On the Stability of Volterra Difference Equations of Convolution Type

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ABSTRACT. In [4], S. Elaydi obtained a characterization of the stability of the null solution of the Volterra difference equation

$$x_n = \sum_{i=0}^{n-1} a_{n-i} x_i, \quad n \geq 1,$$

by localizing the roots of its characteristic equation

$$1 - \sum_{n=1}^{\infty} a_n z^n = 0.$$

The assumption that $(a_n) \in \ell^1$ was the single hypothesis considered for the validity of that characterization, which is an insufficient condition if the ratio R of convergence of the power series of the previous equation equals one. In fact, when $R = 1$, this characterization conflicts with a result obtained by Erdös et al. in [8]. Here, we analyze the $R = 1$ case and show that some parts of that characterization still hold. Furthermore, studies on stability for the $R < 1$ case are presented. Finally, we study some results related to stability via finite approximation.

Keywords: difference equation, stability, convolution.

1 INTRODUCTION

In the present work, we analyze the stability of the null solution of Volterra difference equations of convolution type,

$$x_n = \sum_{i=0}^{n-1} a_{n-i} x_i, \quad n \geq 1, \tag{1.1}$$

whose recursive process starts at $x_0 \in \mathbb{R}$. Several results related to this subject matter circulates in the specialized scientific literature. One of most well-known is the following theorem:

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Theorem 1 (See [10]). *If* $\sum_{n=1}^{\infty} |a_n| < 1$, *then the null solution of (1.1) is asymptotically stable.*

We now present another important characterization of the stability of the null solution of (1.1) as obtained by S. Elaydi in [4]. Let (x_n) be a solution of (1.1) with initial condition $x_0 = 1$. Consider the two power series

$$x(z) := \sum_{n=0}^{\infty} x_n z^n, \quad a(z) := \sum_{n=1}^{\infty} a_n z^n.$$

Then, formally, such series satisfies

$$x(z)(1 - a(z)) = 1. \tag{1.2}$$

Thus the coefficients of $x(z)$ can be found by determining the coefficients of the power series representation of the function $(1 - a(z))^{-1}$. Hence the roots of the characteristic equation

$$1 - \sum_{n=1}^{\infty} a_n z^n = 0 \tag{1.3}$$

play an important role in this sense. By this reasoning, S. Elaydi obtained necessary and sufficient conditions for the stability of the null solution by localizing the roots of $1 - a(1/z) = 0$ with respect to the set $\{z \in \mathbb{C} : \|z\| \geq 1\}$. We now enunciate the result obtained by Elaydi with a variable change by writing z in place of $1/z$ (as considered in [4]). In the following, we use the notation:

$$B_r(z_0) = \{z \in \mathbb{C} : |z - z_0| < r\}, \quad r > 0.$$

Theorem 2 (See [4, 6]). *Let* $(a_n) \in \ell^1$. *Then:*

(a) *The null solution of (1.1) is stable if, and only if, the characteristic equation (1.3) has no roots in $B_1(0)$ and its possible roots in $|z| = 1$ are of order* 1.

(b) *The null solution of (1.1) is asymptotically stable if, and only if, the characteristic equation (1.3) has no roots in $\overline{B_1(0)}$.*

It is worth mentioning that the previous theorem has also appeared as theorems 6.16 and 6.17 in [5]. Furthermore, as a consequence of theorem 2, Elaydi set the following result on asymptotic instability:

Theorem 3 (See [4, 6]). *If $(a_n) \in \ell^1$ is a sequence whose terms do not change signs for $n \geq 1$, then the null solution of (1.1) is not asymptotically stable if one of the following conditions is satisfied:*

(a) $\sum_{n=1}^{\infty} a_n \geq 1$;

(b) $\sum_{n=1}^{\infty} a_n \leq -1$ *and* $a_n > 0$ *for some* $n \geq 1$;

(c) $\sum_{n=1}^{\infty} a_n \leq -1$ *and* $a_n < 0$ *for some* $n \geq 1$ *and* $\sum_{n=1}^{\infty} a_n$ *is sufficiently small.*

