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Two iterative algorithms for solving coupled matrix equations over reflexive and anti-reflexive matrices

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Abstract. An $n \times n$ real matrix P is said to be a generalized reflection matrix if $P^T = P$ and $P^2 = I$ (where P^T is the transpose of P). A matrix $A \in \mathbb{R}^{n\times n}$ is said to be a reflexive (anti-reflexive) matrix with respect to the generalized reflection matrix P if $A = PAP$ ($A = -PAP$). The reflexive and anti-reflexive matrices have wide applications in many fields. In this article, two iterative algorithms are proposed to solve the coupled matrix equations

$$\begin{cases} A_1XB_1 + C_1X^TD_1 = M_1, \\ A_2XB_2 + C_2X^TD_2 = M_2, \end{cases}$$

over reflexive and anti-reflexive matrices, respectively. We prove that the first (second) algorithm converges to the reflexive (anti-reflexive) solution of the coupled matrix equations for any initial reflexive (anti-reflexive) matrix. Finally two numerical examples are used to illustrate the efficiency of the proposed algorithms.

Mathematical subject classification: 15A06, 15A24, 65F15, 65F20.

Key words: iterative algorithm, matrix equation, reflexive matrix, anti-reflexive matrix.

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1 Introduction

In this paper we use the following notation. Let $\mathbb{R}^{m\times n}$ be the set of all $m \times n$ real matrices. We use $\mathrm{tr}(A)$, A^T, $\rho(A)$, $\lambda(A)$ and $\lambda_{\max}(A)$ to denote the trace, the transpose, the spectral radius, the eigenvalue set and the maximum eigenvalue of the matrix A respectively. We denote by I_k and $O_{m\times n}$ the $k \times k$ identity matrix and the $m \times n$ zero matrix, respectively. We also write them as I and O, respectively, when the dimensions of these matrices are clear. We define an inner product as $\langle A, B\rangle = \mathrm{tr}(B^T A)$, then the norm of a matrix A generated by this inner product is Frobenius norm and is denoted by $\langle A, A\rangle = ||A||^2$.

An $n \times n$ real matrix P is said to be a real generalized reflection matrix if $P^T = P$ and $P^2 = I$. An $n \times n$ real matrix A is said to be a reflexive (anti-reflexive) matrix with respect to the generalized reflection matrix P if $A = PAP$ ($A = -PAP$). $\mathbb{R}_r^{n\times n}(P)$ ($\mathbb{R}_a^{n\times n}(P)$) denotes the subspace reflexive (anti-reflexive) matrices with respect to the $n \times n$ generalized reflection matrix P. The reflexive and anti-reflexive matrices have practical applications in many areas such as the numerical solution of certain differential equations [1], pattern recognition [6], Markov processes [42], various physical and engineering problems [7] and so on (e.g. [20, 32, 43]). Chen [3] proposed three applications of reflexive and anti-reflexive matrices obtained from the altitude estimation of a level network, an electric network and structural analysis of trusses. The symmetric Toeplitz matrices, an important subclass of the class of symmetric reflexive matrices, appear naturally in digital signal processing applications and other areas [21].

The linear matrix equations, such as $AXB = C$, $AXB + CXD = E$ and $AXB + CX^T D = M$, play an important role in linear system theory therefore a large number of papers have presented several methods for solving these matrix equations [2, 9, 15, 36]. Research on solving of linear matrix equations has been actively ongoing for past years. In [5], Dai studied the linear matrix equation

$$AXB = C, \tag{1.1}$$

over symmetric matrix X. By using g-inverse, Mitra [38] obtained the common solution of simultaneous matrix equations

$$\begin{cases} A_1XB_1 = C_1, \\ A_2XB_2 = C_2. \end{cases} \tag{1.2}$$

Navarra et al. [39] studied a representation of the solution X to the system of matrix equations (1.2). The matrix equation

$$AX + X^T C = B, \tag{1.3}$$

plays important roles in system theory, such as eigenstructure assignment [29], observer design [4], control of system with input constraint [28], and fault detection [30].

