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Numerical calculations of Hölder exponents for the Weierstrass functions with (min, +)-wavelets

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ABSTRACT. One reminds for all function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ the so-called (min, +)-wavelets which are lower and upper hulls build from (min, +) analysis [12, 13]. One shows that this analysis can be applied numerically to the Weierstrass and Weierstrass-Mandelbrot functions, and that one recovers their theoretical Hölder exponents and fractal dimensions.

Keywords: (min, +)-wavelets , Hölder exponents, Weierstrass functions.

1 INTRODUCTION

Genesis of wavelets theory started in 1946 with D. Gabor [9], who introduced the Windowed Fourier Transform (WFT)

$$\hat{f}(\omega, \tau) = \int_{-\infty}^{\infty} \exp(-\imath\omega t) f(t) g(t-\tau) dt \tag{1.1}$$

for the *local* spectral analysis of radar signals. The localization is reached due to fast decaying window function $g(x) \overset{|x|\to\infty}{\to} 0$. Even if WFT exhibits many powerful and practical features, there are some defects compared to Fourier Transform. The transform (1.1) can not resolve efficiently wavelengths longer than the window $g(x)$ width. Conversely, for signal with high frequencies, short decomposition needs a broad window with a large number of periods. Thus, signal reconstruction in this case adds a large number of terms with comparable amplitudes and hence becomes numerically unstable. Finally, one needs a scheme with a wide window for low frequency signals and a narrow window for high frequency ones. Such a scheme, was independently suggested as a tool for geophysical studies by several authors at the beginning of 1980s [19, 23]. Wavelets Theory (WT) word was introduced in analysis by J. Morlet [17, 11]. It is considered nowadays as a preferable alternative to the Fourier analysis, used where and

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when the signals are random and comprised of fluctuations of different scales, such as in turbulence phenomena [3]. WT has been immediately followed by several applications in science and engineering, such as signal processing and detection, fractals, self-similar objects, self-similar random processes, like turbulence and Brownian motion [2]. WT was then mathematically formalized by Grossman & Morlet [11], Goupillaud et al. [10], Daubechies [6] and some other authors. Practically, WT is a separate convolution of the signal in question with a family of functions obtained from some basic one, the basic wavelet called mother wavelet or analysing function, by shifts τ and dilatations a:

$$W_{\psi}(\tau, a)f = \int_{-\infty}^{\infty} \frac{1}{\sqrt{a}} \bar{\psi}\left(\frac{t-\tau}{a}\right) f(t)dt. \tag{1.2}$$

An interesting point of view is to consider WT as a realization of a function decomposition with respect to the representation of the affine group [2]: $x \mapsto ax + b$.

We refer the reader to useful and interesting articles [1, 8, 20, 4, 7, 5] and both theoretical and practical book [2].

Nevertheless, wavelets decompositions are limited by their linear features. This present article aims to apply for pathological functions such as Weierstrass functions, a non-linear transform, called (min, +) transform, which has been already defined within (min, +) analysis [12, 13, 18]. This one consists to replace in the scalar product definition of two real-valued functions f and g defined on a domain X, the real number field $(\mathbb{R}, +, \times)$ with the (min, +) dioid $(\mathbb{R} \cup \{+\infty\}, \min, +)$. The classical scalar product $\langle f, g\rangle = \int_{x\in X} f(x)g(x)dx$ becomes then the (min, +) scalar product [12]:

$$\langle f, g\rangle_{(\min,+)} = \inf_{x\in X}\{f(x) + g(x)\}.$$

The demonstration that it is a scalar product within the (min, +) dioid is straightforward and easy excepted for its linearity.

One has to show that $\langle f, g\rangle_{\min +}$ is distributive according to min, which means

$$\langle f, \min\{g_1, g_2\}\rangle_{(\min,+)} = \min\{\langle f, g_1\rangle_{(\min,+)}, \langle f, g_2\rangle_{(\min,+)}\},$$

and linear according to the addition of a scalar λ : $\langle f(x), \lambda + g(x)\rangle_{(\min,+)} = \lambda + \langle f, g\rangle_{(\min,+)}$. The linearity is obvious since $\inf_{x\in X}\{f(x)+\lambda+g(x)\} = \lambda+\inf_{x\in X}\{f(x)+g(x)\}$. Distributivity is obtained in two steps. One has first to prove this equality with mean of two inequalities. We start first with the simple relations:

$$\langle f, g_1\rangle_{(\min,+)} \leqslant f(x) + g_1(x), \text{ and } \langle f, g_2\rangle_{(\min,+)} \leqslant f(x) + g_2(x), \ \forall x.$$