At this point, if we consider the sequence

$$a_n = \frac{1}{n(n+1)}, \quad n \geq 1, \tag{1.4}$$

the null solution of (1.1) is not asymptotically stable by the item (b) of theorem 2 or the item (a) of theorem 3. On the other hand, (1.4) satisfies the conditions for asymptotic stability of the null solution of (1.1) as given by the following theorem due to Erdös, Feller e Pollard:

Theorem 4 (See [8]). *Let* (a_n) *be a sequence of nonnegative terms such that*

$$\gcd\{n \in \mathbb{N} : a_n > 0\} = 1, \quad \sum_{n=1}^{\infty} a_n = 1 \quad and \quad \sum_{n=1}^{\infty} n a_n = \infty.$$

Then the null solution of (1.1) is asymptotically stable.

Therefore there exists a contradiction between the previous theorem and theorems 2 and 3. An analysis of the proof of theorem 2 makes clear that the analyticity of the power series $a(z)$ on the circumference $|z| = 1$ was strongly used. But this fact is not a consequence of the assumption that $(a_n) \in \ell^1$, as we can see in the example (1.4). Hence a simple correction can be made by introducing the radius of convergence of the series $a(z)$,

$$\frac{1}{R} = \limsup_{n \to \infty} \sqrt[n]{|a_n|}, \tag{1.5}$$

and replacing the hypothesis that $(a_n) \in \ell^1$ by $R > 1$. In fact, if $R > 1$, then the function $a(z)$ is analytic in $|z| = 1$ and $(a_n) \in \ell^1$, which are conditions that assure us of the validity of theorem 2. Furthermore, by applying this new hypothesis, there is no contradiction between theorem 4 and theorems 2 and 3 since we may easily show that the conditions of theorem 4 implies that $R = 1$. In this sense, in order to avoid conflicts, we rewrite theorem 2 as follows:

Theorem 2′. *Replace* $(a_n) \in \ell^1$ *by* $R > 1$. *Then the items (a) and (b) of theorem 2 are valid.*

Therefore, if $R = 1$, the validity of theorem 2 is an open problem since the analyticity on the unit circumference can not be applied. Besides theorem 4, some other results give us some sufficient conditions for the asymptotic stability of the null solution of (1.1) and can be found in [1, 7, 11, 9, 2, 14, 12, 13].

Finally, note that the stability of the null solution of (1.1) depends on the behavior of the particular solution (x_n) with initial condition $x_0 = 1$. In fact, an arbitrary solution to the equation (1.1) with initial condition β is given by $(x_n \beta)$. Thus, in what follows, (x_n) denotes the solution of (1.1) with initial condition $x_0 = 1$. Therefore, just for future reference, we have the following elementary result, which is already known in a more general case (see e.g. theorem 3.1 in [3]):

Theorem 5. *The null solution of (1.1) is:*

1. *stable if, and only if, (x_n) is bounded;*

2. *asymptotically stable if, and only if, $x_n \to 0$.*

The rest of this paper is divided as follows: In section 2 we analyze the stability for $R = 1$. Furthermore, we study possible characterizations for the null solution of (1.1) to be stable if $R < 1$. In section 3 we analyze the stability via finite approximations.

2   RESULTS ON STABILITY WHEN $R = 1$ OR $R < 1$

In this section, we provide some results concerning the stability/instability for the null solution of (1.1) when $R = 1$ and $R < 1$. It is worth pointing out that the arguments to be used are still valid in the case where $R > 1$. Initially we give two examples on stability/instability for the $R < 1$ case.

Example.   Let $p > 1$. At first consider the sequence $a_n = -p^n$. Then $R < 1$ and the solution of (1.1) with initial condition $x_0 = 1$ is $(x_n) = (1, -p, 0, 0, \ldots)$, which converges to zero. So the null solution is asymptotically stable. On the other hand, if we consider

$$a_n = \frac{p^{n-1}}{n(n+1)}, \quad n \geq 1, \tag{2.1}$$

we also have $R < 1$. However, since each term in this sequence is positive, it follows (by induction) that each term of (x_n) is positive. Hence $x_n \geq a_n$ for each $n \geq 1$. Thus (x_n) is unbounded and consequently the null solution of (1.1) is unstable by theorem 5.