In [40], the necessary and sufficient condition for the existence of the solution to the matrix equation (1.3) and its solution expression was investigated by the generalized inverse matrix. In [37], Cramer's rules for some quaternion matrix equations were obtained within the framework of the theory of the column and row determinants. Kyrchei [35] considered systems of linear quaternionic equations and obtained Cramer's rules for right and left quaternionic systems of linear equations. In [44, 45, 46], the solutions of the several generalized Sylvester matrix equations were established. In [24], a family of iterative methods for linear systems is presented and a least-squares iterative solution to coupled matrix equations are studied by using the hierarchical identification principle and the star product. In [26], gradient iterative algorithms for solving Sylvester coupled matrix equations and general coupled matrix equations are studied by using the gradient search principle. In [22, 25], Ding and Chen applied a hierarchical identification principle to study solving the Sylvester and Lyapunov matrix equations. Also Ding and Chen [23] proposed a hierarchical gradient iterative algorithm and a hierarchical stochastic gradient algorithm and prove that the parameter estimation errors given by the algorithms converge to zero for any initial values under persistent excitation. In [8, 10, 11, 12, 13, 14, 17, 18], Dehghan and Hajarian introduced some efficient iterative methods for solving Sylvester and Lyapunov matrix equations.

In this paper, we introduce two iterative algorithms, respectively, for the finding reflexive and anti-reflexive solutions of the coupled matrix equations

$$\begin{cases} A_1XB_1 + C_1X^T D_1 = M_1, \\ A_2XB_2 + C_2X^T D_2 = M_2, \end{cases} \tag{1.4}$$

(including the matrix equations (1.1)-(1.3) as special cases).

The rest of the paper is structured as follows. In Section 2, first we propose two iterative algorithms for solving (1.4) over reflexive and anti-reflexive matrices.

Then we study the convergence properties of the iterative algorithms. Two examples verify the efficiency of the algorithms in Section 3. Section 4 concludes the paper.

2 Main results

In this section, first we give two systems of matrix equations equivalent to (1.4) over reflexive and anti-reflexive matrices, respectively. Then we will propose two efficient iterative algorithms for solving (1.4).

Lemma 2.1. *The coupled matrix equations* (1.4) *have the reflexive solution* $X \in \mathbb{R}_r^{n\times n}(P)$ *if and only if the system of matrix equations*

$$\begin{cases} A_1XB_1 + C_1X^TD_1 = M_1, \\ A_2XB_2 + C_2X^TD_2 = M_2, \\ A_1PXPB_1 + C_1PX^TPD_1 = M_1, \\ A_2PXPB_2 + C_2PX^TPD_2 = M_2, \end{cases} \tag{2.1}$$

is consistent.

Proof. First, we suppose that the coupled matrix equations (1.4) have the reflexive solution $X^* \in \mathbb{R}_r^{n\times n}(P)$. By using $X^* = PX^*P$ and $A_iX^*B_i + C_i{X^*}^TD_i = M_i$, we have

$$\begin{aligned} A_iPX^*PB_i + C_iPX^{*T}PD_i &= A_iX^*B_i + C_i(PX^*P)^TD_i \\ &= A_iX^*B_i + C_i{X^*}^TD_i \\ &= M_i, \end{aligned} \tag{2.2}$$

for $i = 1, 2$. It is follows from (2.2) that the reflexive matrix X^* is a solution of the system of matrix equations (2.1).

Conversely assume that the system of matrix equations (2.1) is consistent. Let X be a solution of the system of matrix equations (2.1). Set

$$\widetilde{X} = \frac{X + PXP}{2}. \tag{2.3}$$

Therefore $\widetilde{X} \in \mathbb{R}_r^{n\times n}(P)$ and we can get

$$\begin{aligned}
A_i\widetilde{X}B_i + C_i\widetilde{X}^T D_i &= A_i\left(\frac{X+PXP}{2}\right)B_i + C_i\left(\frac{X+PXP}{2}\right)^T D_i \\
&= \frac{A_iXB_i + A_iPXPB_i}{2} + \frac{C_iX^TD_i + C_iPX^TPD_i}{2} \\
&= \frac{A_iXB_i + C_iX^TD_i}{2} + \frac{A_iPXPB_i + C_iPX^TPD_i}{2} \qquad (2.4)\\
&= \frac{M_i + M_i}{2} \\
&= M_i,
\end{aligned}$$

for $i = 1, 2$. Hence $\widetilde{X}$ is a reflexive solution of the coupled matrix equations (1.4). The proof is completed. □

Similarly to the above lemma, we can obtain the following lemma.

Lemma 2.2. *The coupled matrix equations* (1.4) *have the anti-reflexive solution $X \in \mathbb{R}_a^{n\times n}(P)$ $(P \neq I)$ if and only if the system of matrix equations*

$$\begin{cases}
A_1XB_1 + C_1X^TD_1 = M_1, \\
A_2XB_2 + C_2X^TD_2 = M_2, \\
-A_1PXPB_1 - C_1PX^TPD_1 = M_1, \\
-A_2PXPB_2 - C_2PX^TPD_2 = M_2,
\end{cases} \qquad (2.5)$$

is consistent.