This gives $\min\{\langle f, g_1\rangle_{(\min,+)}, \langle f, g_2\rangle_{(\min,+)}\} \leqslant \min\{f(x)+g_1(x), f(x)+g_2(x)\} \ \forall x$. And since

$$\min\{f(x) + g_1(x), f(x) + g_2(x)\} = f(x) + \min\{g_1(x), g_2(x)\},$$

one has $\min\{\langle f, g_1\rangle_{(\min,+)}, \langle f, g_2\rangle_{(\min,+)}\} \leqslant f(x) + \min\{g_1(x), g_2(x)\} \ \forall x$, which yields to the inequality

$$\min\{\langle f, g_1\rangle_{(\min,+)}, \langle f, g_2\rangle_{(\min,+)}\} \leqslant \langle f, \min\{g_1, g_2\}\rangle_{(\min,+)}. \tag{1.3}$$

In a second step, one can write

$$\langle f, \min\{g_1, g_2\}\rangle_{(\min,+)} \leqslant f(x) + \min\{g_1(x), g_2(x)\} \leqslant f(x) + g_1(x)\ \forall x,$$

which becomes

$$\langle f, \min\{g_1, g_2\}\rangle_{(\min,+)} \leqslant \langle f, g_1\rangle_{(\min,+)}. \tag{1.4}$$

and in the same manner

$$\langle f, \min\{g_1, g_2\}\rangle_{(\min,+)} \leqslant f(x) + \min\{g_1(x), g_2(x)\} \leqslant f(x) + g_2(x)\ \forall x,$$

giving now

$$\langle f, \min\{g_1, g_2\}\rangle_{(\min,+)} \leqslant \langle f, g_2\rangle_{(\min,+)}, \tag{1.5}$$

and then from (1.4) and (1.5)

$$\langle f, \min\{g_1, g_2\}\rangle_{(\min,+)} \leqslant \min\{\langle f, g_1\rangle_{(\min,+)}, \langle f, g_2\rangle_{(\min,+)}\}. \tag{1.6}$$

From relations (1.3) and (1.6), one deduces finally the equality and thus the distributivity.

With this (min, +) scalar product, one obtains a distribution-like theory : the operator is linear and continuous according the dioid structure $(\mathbb{R} \cup \{+\infty\}, \min, +)$, non-linear and continuous according to the classical structure $(\mathbb{R}, +, \times)$. The non-linear distribution $\delta_{\min}(\mathbf{x})$ defined as

$$\delta_{(\min,+)}(\mathbf{x}) = \{0 \text{ if } \mathbf{x} = \mathbf{0}, +\infty \text{ else}\}$$

is similar in (min, +) analysis to the classical Dirac distribution. Then, one has

$$\langle \delta_{(\min,+)}, f\rangle_{(\min,+)} = \min_{x\in X}\{\delta_{(\min,+)}(x) + f(x)\} = \min\{f(0), +\infty\} = f(0).$$

In (min, +) analysis, the Legendre-Fenchel transform which permits to get Hamiltonian from Lagrangian and which has an important role in physics is similar in (min, +) analysis to the Fourier transform in the classical one [18].

In this article, we explore how (min, +)-wavelets decomposition and reconstruction could be an interesting signal processing tool, since (min, +) transforms can be applied to a larger class of functions than the functions treated with classical wavelet transforms, especially to lower semi-continuous functions [12], such as $\big(x \mapsto g(x) \cdot \text{Floor(x)}\big)$ for instance, where g is a continuous function.

In this paper, one focus on Weierstrass and Weierstrass-Mandelbrot functions which are classical examples of functions continuous everywhere but differentiable nowhere [22].

One introduces in the following Section 2, the (min, +)-wavelets decomposition and reconstruction of a signal with mean of (min, +) scalar product. in Section 3, we show how the (min, +)-wavelets allow a characterisation of Hölder functions. in Section 4, we apply these results to numerical calculations of Hölder exponents of Weierstrass-like functions and compare them to the theoretical values [21]. This permits to deduce immediately their fractal dimensions.