The previous example shows that the stability or instability of the null solution does not depend on the radius of convergence of $a(z)$. In what follows we present a result on instability which does not depend on the size of R. (In particular, it holds if $R = 1$ or $R < 1$, which is the case where Theorem $2'$ is not applicable.)

Theorem 6. *Let $D_a = \left\{ z \in \mathbb{C} : \sum_{n=1}^{\infty} a_n z^n \text{ converges} \right\}$. If the characteristic equation (1.3) has a root in $D_a \cap B_1(0)$, then the null solution of (1.1) is unstable.*

Proof.   Denote by $\rho_0 e^{i\theta}$, $\rho_0 \in]0, 1[$, one of the zeros of $1 - a(z)$ of smallest modulus. Hence $1 - a(z) \neq 0$ for each $z \in B_{\rho_0}(0)$. Since $x(z) = (1 - a(z))^{-1}$ for every $z \in B_{\rho_0}(0)$, we have that

$$\lim_{\rho \to \rho_0^-} |x(\rho e^{i\theta})| = \infty.$$

Therefore the radius of convergence of $x(z)$ is not greater than ρ_0. Then

$$\limsup \sqrt[n]{|x_n|} \geq \frac{1}{\rho_0} > 1.$$

Thus, if $\frac{1}{\rho_0} > \beta > 1$, there exists a subsequence (x_{n_k}) for which $\sqrt[n_k]{|x_{n_k}|} > \beta$. So we conclude that $|x_{n_k}| > \beta^{n_k}$. Therefore the sequence (x_n) is not bounded. $\qquad\square$

Remark.   The converse of theorem 6 is not valid. In fact, for the sequence given in (2.1), the null solution of (1.1) is unstable. On the other hand, since $D_a = \overline{B_{1/p}(0)}$ and

$$|a(z)| = \left| \sum_{n=1}^{\infty} \frac{p^{n-1}}{n(n+1)} z^n \right| \leq \frac{1}{p} < 1, \quad \forall z \in \overline{B_{1/p}(0)},$$

it follows that $1 - a(z)$ is not zero in $D_a \cup B_1(0)$.

Corollary 7. *If $\displaystyle\sum_{n=1}^{\infty} a_n > 1$ converges, then the null solution of (1.1) is unstable.*

Proof.   Since $\displaystyle\sum_{n=1}^{\infty} a_n$ converges, it follows from Abel's theorem that the power series $a(z)$ is continuous in $[0, 1]$. Specifically, one has that $\displaystyle\lim_{z \to 1^-} a(z) = \sum_{n=0}^{\infty} a_n$. So consider the function $b(z) = 1 - a(z)$. Then $b(0) = 1$ and $b(1) < 0$. Therefore $b(z)$ has a zero in the interval $]0, 1[\subset B_1(0)$. It follows from the previous theorem that the null solution of (1.1) is not stable. $\square$

Remark.   By item (a) of theorem 3, we have that the lack of asymptotic stability takes place when, in particular, the hypothesis of the previous corollary holds, provided that the terms of the sequence (a_n) do not change signs. So, the previous corollary states the lack of stability (and therefore the lack of asymptotic stability) without any sign-preserving condition. Furthermore, that corollary remains valid if we replace the hypothesis $\displaystyle\sum_{n=1}^{\infty} a_n > 1$ by $\displaystyle\sum_{n=1}^{\infty} (-1)^n a_n < 1$.

The following theorem shows that part of what was stated in the item (b) of theorem $2'$ remains valid if $R = 1$.

