002	28/57/29/0.002	8/44/0.001
154	60/164/104/0.003	60/164/104/0.003	86/233/147/0.005	25/57/0.002
155	–/–/–/–	–/–/–/–	–/–/–/–	–/–/–
156	1071/2148/1352/0.03	1230/2468/1548/0.03	1401/2824/1811/0.04	2602/5223/0.08
157	148/342/214/0.235	205/446/260/0.321	271/562/303/0.383	190/393/0.311
158	960/2662/1999/0.026	1529/4203/3153/0.044	555/1517/1133/0.015	–/–/–
159	1/3/2/0	1/3/2/0	1/3/2/0.001	1/3/0

The problems which were discarded by this process were BROYDN7D, CHAINWOO, NONCVXUN and SENSORS.

Figures 1-4 show the performance of the above four methods relative to the CPU time (in second), the total number of iterations, the total number of function evaluations, the total number of gradient evaluations, respectively. For example, the performance of the four algorithms relative to CPU time means that, for each method, we plot the fraction P of problems for which the method is within a factor τ of the best time. The left side of the figure gives the percentage of the test problems for which a method is the fastest; the right side gives the percentage of the test problems that were successfully solved by each of the methods. The top curve is the method that solved the most problems in a time that was within a factor τ of the best time. For convenience, we give the meanings of these methods in the figures.

- "CG_DESCENT" stands for the CG_DESCENT method with the approximate Wolfe line search [13]. Here we use the Fortran77 (version 1.4) code obtained from Hager's web site and use the default parameters there.

- "PRP+" shows the PRP+ method with Wolfe line search proposed in [22].

- "MPRP" means the three term PRP method proposed in [10] with the same line search as "CG_DESCENT" method.

- "Algorithm 2.1" means Algorithm 2.1 with the modified strong Wolfe line search (3.3). We determine the steplength α_k which satisfies the conditions in (3.3) or the following approximate conditions:

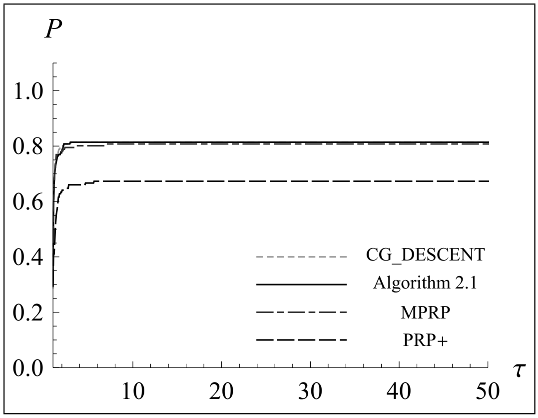
$$\begin{cases} -\min\{M, A\} \leq \phi'(\alpha_k) \leq \min\{M, A, (2\delta - 1)\phi'(0)\}, \\ \phi(\alpha_k) \leq \phi(0) + \epsilon|\phi(0)|, \end{cases} \tag{5.1}$$

where $A = \sigma|\phi'(0)|$, $\phi(\alpha) = f(x_k + \alpha d_k)$, the constant ϵ is the estimate to the error of $\phi(0)$ at iteration x_k. The first inequality in (5.1) is obtained by replacing $\phi(\alpha)$ with its approximation function

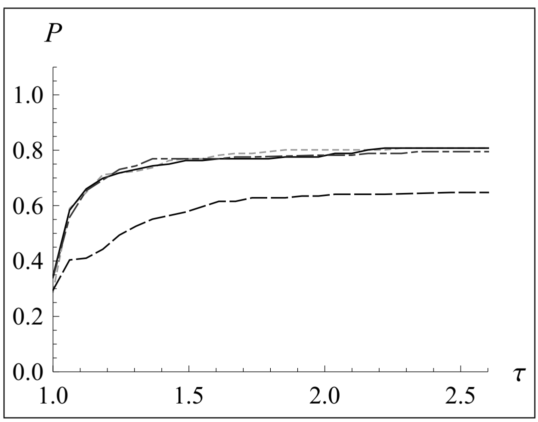
$$q(\alpha) = \phi(0) + \alpha\phi'(0) + \alpha^2 \frac{\phi'(\alpha_k) - \phi'(0)}{2\alpha_k}$$

in the modified strong Wolfe conditions (3.3). The approximate conditions in (5.1) are modifications to the approximate Wolfe conditions proposed by Hager and Zhang in [13]. We refer to [13] for more details. The Algorithm 2.1 code is a modification of the CG_DESCENT subroutine proposed by Hager and Zhang [13]. We use the default parameter there. Moreover, we set $M = 10^{30}$ for (3.3) to ensure that $M \geq \sigma|g_k^T d_k|$ hold for most indices k. If $\sigma|g_k^T d_k| \geq M$, we determine the steplength α_k which satisfies the first inequality in (3.3). The code of Algorithm 2.1 was posted on the web page at: `http://blog.sina.com.cn/liminmath`.

We can see from Figures 1-4 that the curves "Algorithm 2.1", "CG_DESCENT" and "MPRP" are very close. This means that the performances of Algorithm 2.1, CG_DESCENT and MPRP methods are similar. That is to say,



(A)



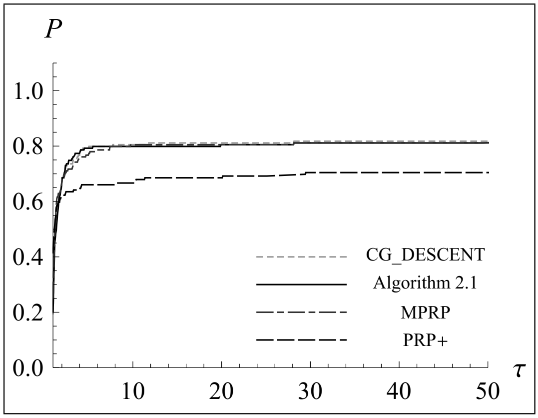
(B)

Figure 1 – Performance profiles relative to the CPU time.