2 (min, +)-WAVELETS

The usual wavelet transform of a function f from $\mathbb{R}^n$ to $\mathbb{R}$ is a linear transform defined for all scales $a \in \mathbb{R}^+$ and points $b \in \mathbb{R}^n$, which can be computed according to the equation (1.2):

$$T_f(a, b) = a^{-n} \int_{-\infty}^{+\infty} f(x)\Psi\Big(\frac{x-b}{a}\Big)dx,$$

where Ψ is a function called mother wavelet or analysing function. It has to be zero average and exhibiting oscillations until a certain order p. This can be written as

$$\int_{-\infty}^{\infty} x^m \Psi(x)dx = 0, \ \forall m, \ 1 \leq m \leq p. \tag{2.7}$$

In (min, +) analysis, a set of non-linear transforms has been introduced for lower semi-continuous functions [12, 18], the so-called (min, +)-wavelets transforms which are defined for a function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ and for all $a \in \mathbb{R}^+$ and $b \in \mathbb{R}^n$ such as:

$$T_f^-(a, b) = \inf_{x \in \mathbb{R}^n} \Big\{ f(x) + h\Big(\frac{x-b}{a}\Big)\Big\}, \tag{2.8}$$

where h is a basis analysing function (upper semi-continuous and inf-compact verifying $h(0) = 0$), like the following functions:

$$h_\alpha(x) = \frac{1}{\alpha}|x|^\alpha \ \text{ with } \ \alpha > 1 \ \text{ and } \ h_\infty(x) = \{0 \ \text{ if } \ |x| < 1, +\infty \ \text{ else}\}.$$

Since $T_f^-(a, x) \leq f(x)$ for all $a > 0$, $T_f^-(a, x)$ is a lower hull of $f(x)$. For any lower bounded and lower semi-continuous function, one has a reconstruction formula like in the linear wavelets theory [2]:

$$f(x) = \sup_{a \in \mathbb{R}^+, b \in \mathbb{R}^n} \Big\{ T_f^-(a, b) - h\Big(\frac{x-b}{a}\Big)\Big\}, \tag{2.9}$$

which can be simplified within the (min, +) theory in

$$f(x) = \sup_{a \in \mathbb{R}^+} T_f^-(a, x). \tag{2.10}$$

The (min, +)-wavelets analysis will be based on simultaneous analysis of lower hulls $T_f^-(a, b)$, and upper hulls of f represented by $T_f^+(a, b)$ defined by:

$$T_f^+(a, b) = \sup_{x \in \mathbb{R}^n} \Big\{ f(x) - h\Big(\frac{x-b}{a}\Big)\Big\}. \tag{2.11}$$

For the upper hulls $T_f^+(a, b)$, we have a reconstruction formula which is symmetric to lower hulls $T_f^-(a, b)$ (2.9, 2.8):

$$f(x) = \inf_{a \in \mathbb{R}^+, b \in \mathbb{R}^n} \Big\{ T_f^+(a, b) + h\Big(\frac{x-b}{a}\Big)\Big\}, \tag{2.12}$$

which simplifies as well as:

$$f(x) = \inf_{a \in \mathbb{R}^+} T_f^+(a, x). \tag{2.13}$$

For each analysing function h, one has [13]:

$$T_f^-(a, x) \leq f_*(x) \leq f(x) \leq f^*(x) \leq T_f^+(a, x), \tag{2.14}$$

because $T_f^-(a, x)$ (respectively $T_f^+(a, x)$) are functions decreasing with scales (respectively increasing) and converging to $f_*(x)$ (respectively $f^*(x)$), the lower semi-continuous closure of f (respectively upper semi-continuous closure) when the scale tends to 0.

Remark 1. We use the word "wavelet" by analogy with linear wavelets since the decomposition and reconstruction formula are very similar and since one just replaces the usual real number field $(\mathbb{R}, +, \times)$ with the (min, +) dioid $(\mathbb{R} \cup \{+\infty\}, \min, +)$. Another name can be (min, +) pen or (min, +) hulls.

Remark 2. The shift and scale parameters have the same meaning as in Linear Wavelet Theory: for high frequencies, one needs small scales, and the inverse as well. But the relation between them is not simply proportionally inverse as in linear theory, because it depends on the choice of analyzing function h_α, and this introduces non-linear dependency between scale and frequency. This leads to a relation such as $\nu = \gamma(a, \alpha)$, where ν, and a are respectively the frequency and the scale, and γ a non-linear function decreasing with a.

Definition 1. (min, +)-*wavelet* *is defined as the couple* $\{T_f^-(a, x), T_f^+(a, x)\}$. *For all* $\mathbb{R}^+$, *the* ***a-oscillation*** *of f is defined:*

$$\Delta T_f(a, x) = T_f^+(a, x) - T_f^-(a, x). \tag{2.15}$$

In the case of analysing function h_∞, one has

$$T_f^+(a, x) = \sup_{|x-y| \leq a} f(y), \;\; T_f^-(a, x) = \inf_{|x-y| \leq a} f(y)$$

and $\Delta T_f(a, x) = \sup_{|x-y| \leq a} f(y) - \inf_{|x-z| \leq a} f(z)$ corresponds to the a-oscillation defined in one dimension by Tricot [21]: $\mathrm{osc}_a f(x) = \sup_{y,z \in [x-a, x+a]} [f(y) - f(z)]$.