Theorem 8. *If $1 - a(z)$ is continuous and not zero in $\overline{B_1(0)}$, then the null solution of (1.1) is asymptotically stable.*

Proof.   First we observe that the function $x(z) = (1 - a(z))^{-1}$ is uniformly continuous on $\overline{B_1(0)}$. Now, for each $\rho \in [0, 1[$ and $n \in \mathbb{N}$, define the following two functions on the interval $[0, 2\pi]$:

$$f_\rho(\theta) = x(\rho e^{i\theta}) e^{-in\theta}, \quad f(\theta) = x(e^{i\theta}) e^{-in\theta}.$$

It follows from the uniform continuity of $x(z)$ on $\overline{B_1(0)}$ that $f_\rho \to f$ uniformly on $[0, 2\pi]$ as $\rho \to 1^-$. Therefore

$$\lim_{\rho \to 1^-} \int_0^{2\pi} x(\rho e^{i\theta})e^{-in\theta}\, d\theta = \int_0^{2\pi} x(e^{i\theta})e^{-in\theta}\, d\theta.$$

From the Cauchy's Integral Formula, we have that, for every $\rho \in]0, 1[$,

$$x_n = \frac{1}{2\pi i} \int_{|z|=\rho} \frac{x(z)}{z^{n+1}}\, dz = \frac{1}{2\pi \rho^n} \int_0^{2\pi} x(\rho e^{i\theta})e^{-in\theta}\, d\theta.$$

Applying the limit when $\rho \to 1^-$, it follows that

$$x_n = \frac{1}{2\pi} \int_0^{2\pi} x(e^{i\theta})e^{-in\theta}\, d\theta.$$

So, by the Riemann-Lebesgue Lemma, one has $x_n \to 0$, which shows that the null solution of (1.1) is asymptotically stable. $\qquad\square$

Remark.   The conclusion of the preceding theorem is not valid if the radius of convergence of $a(z)$, R, is less than one. In other words, even if $1 - a(z)$ is continuous and not zero in $\overline{B_R(0)}$, the null solution may not be asymptotically stable. To illustrate this statement, it suffices to consider the sequence given in (2.1). Additionally, note that, if $R = 1$, the converse of the preceding theorem is not valid, as shown in example (1.4).

To finalize this section, we enunciate an auxiliary lemma for the characterization of the stability (not necessarily an asymptotic one) of the null solution when $R = 1$.

Lemma 9. *Consider the following power series*

$$y(z) = \sum_{n=0}^{\infty} y_n z^n, \quad p(z) = \sum_{n=0}^{\infty} p_n z^n.$$

Suppose that (y_n) is bounded. Then:

1. *If $p(z) = (1 - e^{-i\theta} z)y(z)$, then (p_n) is bounded.*

2. *If $p(z) = (1 - z)y(z)$, then $p(z)$ is bounded on the interval $[0, 1[$.*

Proof.   First note that $p_n = y_n - e^{-i\theta} y_{n-1}$ for each $n \geq 1$. Hence, if $C = \sup_{n\geq 0} |y_n|$, then $|p_n| \leq 2C$ for every $n \geq 1$, which demonstrates the item 1. Consider now that the hypothesis of the item 2 is valid. It follows that $z \in [0, 1[$ implies

$$|p(z)| \leq (1 - z) \sum_{n=0}^{\infty} |y_n| z^n \leq C(1 - z) \sum_{n=0}^{\infty} z^n = C. \qquad\square$$

In what follows, suppose that the power series $a(z)$ converges on $\overline{B_1(0)}$ and the possible zeros of $1 - a(z)$ occur at $e^{i\theta_1}, \ldots, e^{i\theta_s}$, in other words,

$$1 - a(z) = (1 - e^{-i\theta_1} z)^{m_1} \cdots (1 - e^{-i\theta_s} z)^{m_s} q(z), \tag{2.2}$$

where $q(z)$ is not zero in $\overline{B_1(0)}$. Furthermore, consider the space

$$\mathbb{L}^1 := \left\{ q(z) = \sum_{n=0}^{\infty} q_n z^n \; : \; (q_n) \in \ell^1 \right\}.$$

Next, it follows a characterization of the stability for $R = 1$:

Theorem 10. *Assume that the power series $a(z)$ converges on $\overline{B_1(0)}$ and $q \in \mathbb{L}^1$. The null solution of (1.1) is stable if, and only if, the possible zeros of $1 - a(z)$ as given in (2.2) are of order 1.*