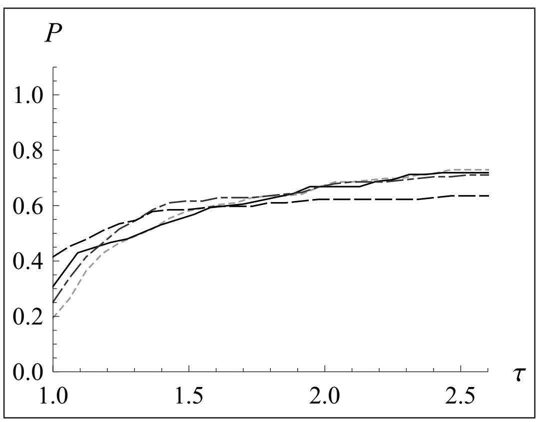
Algorithm 2.1 is efficient for solving unconstrained optimal problems since CG_DESCENT has been regarded as one of the most efficient nonlinear conjugate gradients. We also note that the performances of Algorithm 2.1, CG_DESCENT and MPRP methods are much better than that of the PRP+ method.

6 Conclusion

We have established the convergence of a sufficient descent PRP conjugate gradient method with a modified strong Wolfe line search method and an Armijo-type line search. It was shown from the numerical results that Algorithm 2.1



(A)



(B)

Figure 2 – Performance profiles relative to the number of iterations.

was efficient for solving the unconstrained problems in CUTEr library, and the numerical performance was similar to that of the wellknown CG_DESCENT and MPRP methods.

Acknowledgements. The authors would like to thank the referees and Prof. DongHui Li for giving us many valuable suggestions and comments, which improves this paper greatly. We are very grateful to Prof. W.W. Hager, H. Zhang and Prof. J. Nocedal for providing their codes, and Zhang, Zhou and Li for the MPRP method as well.

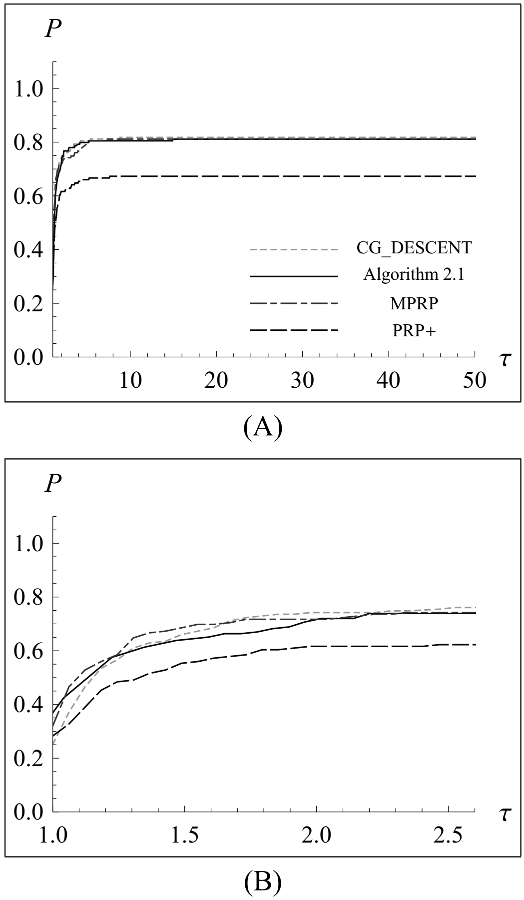


Figure 3 – Performance profiles relative to the number of function evaluated.

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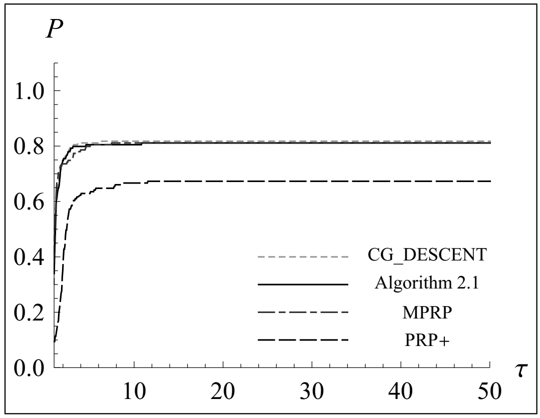
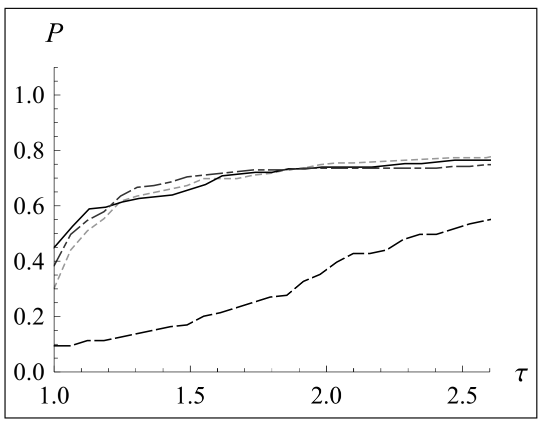


(A)



(B)

Figure 4 – Performance profiles relative to the number of gradient evaluated.

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