3 CHARACTERISATION OF HÖLDERIAN FUNCTIONS WITH (min, +)-WAVELETS ANALYSIS

The calculations of oscillations according to the analysing function and the scale will permit to study the global and local regularity of a function.

First, let's start with the case of global regularity of a Hölderian function for which it exists H $(0 < H \leq 1)$ and a constant K such as

$$|f(x) - f(y)| \leq K|x - y|^H \quad \forall\, x, y \in \mathbb{R}^n. \tag{3.16}$$

It is a sufficient but not necessary condition for a function to be continuous. In the case of fractal function, K is related to its fractal dimension.

Theorem 1. *The function f is Hölderian with exponent H, $0 < H \leq 1$, if and only if it exists a constant C such as for all a, one of the following condition is verified:*

$$\Delta T_f(a,x) \leq Ca^H \qquad \text{if } h = h_\infty, \tag{3.17}$$

$$\Delta T_f(a,x) \leq Ca^{\frac{\alpha H}{\alpha - H}} \qquad \text{if } h = h_\alpha \text{ and } \alpha > 1. \tag{3.18}$$

Demonstration:

- Demonstration for the case of analysing function h_∞ is classic [21]: if f verifies (3.17), let's consider some x and y in $\mathbb{R}^n$. One can assume that $f(x) \geq f(y)$. Then, one has

$$a = |x-y|, \sup_{|x-z|\leq a} f(z) \geq f(x) \geq f(y) \geq \inf_{|x-z|\leq a} f(z),$$

this yields to

$$|f(x) - f(y)| \leq \Delta T_f(a,x) \leq Ka^H \leq K|x-y|^H.$$

Conversely, let's assume that $|f(x) - f(y)| \leq K|x-y|^H$ for all y. Let y_1 such as $f(y_1) = \sup_{|x-z|\leq a} f(z)$ and y_2 such as $f(y_2) = \inf_{|x-z|\leq a} f(z)$. One has then

$$\Delta T_f(a,x) = f(y_1) - f(y_2) = f(y_1) - f(y) + f(y) - f(y_2),$$

which yields to

$$\Delta T_f(a,x) \leq |f(y_1) - f(y)| + |f(y) - f(y_2)| \leq 2Ka^H.$$

In the case of analysing functions h_α, $\alpha > 1$, let's suppose first that f verifies (3.18). We consider x and y in $\mathbb{R}^n$ with $f(x) \geq f(y)$. Reconstruction equation (2.12) of $f(x)$ can be written as

$$f(x) = \inf_{a\in\mathbb{R}^+, b\in\mathbb{R}^n} \left\{ T_f^+(a,b) + h\Big(\frac{x-b}{a}\Big) \right\},$$

and the equation of reconstuction for $f(y)$ (2.8)

$$f(y) = \sup_{a\in\mathbb{R}^+} T_f^-(a,y).$$

One deduces

$$f(x) - f(y) = \inf_{a\in\mathbb{R}^+, b\in\mathbb{R}^n} \left\{ (T_f^+(a,b) + h\Big(\frac{x-b}{a}\Big) - T_f^-(a,y) \right\},$$

thus

$$f(x) - f(y) \le \inf_{a\in\mathbb{R}^+} \left\{T_f^+(a, y) + h\Big(\frac{x-y}{a}\Big) - T_f^-(a, y)\right\}$$
$$\le \inf_{a\in\mathbb{R}^+} \left\{Ca^{\frac{\alpha H}{\alpha - H}} + h\Big(\frac{x-y}{a}\Big)\right\}$$

and the optimisation on the scale a implies that it exists K such as $|f(x) - f(y)| \le K|x-y|^H$.