Proof.   First consider that $m_1 = \cdots = m_s = 1$. Since $q \in \mathbb{L}^1$ and q is not zero in $\overline{B_1(0)}$, by Wiener's Theorem, we have that

$$[q(z)]^{-1} = \hat{q}(z) = \sum_{n=0}^{\infty} \hat{q}_n z^n \in \mathbb{L}^1. \tag{2.3}$$

On the other hand,

$$\left[\prod_{j=1}^{s} (1 - e^{-i\theta_j} z) \right]^{-1} = \sum_{n=0}^{\infty} \alpha_n z^n \quad \text{with} \quad \alpha_n = \sum_{j=1}^{s} \left(\frac{e^{i\left(\sum_{\mu \neq j} \theta_\mu \right)}}{\prod_{\mu \neq j} (e^{i\theta_\mu} - e^{i\theta_j})} \right) e^{-i\theta_j n}. \tag{2.4}$$

Since $x(z) = (1 - a(z))^{-1}$ for $z \in B_1(0)$, it follows from (2.3) and (2.4) that

$$x(z) = \left(\sum_{n=0}^{\infty} \alpha_n z^n \right) \left(\sum_{n=0}^{\infty} \hat{q}_n z^n \right).$$

As a result of equating coefficients, we have that $x_n = \sum_{k=0}^{n} \alpha_{n-k} \hat{q}_k$. Therefore

$$|x_n| \leq C \sum_{k=0}^{\infty} |\hat{q}_k|,$$

where C is a constant which is an upper bound for the sequence $(|\alpha_n|)$. So (x_n) is bounded and then, by theorem 5, the null solution of (1.1) is stable. Conversely, suppose that the null solution of (1.1) is stable and $m_1 \geq 2$. Replacing z by $e^{i\theta_1} z$ in $x(z)(1 - a(z)) = 1$, it follows that

$$p(z)(1 - z)^{m_1 - 1} q\left(e^{i\theta_1} z \right) = 1, \quad \forall z \in B_1(0),$$

with

$$p(z) = (1 - z)\left(1 - e^{i(\theta_1 - \theta_2)} z\right)^{m_2} \cdots \left(1 - e^{i(\theta_1 - \theta_s)} z\right)^{m_s} x \left(e^{i\theta_1} z\right).$$

Since, by theorem 5, the sequence (x_n) is bounded, by applying the item 1 several times and finally the item 2 of the preceding lemma, one has that $p(z)$ is bounded on $[0, 1[$. Therefore

$$1 = \lim_{\substack{z \to 1^- \\ z \in \mathbb{R}}} x(e^{i\theta_1} z)(1 - a(e^{i\theta_1} z)) = \lim_{\substack{z \to 1^- \\ z \in \mathbb{R}}} p(z)(1 - z)^{m_1 - 1} q(e^{i\theta_1} z) = 0,$$

which is absurd. Hence $m_1 = 1$. □

Corollary 11. *Let* $\sum_{n=1}^{\infty} n|a_n| < \infty$. *If* $1 - a(z)$ *is not zero in* $B_1(0)$ *and has a finite number of zeros of order one in* $|z| = 1$, *then the null solution of (1.1) is stable.*

Proof. Suppose that $z = 1$ is a zero of $1 - a(z)$ and consider

$$q(z) = \sum_{n=0}^{\infty} q_n z^n := \frac{1 - a(z)}{1 - z} = (1 - a(z))\left(\sum_{n=0}^{\infty} z^n\right).$$

Then $q_0 = 1$ and, for $n \geq 1$, one has that

$$q_n = 1 - \sum_{k=1}^{n} a_k = \sum_{k=n+1}^{\infty} a_k.$$

It is easy to verify that, for $n \geq 1$, we have

$$\sum_{k=0}^{n-1} q_k = \sum_{k=1}^{n} k a_k + n \sum_{k=n+1}^{\infty} a_k.$$

So

$$\sum_{k=0}^{n-1} |q_k| \leq \sum_{k=1}^{n} k|a_k| + n \sum_{k=n+1}^{\infty} |a_k| \leq \sum_{k=1}^{\infty} k|a_k| \quad \forall n \in \mathbb{N}.$$