According to Theorem 4.3.8 and Corollary 4.3.10 in [33], the systems (2.1) and (2.5), respectively, are equivalent to

$$\begin{pmatrix}
\left(B_1^T \otimes A_1\right) + \left(D_1^T \otimes C_1\right)P(n,n) \\
\left(B_2^T \otimes A_2\right) + \left(D_2^T \otimes C_2\right)P(n,n) \\
\left(B_1^TP \otimes A_1P\right) + \left(D_1^TP \otimes C_1P\right)P(n,n) \\
\left(B_2^TP \otimes A_2P\right) + \left(D_2^TP \otimes C_2P\right)P(n,n)
\end{pmatrix} \qquad (2.6)$$

$$\times \operatorname{vec}(X) = \operatorname{vec}\big(M_1, M_2, M_1, M_2\big),$$

and

$$\begin{pmatrix} \left(B_1^T \otimes A_1\right) + \left(D_1^T \otimes C_1\right) P(n,n) \\ \left(B_2^T \otimes A_2\right) + \left(D_2^T \otimes C_2\right) P(n,n) \\ -\left(B_1^T P \otimes A_1 P\right) - \left(D_1^T P \otimes C_1 P\right) P(n,n) \\ -\left(B_2^T P \otimes A_2 P\right) - \left(D_2^T P \otimes C_2 P\right) P(n,n) \end{pmatrix} \times \operatorname{vec}(X) = \operatorname{vec}\left(M_1, M_2, M_1, M_2\right), \tag{2.7}$$

where $P(n, n)$ is a permutation matrix [33]. Now by using the above results and considering

$$Z_1 := \begin{pmatrix} \left(B_1^T \otimes A_1\right) + \left(D_1^T \otimes C_1\right) P(n,n) \\ \left(B_2^T \otimes A_2\right) + \left(D_2^T \otimes C_2\right) P(n,n) \\ \left(B_1^T P \otimes A_1 P\right) + \left(D_1^T P \otimes C_1 P\right) P(n,n) \\ \left(B_2^T P \otimes A_2 P\right) + \left(D_2^T P \otimes C_2 P\right) P(n,n) \end{pmatrix}, \tag{2.8}$$

and

$$Z_2 := \begin{pmatrix} \left(B_1^T \otimes A_1\right) + \left(D_1^T \otimes C_1\right) P(n,n) \\ \left(B_2^T \otimes A_2\right) + \left(D_2^T \otimes C_2\right) P(n,n) \\ -\left(B_1^T P \otimes A_1 P\right) - \left(D_1^T P \otimes C_1 P\right) P(n,n) \\ -\left(B_2^T P \otimes A_2 P\right) - \left(D_2^T P \otimes C_2 P\right) P(n,n) \end{pmatrix}, \tag{2.9}$$

the following lemmas are well known [31, 33, 34].

Lemma 2.3. *The coupled matrix equations* (1.4) *have a unique reflexive solution with respect to the generalized reflection matrix P if and only if*

$$\operatorname{rank}((Z_1, \operatorname{vec}(M_1, M_2, M_1, M_2))) = \operatorname{rank}(Z_1)$$

and Z_1 has a full column rank. In that case, the reflexive solution of (1.4) *can be expressed by the following form*

$$\begin{aligned} X &= \frac{X_1 + P X_1 P}{2} \quad \textit{where} \\ \operatorname{vec}(X_1) &= (Z_1^T Z_1)^{-1} Z_1^T \operatorname{vec}(M_1, M_2, M_1, M_2), \end{aligned} \tag{2.10}$$

and the homogenous coupled matrix equations

$$\begin{cases} A_1XB_1 + C_1X^TD_1 = 0, \\ A_2XB_2 + C_2X^TD_2 = 0, \end{cases} \tag{2.11}$$

have a unique reflexive solution $X = 0$.

Lemma 2.4. *The coupled matrix equations* (1.4) *have a unique anti-reflexive solution with respect to the generalized reflection matrix* $P \neq I$ *if and only if* $\mathrm{rank}((Z_2, \mathrm{vec}(M_1, M_2, M_1, M_2))) = \mathrm{rank}(Z_2)$ *and* Z_2 *has a full column rank. In that case, the anti-reflexive solution of* (1.4) *can be expressed by the following form*

$$X = \frac{X_1 - PX_1P}{2} \quad \textit{where}$$
$$\mathrm{vec}(X_1) = (Z_2^TZ_2)^{-1}Z_2^T\mathrm{vec}(M_1, M_2, M_1, M_2), \tag{2.12}$$

and the homogenous coupled matrix equations (2.11) *have a unique anti-reflexive solution* $X = 0$.