Conversely, let's assume that $|f(x) - f(y)| \le K|x-y|^H$ for all x and y. Using (2.8) and (2.11), one has

$$\Delta T_f(a,b) = \sup_{x,y}\left\{f(x) - f(y) - h\Big(\frac{x-b}{a}\Big) - h\Big(\frac{y-b}{a}\Big)\right\}.$$

We deduce that

$$\Delta T_f(a,b) \le \sup_{x,y}\left\{K|x-y|^H - h\Big(\frac{x-b}{a}\Big) - h\Big(\frac{y-b}{a}\Big)\right\},$$

whose optimisation gives (3.18). □

Let's consider now the case of local irregularity at x_0 where the function is Hölderian: it exists H $(0 < H \le 1)$ and a constant K such as

$$|f(x) - f(x_0)| \le K|x - x_0|^H \quad \forall x \in \mathbb{R}^n. \tag{3.19}$$

Theorem 2. *The function f is Hölderian at point x_0, with exponent H, $0 < H \le 1$, if and only if it exists a constant C such as for all a, one has one of the following conditions:*

$$\Delta T_f(a,x) \le C(a^H + |x-x_0|^H), \ \text{if } h = h_\infty. \tag{3.20}$$

$$\Delta T_f(a,x) \le C(a^{\frac{\alpha H}{\alpha-H}} + |x-x_0|^H), \ \text{if } h = h_\alpha \text{ and } \alpha > H. \tag{3.21}$$

Demonstration:

- In the case of the analysing function h_∞, if f verifies (3.20) for all x, let $a = |x - x_0|$. One has then inequations $\sup_{|x-x_0|\le a} f(z) \ge f(x) \ge f(x_0) \ge \inf_{|x-z|\le a} f(z)$ or $\sup_{|x-x_0|\le a} f(z) \ge f(x_0) \ge f(x) \ge \inf_{|x-z|\le a} f(z)$. In both cases one gets

$$|f(x) - f(x_0)| \le \Delta T_f(a,x) \le 2C|x-x_0|^H.$$

  Conversely, let suppose $|f(x) - f(x_0)| \le K|x-x_0|^H$ for all x, and y_1 such as $f(y_1) = \sup_{|x-z|\le a} f(z)$ and y_2 such as $f(y_2) = \inf_{|x-z|\le a} f(z)$; One has then

$$\begin{aligned}\Delta T_f(a,x) &= f(y_1) - f(y_2)\\ &= f(y_1) - f(x_0) + f(x_0) - f(y_2)\\ &\le K(|y_1 - x_0|^H + |y_2 - x_0|^H),\end{aligned}$$

that means

$$\Delta T_f(a,x) \leq K(|y_1 - x|^H + |x - x_0|^H + |y_2 - x|^H + |x - x_0|^H),$$

this yields to

$$\Delta T_f(a,x) \leq 2K(a^H + |x - x_0|^H).$$

- For analysing functions h_α, $\alpha > 1$, we assume first that f verifies (3.21). Let's consider x in $\mathbb{R}^n$ and the two cases, $f(x) \geq f(x_0)$, and $f(x) \leq f(x_0)$. In the first case, one uses the reconstruction equations

$$f(x) = \inf_{a\in\mathbb{R}^+, b\in\mathbb{R}^n} \Big\{ T_f^+(a,b) + h\Big(\frac{x-b}{a}\Big) \Big\}, \tag{3.22}$$

and

$$f(x_0) = \sup_{a\in\mathbb{R}^+} T_f^-(a, x_0).$$

For the second case, one uses a symmetric reconstruction method. This yields to

$$f(x) = \inf_{a\in\mathbb{R}^+, b\in\mathbb{R}^n} \Big\{ T_f^+(a,b) + h\Big(\frac{x-b}{a}\Big) - T_f^-(a,x_0) \Big\}, \tag{3.23}$$

which gives

$$|f(x) - f(x_0)| \leq \inf_{a\in\mathbb{R}^+} \Big\{ T_f^+(a,x_0) + h\Big(\frac{x-x_0}{a}\Big) - T_f^-(a,x_0) \Big\},$$

$$|f(x) - f(x_0)| \leq \inf_{a\in\mathbb{R}^+} \Big\{ Ca^{\frac{\alpha H}{\alpha - H}} + C|x - x_0|^H + h\Big(\frac{x-x_0}{a}\Big) \Big\}.$$

This implies that it exists a constant K such as

$$|f(x) - f(x_0)| \leq K|x - x_0|^H.$$

Conversely, let suppose that $|f(x) - f(x_0)| \leq K|x - x_0|^H$ for all x. With mean of (2.8) and (2.11), one has

$$\Delta T_f(a,b) = \sup_{x,y} \Big\{ f(x) - f(y) - h\Big(\frac{x-b}{a}\Big) \Big\}.$$

Since

$$f(x) - f(y) = f(x) - f(x_0) + f(x_0) - f(y),$$

one deduces

$$\Delta T_f(a,b) \leq \sup_{x,y} \Big\{ K|x - x_0|^H + K|y - x_0|^H - h\Big(\frac{x-b}{a}\Big) - h\Big(\frac{y-b}{a}\Big) \Big\},$$

which yields to

$$\begin{aligned} \Delta T_f(a,b) &\leq 2\sup_x \Big\{ K|x - x_0|^H - h\Big(\frac{x-b}{a}\Big) \Big\} \\ &\leq 2\sup_x \Big\{ K|x - b|^H + K|b - x_0|^H - h\Big(\frac{x-b}{a}\Big) \Big\}. \end{aligned}$$

whose optimisation gives (3.21). □

One gets here a reciprocal relation which is not fully obtained with linear wavelets [15].