Thus $1 - a(z) = (1 - z)q(z)$ with $q \in \mathbb{L}^1$. On the other hand, if $z = e^{i\theta}$ is a zero of $1 - a(z)$, one has that $z = 1$ is a zero of $1 - \tilde{a}(z)$, $\tilde{a}(z) = a(e^{i\theta} z)$, which satisfies the hypotheses of this corollary. Hence $1 - \tilde{a}(z) = (1 - z)\tilde{q}(z)$ with $\tilde{q} \in \mathbb{L}^1$ or, equivalently, $(1 - a(z)) = (1 - e^{-i\theta} z)q(z)$, where $q(z) = \tilde{q}(e^{-i\theta} z)$ and therefore $q \in \mathbb{L}^1$. Now consider the set $\{e^{i\theta_j} : j = 1, \ldots, s\}$ consisting of all zeros of $1 - a(z)$. Then, by partial fractions, we have

$$\frac{1 - a(z)}{(1 - e^{-i\theta_1} z) \cdots (1 - e^{-i\theta_s} z)} = \sum_{j=1}^{s} \frac{\beta_j (1 - a(z))}{(1 - e^{-i\theta_j} z)} = \sum_{j=1}^{s} \beta_j q_j(z),$$

where each $q_j \in \mathbb{L}^1$. So, if $q = \sum_{j=1}^{s} \beta_j q_j \in \mathbb{L}^1$, then

$$1 - a(z) = (1 - e^{-i\theta_1}z) \cdots (1 - e^{-i\theta_s}z)q(z).$$

It follows from the preceding theorem that the null solution of (1.1) is stable.   □

Example.   The sequence $a_n = c_0 \dfrac{(-1)^n}{n^3}$, $c_0 := \left(\sum_{n=1}^{\infty} \dfrac{1}{n^3} \right)^{-1}$, satisfies the hypotheses of the previous corollary. So the null solution of (1.1) is stable. Note that, in this case, theorem $2'$ cannot be used for obtaining this result since $R = 1$.

3   STABILITY VIA APPROXIMATION

In this final section we state some conditions for stability via polynomial approximation by applying the following theorem (known as Rouché's Theorem):

Theorem 12. *If f and $f + h$ are analytic functions on $\overline{B_\rho(z_0)}$ such that*

$$|h(z)| < |f(z)| \quad in \quad |z| = \rho,$$

then f and $f + h$ have the same number of zeros in $B_\rho(z_0)$.

Now, for each $n \in \mathbb{N}$, consider the polynomial

$$p_n(z) = z^n - a_1 z^{n-1} - \cdots - a_{n-1}z - a_n,$$

where $a_1, \ldots, a_n$ are the first n coefficients of the power series $a(z)$. Define

$$r_n := \max\{|z| \ : \ p_n(z) = 0\}$$

and $z_1, \ldots, z_n$ as the n zeros of $p_n(z)$, that is,

$$p(z) = (z - z_1) \ldots (z - z_n).$$

In what follows we enunciate some results of stability via finite approximations of the characteristic equation.

Theorem 13. *If there exists an index n such that*

$$r_n < 1 \quad and \quad \sum_{i=n+1}^{\infty} |a_i| < (1 - r_n)^n,$$

then the null solution of (1.1) is asymptotically stable.