If Lemma 2.3 (Lemma 2.4) is applied for finding the reflexive (anti-reflexive) solution of the coupled matrix equations (1.4), we need to take the inverse of the large matrix $Z_1^TZ_1$ ($Z_2^TZ_2$). The above method may turn out to be numerically expensive and are not practical for equations of large systems. Our purpose in this paper is to obtain two iterative methods without any inverse for solving the coupled matrix equations (1.4) over reflexive and anti-reflexive matrices. We extend the idea of the Jacobi and the Gauss-Seidel iterations to solve the coupled matrix equations (1.4) over reflexive and anti-reflexive matrices.

Suppose that $A = M - N$ is a splitting of the matrix A. The Jacobi and Gauss-Seidel procedures for solving the linear system $Ax = b$ are typical members of a large family of iterations that have the form

$$Mx^{(k+1)} = Nx^{(k)} + b, \tag{2.13}$$

with $M = D$, $N = -(L + U)$ for Jacobi and $M = D + L$, $N = -U$ for Gauss-Seidel [32]. Here by extending the Jacobi and the Gauss-Seidel iterations and by applying the hierarchical identification principle [23, 25], we present two iterative methods for solving the coupled matrix equations (1.4) over reflexive and anti-reflexive matrices. These iterative methods are derived as follows:

Algorithm 2.1. To solve (1.4) over reflexive matrix X:

Step 2.1.1. Input matrices $A, C \in \mathbb{R}^{r\times n}$, $B, D \in \mathbb{R}^{n\times s}$ and $M \in \mathbb{R}^{r\times s}$;

Step 2.1.2. Choose arbitrary $X(1) \in \mathbb{R}_r^{n\times n}(P)$ where P is an n-by-n arbitrary generalized reflection matrix and a parameter $\omega \in \mathbb{R}^+$;

Step 2.1.3. Calculate

$$R_i(1) = M_i - A_i X(1) B_i - C_i X(1)^T D_i, \quad i = 1, 2;$$

$$k := 1;$$

Step 2.1.4. If $||R_1(k)|| + ||R_2(k)|| = 0$, then stop; Else go to step 2.1.5;

Step 2.1.5.

$$X(k+1) = X(k)$$
$$+\frac{\omega}{4}\sum_{i=1}^{2}\Big(A_i^T R_i(k) B_i^T + D_i R_i(k)^T C_i + P A_i^T R_i(k) B_i^T P + P D_i R_i(k)^T C_i P\Big);$$
$$R_i(k+1) = M_i - A_i X(k+1) B_i - C_i X(k+1)^T D_i, \quad i = 1, 2;$$

Step 2.1.6. If $||R_1(k+1)|| + ||R_2(k+1)|| = 0$, then stop; Else, let $k := k+1$, go to step 2.1.5.

Algorithm 2.2. To solve (1.4) over anti-reflexive matrix X:

Step 2.2.1. Input matrices $A, C \in \mathbb{R}^{r\times n}$, $B, D \in \mathbb{R}^{n\times s}$ and $M \in \mathbb{R}^{r\times s}$;

Step 2.2.2. Choose arbitrary $X(1) \in \mathbb{R}_a^{n\times n}(P)$ where $P \neq I$ is an n-by-n arbitrary generalized reflection matrix and a parameter $\omega \in \mathbb{R}^+$;

Step 2.2.3. Calculate

$$R_i(1) = M_i - A_i X(1) B_i - C_i X(1)^T D_i, \quad i = 1, 2;$$

$$k := 1;$$

Step 2.2.4. If $||R_1(k)|| + ||R_2(k)|| = 0$, then stop; Else go to step 2.2.5;

Step 2.2.5.

$$X(k+1) = X(k)$$
$$+\frac{\omega}{4}\sum_{i=1}^{2}\Big(A_i^T R_i(k) B_i^T + D_i R_i(k)^T C_i - P A_i^T R_i(k) B_i^T P - P D_i R_i(k)^T C_i P\Big);$$
$$R_i(k+1) = M_i - A_i X(k+1) B_i - C_i X(k+1)^T D_i, \quad i = 1, 2;$$

Step 2.2.6. If $||R_1(k+1)|| + ||R_2(k+1)|| = 0$, then stop; Else, let $k := k+1$, go to step 2.2.5.

Now convergence properties of Algorithms 2.1 and 2.2 are presented.