4 HÖLDER EXPONENTS CALCULATION FOR WEIERSTRASS FUNCTIONS

We exhibit an application of the (min, +)-wavelets analysis to the Weierstrass function in order to compute its Hölder exponent H and its fractal dimension D. This one is a typical example of function continuous everywhere but nowhere differentiable [22]. One consider the general form of Weierstrass functions on $[0, 2\pi]$

$$W(t) = \sum_{m=0}^{\infty} (\omega^{-H})^m \cos(\omega^m t + \varphi_m), \tag{4.24}$$

with $\omega^H > 1$ and $\{\varphi_m\}_{m\geq 0}$, constant or randomly distributed variable.

Those functions are Hölderian (and anti-Hölderian) with coefficient H and fractal dimension [16, 14]:

$$D = 2 + \frac{\log \omega^{-H}}{\log \omega} = 2 - H. \tag{4.25}$$

One calculates for all scales $s = k \cdot \text{scale}_{\min}$ with k an integer from 1 to 10 and $\text{scale}_{\min} = 10^{-2}$, the following function of scale for h_2 and h_∞

$$\overline{\Delta T}_f(s) = \int_{\mathcal{T}} \Delta T_f(s, t) dt.$$

For the Weierstrass function, the upper bound of the sum is replaced with a finite constant $M = 15$ which is sufficient for our tests. Thus, the truncated Weierstrass function can be written as

$$W(t) = \sum_{m=0}^{15} 2^{-\frac{m}{2}} \cos(2^m t + \varphi_m),$$

and is represented with its (min, +)-wavelets decomposition on Figure 1 for $\varphi_m = 0$.

We made numerical calculations to determine Hölder exponents. The fractal dimension is then directly given by equation (4.25). Computations were performed for $H \in \{\frac{1}{2}, \frac{1}{4}\}$, $\omega = 2$, for analysing functions h_∞ and h_2, for both cases of zero and random φ_m with a uniform probability measure in $[0, 2\pi]$.

The slope of the linear part of curves for small scales gives the value of Hölder exponent.

Hölder exponents calculations for random phase Weierstrass functions are summarized on Tables 1 and 2. According to equations (3.17, 3.18), the slopes and Hölder exponents are very close to the theoretical value $H = \frac{1}{2}$ for h_∞ and $\frac{2H}{2-H} = \frac{2}{3}$ for h_2 [21, 14]. The fractal dimension is then given by $D = 2 - H = \frac{3}{2}$. Same result for $H = \frac{1}{4}$ with a slope of $H = \frac{1}{4}$ for h_∞ and $\frac{2}{7}$ for h_2. They confirm that the Hölder exponents and fractal dimensions of Weierstrass function remain the same in the case of a uniform random phase [16, 14].

The Weierstrass function $W(t) = \sum\limits_{m=0}^{\infty} (\omega^{-H})^m \cos(\omega^m t)$, verifies

$$W(\omega t) = \omega^H \{W(t) - \cos(t)\}, \tag{4.26}$$

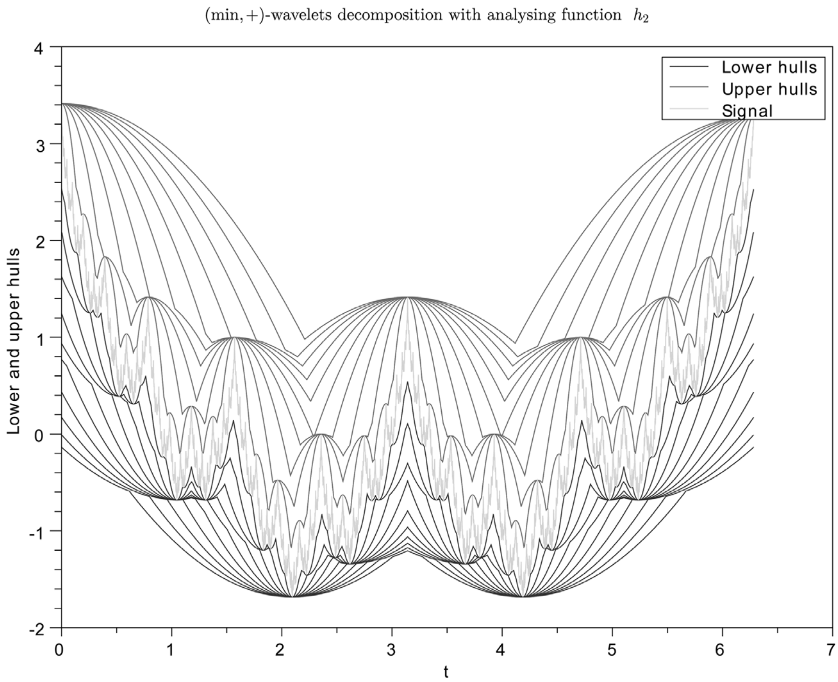

Figure 1: (min, +)-wavelets decomposition for the Weierstrass truncated function $W(t) = \sum_{m=0}^{15} 2^{-\frac{m}{2}} \cos(2^m t)$ with the analysing function h_2 for scales $k \cdot 10^{-1}$ with k from 1 to 10.

Table 1: Numerical results for random phase Weierstrass function with $\omega = 2$ and (min, +)-wavelets decomposition performed with h_∞.

Theoretical Hölder exponent H	$\frac{1}{4} = 0.250$	$\frac{1}{2} = 0.500$
Theoretical slope	$\frac{1}{4} = 0.250$	$\frac{1}{2} = 0.500$
Numerical Hölder exponent H	0.253	0.507
Numerical slope	0.253	0.507
Slope relative error (%)	1.2	1.4

Table 2: Numerical results for random phase Weierstrass function with $\omega = 2$ and (min, +)-wavelets decomposition performed with h_2.

Theoretical Hölder exponent H	$\frac{1}{4} = 0.250$	$\frac{1}{2} = 0.500$
Theoretical slope	$\frac{2}{7} \simeq 0.286$	$\frac{2}{3} \simeq 0.667$
Numerical Hölder exponent H	0.246	0.497
Numerical slope	0.280	0.661
Slope relative error (%)	2.0	0.9

which is not a scaling invariance property [21]. In order to circumvent that, one can build the Weierstrass-Mandelbrot function

$$WM(t) = \sum_{m=-\infty}^{\infty} (\omega^{-H})^m \{1 - \cos(\omega^m t)\}. \tag{4.27}$$

Since

$$WM(\omega t) = \sum_{m=-\infty}^{\infty} (\omega^{-H})^m \{1 - \cos(\omega^{m+1} t)\},$$

the change of variable $m' = m+1$ leads to $WM(\omega t) = \omega^H WM(t)$, which has scaling invariance property. Hölder exponents calculations for a truncated version of this function are exhibited on Figures 2 and 3, confirming thus the validity of (min, +)-wavelets decomposition for its Hölder exponents computation.