Theorem 2.1. *If the coupled matrix equations* (1.4) *have a unique reflexive solution X, then iterative solution $X(k)$ given by Algorithm* 2.1 *converges to X for any initial reflexive matrix $X(1)$, if the parameter ω satisfies the inequality*

$$0 < \omega < 2\Big[\sum_{i=1}^{2}\Big(||A_i||^2||B_i||^2 + ||C_i||^2||D_i||^2\Big)\Big]^{-1}. \tag{2.14}$$

Proof. We define the estimation error matrix in the form

$$\varepsilon(k) = X(k) - X, \quad \text{for} \quad k = 1, 2, \ldots \tag{2.15}$$

By applying (2.15), we can get

$$R_i(k) = -A_i\varepsilon(k)B_i - C_i\varepsilon(k)^T D_i, \quad \text{for} \quad i = 1, 2, \tag{2.16}$$

Also it is not difficult to obtain

$$\varepsilon(k+1) = \varepsilon(k)$$
$$-\frac{\omega}{4}\sum_{i=1}^{2}\Big[A_i^T\Big(A_i\varepsilon(k)B_i + C_i\varepsilon(k)^T D_i\Big)B_i^T + D_i\Big(A_i\varepsilon(k)B_i + C_i\varepsilon(k)^T D_i\Big)^T C_i$$
$$+P A_i^T\Big(A_i\varepsilon(k)B_i + C_i\varepsilon(k)^T D_i\Big)B_i^T P + P D_i\Big(A_i\varepsilon(k)B_i + C_i\varepsilon(k)^T D_i\Big)^T C_i P\Big]$$

Now we can write

$$
\begin{aligned}
&||\varepsilon(k+1)||^2 = \mathrm{tr}\Big(\varepsilon(k+1)^T\varepsilon(k+1)\Big)\\
&= ||\varepsilon(k)||^2 - \frac{\omega}{2}\mathrm{tr}\Big(\varepsilon(k)^T\sum_{i=1}^{2}\Big[A_i^T\Big(A_i\varepsilon(k)B_i + C_i\varepsilon(k)^T D_i\Big)B_i^T\\
&\quad + D_i\Big(A_i\varepsilon(k)B_i + C_i\varepsilon(k)^T D_i\Big)^T C_i\Big)\\
&\quad + PA_i^T\Big(A\varepsilon(k)B_i + C_i\varepsilon(k)^T D_i\Big)B_i^T P\\
&\quad + PD_i\Big(A_i\varepsilon(k)B_i + C_i\varepsilon(k)^T D_i\Big)^T C_i P\Big]\Big)\\
&\quad + \frac{\omega^2}{16}\Big|\Big|\sum_{i=1}^{2}\Big[A_i^T\Big(A_i\varepsilon(k)B_i + C_i\varepsilon(k)^T D_i\Big)B_i^T\\
&\quad + D_i\Big(A_i\varepsilon(k)B_i + C_i\varepsilon(k)^T D_i\Big)^T C_i\\
&\quad + PA_i^T\Big(A_i\varepsilon(k)B_i + C_i\varepsilon(k)^T D_i\Big)B_i^T P\\
&\quad + PD_i\Big(A_i\varepsilon(k)B_i + C_i\varepsilon(k)^T D_i\Big)^T C_i P\Big]\Big|\Big|^2\\
&\le ||\varepsilon(k)||^2 - \frac{\omega}{2}\sum_{i=1}^{2}\mathrm{tr}\Big(A_i\varepsilon(k)B_i\Big(A_i\varepsilon(k)B_i + C_i\varepsilon(k)^T D_i\Big)^T\\
&\quad + C_i\varepsilon(k)^T D_i\Big(A_i\varepsilon(k)B_i + C_i\varepsilon(k)^T D_i\Big)^T\\
&\quad + A_iP\varepsilon(k)PB_i\Big(A_i\varepsilon(k)B_i + C_i\varepsilon(k)^T D_i\Big)^T\\
&\quad + C_iP\varepsilon(k)^T PD_i\Big(A_i\varepsilon(k)B_i + C_i\varepsilon(k)^T D_i\Big)^T\Big)\\
&\quad + \frac{\omega^2}{4}\Big\{\Big|\Big|\sum_{i=1}^{2}A_i^T\Big(A_i\varepsilon(k)B_i + C_i\varepsilon(k)^T D_i\Big)B_i^T\Big|\Big|^2\\
&\quad + \Big|\Big|\sum_{i=1}^{2}D_i\Big(A_i\varepsilon(k)B_i + C_i\varepsilon(k)^T D_i\Big)^T C_i\Big|\Big|^2\\
&\quad + \Big|\Big|\sum_{i=1}^{2}PA_i^T\Big(A_i\varepsilon(k)B_i + C_i\varepsilon(k)^T D_i\Big)B_i^T P\Big|\Big|^2\\
&\quad + \Big|\Big|\sum_{i=1}^{2}PD_i\Big(A_i\varepsilon(k)B_i + C_i\varepsilon(k)^T D_i\Big)^T C_i P\Big|\Big|^2\Big\}
\end{aligned}
\tag{2.17}
$$

$$\begin{aligned}
&\leq ||\varepsilon(k)||^2 - \omega \sum_{i=1}^{2} \mathrm{tr}\Big(\Big(A_i\varepsilon(k)B_i + C_i\varepsilon(k)^T D_i\Big) \\
&\quad \times \Big(A_i\varepsilon(k)B_i + C_i\varepsilon(k)^T D_i\Big)^T\Big) \\
&\quad + \frac{\omega^2}{2}\sum_{i=1}^{2}\Big\{||A_i||^2||B_i||^2 + ||D_i||^2||C_i||^2\Big\}||A_i\varepsilon(k)B_i + C_i\varepsilon(k)^T D_i||^2 \\
&\leq ||\varepsilon(k)||^2 - \omega\Big[1 - \frac{\omega}{2}\sum_{i=1}^{2}(||A_i||^2||B_i||^2 + ||D_i||^2||C_i||^2)\Big] \qquad (2.17) \\
&\quad \times \sum_{i=1}^{2}||A_i\varepsilon(k)B_i + C_i\varepsilon(k)^T D_i||^2 \\
&\leq ||\varepsilon(1)||^2 - \omega\Big[1 - \frac{\omega}{2}\sum_{i=1}^{2}(||A_i||^2||B_i||^2 + ||D_i||^2||C_i||^2)\Big] \\
&\quad \times \sum_{t=1}^{k}\sum_{i=1}^{2}||A_i\varepsilon(t)B_i + C_i\varepsilon(t)^T D_i||^2.
\end{aligned}$$

From (2.14) and (2.17), it is not difficult to get

$$\sum_{t=1}^{\infty}\sum_{i=1}^{2}||A_i\varepsilon(t)B_i + C_i\varepsilon(t)^T D_i||^2 < \infty. \qquad (2.18)$$

The necessary condition of the series convergence (2.18) implies that

$$\lim_{t\to\infty}\Big[A_i\varepsilon(t)B_i + C_i\varepsilon(t)^T D_i\Big] = A_i\Big(\lim_{t\to\infty}\varepsilon(t)\Big)B_i + C_i\Big(\lim_{t\to\infty}\varepsilon(t)^T\Big)D_i = 0, \quad \text{for} \quad i = 1, 2.$$

By considering Lemma 2.3, we have

$$\lim_{t\to\infty}\varepsilon(t) = 0.$$

The proof of theorem is completed. □

Similar to the proof of the above theorem, we can prove the following theorem.

Theorem 2.2. *If the coupled matrix equations* (1.4) *have a unique anti-reflexive solution X, then iterative solution $X(k)$ given by Algorithm* 2.2 *converges to*

X for any initial anti-reflexive matrix $X(1)$, if the parameter ω satisfies the inequality

$$0 < \omega < 2\Big[\sum_{i=1}^{2}\Big(||A_i||^2||B_i||^2 + ||C_i||^2||D_i||^2\Big)\Big]^{-1}. \tag{2.19}$$

Remark 2.1. The convergence factor in (2.14) and (2.19) may also be taken as:

$$0 < \omega < 2 \times\Big[\sum_{i=1}^{2}\Big(\lambda_{\max}(A_iA_i^T)\lambda_{\max}(B_iB_i^T) + \lambda_{\max}(C_iC_i^T)\lambda_{\max}(D_iD_i^T)\Big)\Big]^{-1}. \tag{2.20}$$

3 Numerical examples

In this section, we give two examples to illustrate the convergence of Algorithms 2.1 and 2.2, respectively. All the tests are performed by MATLAB.

Example 3.1. As the first example we consider the linear matrix equation $AXB + CX^TD = M$ with

$$A = \begin{pmatrix} -3.7972 & 0.7621 & 0.6154 & 0.4057 & 0.0579 \\ 0 & -3.2532 & 0.7919 & 0.9355 & 0.3529 \\ 0 & 0 & -3.4748 & 0.9169 & 0.8132 \\ 0 & 0 & 0 & -3.6205 & 0.0099 \\ 0 & 0 & 0 & 0 & -3.8103 \end{pmatrix},$$

$$B = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix},$$

$$C = \begin{pmatrix} -4.3881 & 0.2259 & 0.2091 & 0 & 0 \\ 0.4235 & -4.0637 & 0.3798 & 0.7942 & 0 \\ 0.5155 & 0.7604 & -3.4846 & 0.0592 & 0.8744 \\ 0.3340 & 0.5298 & 0.6808 & -4.0259 & 0.0150 \\ 0.4329 & 0.6405 & 0.4611 & 0.0503 & -3.8072 \end{pmatrix},$$