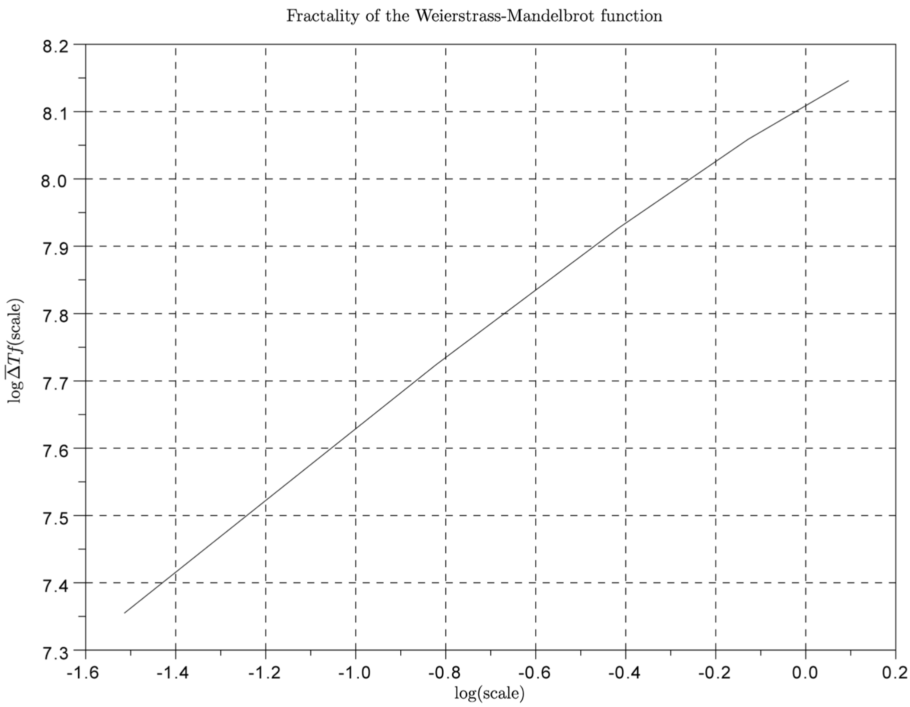

Figure 2: Logarithm of $\overline{\Delta T}_{WM}(s)$ according to scale logarithm with h_∞ decomposition of the Weierstrass-Mandelbrot function, $H = \frac{1}{2}$, $\omega = 2$. The slope is obtained with mean of linear regression and its value is 0.496. The theoretical value is $\frac{1}{2}$. That is a relative error of 0.5%.

5 CONCLUSION AND PERSPECTIVES

We have presented in this article a promising tool to determine numerically Hölder exponents of Weierstrass-like functions which are exhibiting fractal properties. It is based on (min, +) analysis

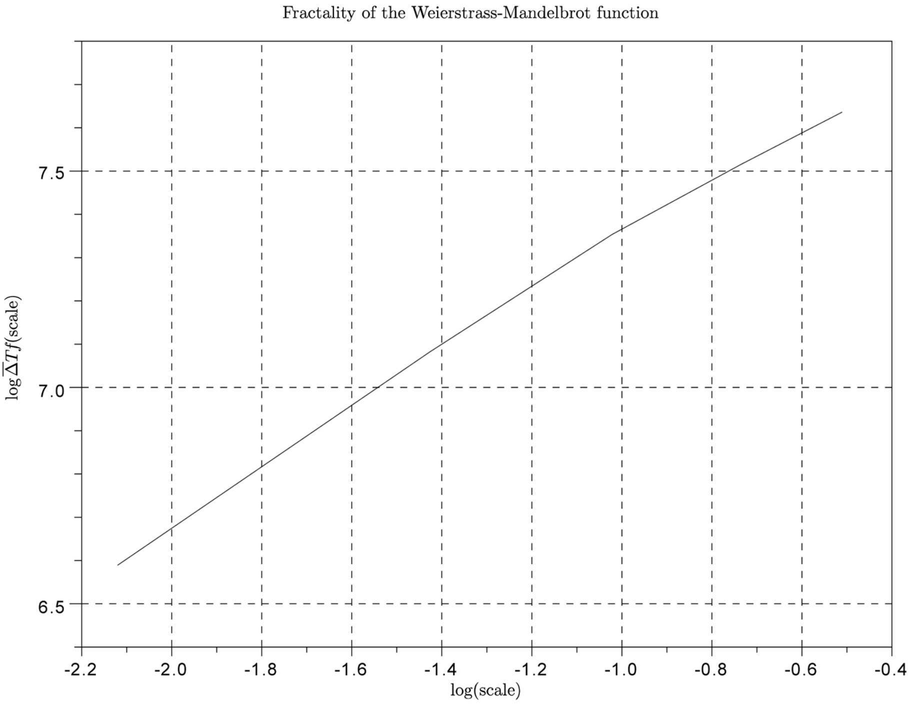

Figure 3: Logarithm of $\overline{\Delta T}_{WM}(s)$ according to scale logarithm with h_2 decomposition of the Weierstrass-Mandelbrot function, $H = \frac{1}{2}$, $\omega = 2$. The slope is obtained with mean of linear regression and its value is 0.655. The theoretical value is $\frac{2}{3}$. That is a relative error of 1.8%.

and proposes a signal decomposition using the (min, +) scalar product. By analogy with Linear Wavelet Theory, this permits to define (min, +)-wavelets , which are lower and upper hulls of a signal at a certain scale.

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RESUMO. Lembrando que para todas as funções $f : \mathbb{R}^n \to \overline{\mathbb{R}}$, as chamadas (min, +)-wavelets são construções do fecho inferior e superior, vindos da análise (min, +) [12, 13]. Mostra-se que esta análise pode ser aplicada numericamente às funções de Weierstrass e Weierstrass-Mandelbrot, e que recupera-se os seus expoentes de Hölder teóricos e dimensões fractais.

Palavras-chave: (min, +)-wavelets , expoentes de Hölder, funções de Weierstrass.

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