$$D = \begin{pmatrix} 4.7027 & -0.6979 & -0.4966 & -0.6602 & -0.7271 \\ 0 & 4.9568 & -0.8998 & -0.3420 & -0.3093 \\ 0 & 0 & 4.2523 & -0.2897 & -0.8385 \\ 0 & 0 & 0 & 4.1991 & -0.5681 \\ 0 & 0 & 0 & 0 & 4.9883 \end{pmatrix},$$

$$M = \begin{pmatrix} 118.0878 & -23.1868 & -53.8639 & -12.1011 & -52.2842 \\ -6.3715 & 105.8088 & -24.0695 & -58.8678 & 9.0772 \\ -64.4487 & -13.9112 & 93.5506 & 1.4371 & -69.5242 \\ 13.4263 & -77.6180 & 16.4496 & 111.4427 & 16.3479 \\ -36.0898 & -0.8957 & -51.6715 & 21.7130 & 141.3938 \end{pmatrix}.$$

It can be verified that this matrix equation is consistent over reflexive matrices and has the reflexive solution

$$X^* = \begin{pmatrix} -5.5945 & 0 & 3.2309 & 0 & 2.1158 \\ 0 & -4.5064 & 0 & 3.8709 & 0 \\ 2.0000 & 0 & -4.9497 & 0 & 3.6263 \\ 0 & 2.0000 & 0 & -5.2410 & 0 \\ 2.0000 & 0 & 2.0000 & 0 & -5.6207 \end{pmatrix} \mathbb{R}_r^{5\times5}(P),$$

with

$$P = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

Choose arbitrary initial iterative matrix $X(1) = 0$. By Algorithm 2.1, we obtain the sequence $X(k)$. In Figure 1, we report the obtained results with several values of ω where

$$\delta(k) = \log_{10} \frac{||X(k) - X^*||}{||X^*||} \quad \text{and} \quad r(k) = \log_{10} ||M - AX(k)B - CX(k)^T D||.$$

It can be observed from Figure 1 that Algorithm 2.1 is effective. The effect of changing the convergence factor ω is illustrated in Figure 1. We see that the larger the convergence factor ω is, the faster the convergence the algorithm.

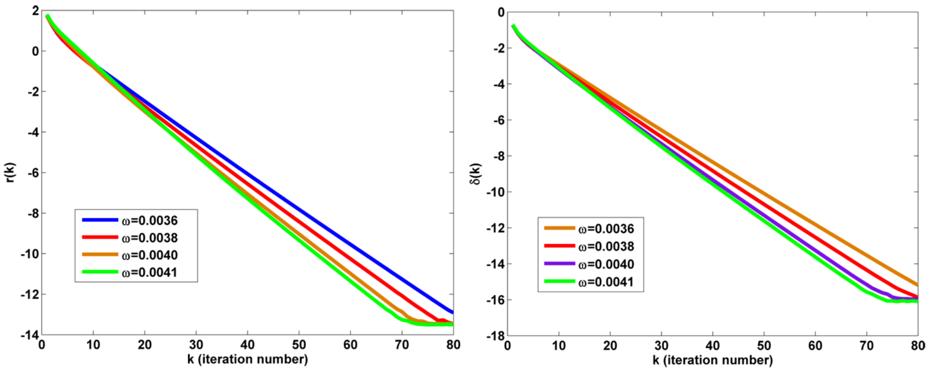

Figure 1 – The results of Example 3.1.

Example 3.2. Consider a pair of matrix equations in the form of (1.2) with the following parameters:

$$A_1 = \begin{pmatrix} -3.7972 & 1.5242 & 1.2309 & 0.8114 & 0.1158 \\ 0 & -3.2532 & 1.5839 & 1.8709 & 0.7057 \\ 0 & 0 & -3.4748 & 1.8338 & 1.6263 \\ 0 & 0 & 0 & -3.6205 & 0.0197 \\ 0 & 0 & 0 & 0 & -3.8103 \end{pmatrix},$$

$$B_1 = \begin{pmatrix} 3.6756 & 0 & 0.6085 & 0.0576 & 0.0841 \\ 0 & 3.4508 & 0 & 0.3676 & 0.4544 \\ 0 & 0 & 3.2324 & 0 & 0.4418 \\ 0 & 0 & 0 & 3.0784 & 0 \\ 0 & 0 & 0 & 0 & 3.9943 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 5.5536 & 0.2259 & 0.2091 & 0 & 0 \\ 0.4235 & 5.2233 & 0.3798 & 0.7942 & 0 \\ 0.5155 & 0.7604 & 5.0513 & 0.0592 & 0.8744 \\ 0.3340 & 0.5298 & 0.6808 & 5.2317 & 0.0150 \\ 0.4329 & 0.6405 & 0.4611 & 0.0503 & 5.3431 \end{pmatrix},$$

$$B_2 = \begin{pmatrix} -5.7027 & -0.6979 & -0.4966 & -0.6602 & -0.7271 \\ 0 & -5.9568 & -0.8998 & -0.3420 & -0.3093 \\ 0 & 0 & -5.2523 & -0.2897 & -0.8385 \\ 0 & 0 & 0 & -5.1991 & -0.5681 \\ 0 & 0 & 0 & 0 & -5.9883 \end{pmatrix},$$

$$C_1 = \begin{pmatrix} 17.1697 & -56.8568 & 33.5482 & -25.9479 & 24.4829 \\ -10.1615 & 15.8018 & -43.9289 & 33.5031 & -33.0153 \\ 13.4809 & -12.7575 & 14.0869 & -51.7608 & 15.1875 \\ -26.6155 & 0.1361 & -27.8120 & -0.2810 & -33.2835 \\ 0 & -26.2971 & 0 & -26.2605 & -3.4625 \end{pmatrix},$$

and

$$C_2 = \begin{pmatrix} -2.5771 & -169.8176 & -31.9603 & -121.1339 & -26.8271 \\ -68.6326 & -25.6598 & -158.7006 & -36.3862 & -152.1132 \\ -9.3473 & -87.2429 & -35.0790 & -174.8441 & -43.6301 \\ -65.7124 & -26.3748 & -77.8285 & -38.9935 & -98.0133 \\ -7.8789 & -83.1321 & -31.0110 & -83.9174 & -30.2760 \end{pmatrix}.$$

We can verify the pair of matrix equations in the form of (1.2) are consistent over anti-reflexive matrix X and have the anti-reflexive solution

$$X^* = \begin{pmatrix} 0 & 5.0484 & 0 & 3.6228 & 0 \\ 2.0000 & 0 & 5.1677 & 0 & 3.4115 \\ 0 & 2.0000 & 0 & 5.6676 & 0 \\ 2.0000 & 0 & 2.0000 & 0 & 2.0394 \\ 0 & 2.0000 & 0 & 2.0000 & 0 \end{pmatrix} \mathbb{R}_a^{5\times 5}(P).$$

Taking $X(1) = 0$, we apply Algorithm 2.2 to compute $X(k)$. The effect of changing the convergence factor ω is illustrated in Figure 2 where

$$\delta(k) = \log_{10} \frac{||X(k) - X^*||}{||X^*||} \quad \text{and} \quad r(k) = \log_{10} \sum_{i=1}^{2} ||C_i - A_i X(k) B_i||.$$

Obviously both $\delta(k)$ and $r(k)$ decrease, and converge to zero as k increases.

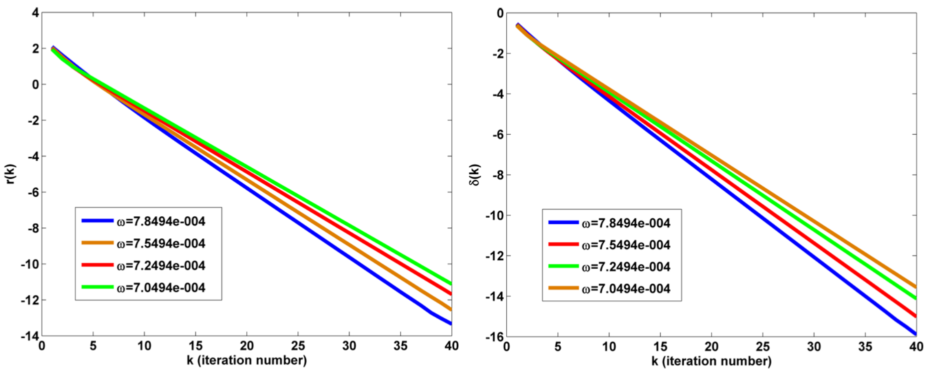

Figure 2 – The results of Example 3.2.

4 Concluding remarks

In this paper, we have considered the coupled matrix equations (1.4) over reflexive and anti-reflexive matrices. First Algorithms 2.1 and 2.2 were introduced for finding reflexive and anti-reflexive solutions of (1.4). Second the convergence theorems of the iterative algorithms were presented. The experiments are encouraging and seem to indicate that Algorithms 2.1 and 2.2 work well for numerical examples. It is interesting to develop the introduced algorithms for solving other linear matrix equations. We leave it as a topic for further research.

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