Proof.   For $|z| = 1$, we have that

$$1 - r_n \leq 1 - |z_i| \leq |z| - |z_i| \leq |z - z_i|, \quad i = 1, \ldots, n.$$

So $(1 - r_n)^n \leq |p_n(z)|$ for $|z| = 1$. Consider the nth partial sum of $1 - a(z)$, that is,

$$s_n(z) := 1 - \sum_{k=1}^{n} a_k z^k.$$

Hence, since $s_n(z) = z^n p_n(1/z)$ for $z \neq 0$, one has that $s_n(z)$ is not zero in $\overline{B_1(0)}$ and $(1 - r_n)^n \leq |s_n(z)|$ for $|z| = 1$. Then

$$|1 - a(z) - s_n(z)| = \left| \sum_{i=n+1}^{\infty} a_i z^i \right| \leq \sum_{i=n+1}^{\infty} |a_i| < (1 - r_n)^n \leq |s_n(z)|.$$

By Rouché's Theorem, $1 - a(z)$ and $s_n(z)$ have the same number of zeros in $\overline{B_1(0)}$. Therefore $1 - a(z)$ is not zero in $\overline{B_1(0)}$. It follows from theorem 8 that the null solution of (1.1) is asymptotically stable.   □

Example.   Let (β_n) be a sequence with $\beta_n \in \{-1, 1\}$. The sequence

$$(a_n) = \left(\frac{3}{2}, -\frac{9}{16}, \frac{\beta_1}{20}, \frac{\beta_2}{20^2}, \frac{\beta_3}{20^3}, \cdots \right)$$

does not satisfy the hypothesis of theorem 1. However, by considering the polynomial $p_2(z)$, we obtain $r_2 = 3/4$ and, since

$$\sum_{i=3}^{\infty} |a_i| = \frac{1}{19} < (1 - 3/4)^2,$$

the null solution of (1.1) is asymptotically stable.

Example.   The sequence

$$(a_n) = \left(1, -\frac{41}{36}, \frac{8}{9}, -\frac{34}{81}, \frac{16}{81}, -\frac{4}{81}, \frac{1}{2 \cdot 4^6}, \frac{1}{2^2 \cdot 4^6}, \frac{1}{2^3 \cdot 4^6}, \cdots \right)$$

does not satisfy the hypothesis of theorem 1. By a computational calculus, the values of r_n and $L_n := \sum_{i=n+1}^{\infty} |a_i|$ are as shown in the table that follows. Note that the hypothesis of the preceding theorem is satisfied for $n = 6$. So the null solution of (1.1) is asymptotically stable.

Theorem 14. *If there exists an index n such that $r_n > 1$ and*

$$\sum_{i=n+1}^{\infty} |a_i| < \delta_n(\rho_0),$$

n	r_n	L_n	$(1 - r_n)^n$
1	1	–	–
2	1.067	–	–
3	1.012	–	–
4	0.913	0.24716	0.00005
5	0.781	0.04963	0.00050
6	0.667	0.00024	0.00137

where ρ_0 is the point that maximizes the function $\delta_n : [r_n^{-1}, 1] \to \mathbb{R}$ defined by

$$\delta_n(\rho) := |(1 - \rho|z_1|)(1 - \rho|z_2|) \cdots (1 - \rho|z_n|)|,$$

then the null solution of (1.1) is unstable.

Proof.   Since $\delta(r_n^{-1}) = 0$, one has $r_n^{-1} < \rho_0 \leq 1$. If $|z| = \rho_0$, then $|1 - zz_i| \geq |1 - \rho_0|z_i||$. So the partial sum considered in the preceding theorem satisfies

$$|s_n(z)| = |1 - zz_1| \cdots |1 - zz_n| \geq \delta_n(\rho_0).$$

Therefore, for $|z| = \rho_0$, we have that

$$|1 - a(z) - s_n(z)| = |\sum_{i=n+1}^{\infty} a_i z^i| \leq \sum_{i=n+1}^{\infty} |a_i| < \delta_n(\rho_0) \leq |s_n(z)|.$$

Now, let j with $|z_j| = r_n$. Then $z_j^{-1} \in B_{\rho_0}(0)$ and $s_n(z_j^{-1}) = 0$. Hence, by Rouché's Theorem, $1 - a(z)$ has at least a zero in $B_{\rho_0}(0) \subset B_1(0)$. Therefore, by theorem 6, the null solution of (1.1) is unstable. $\square$

Remark.   Let us put the moduli of the zeros of p_n in descending order, say $|z_1| \geq \cdots \geq |z_n|$. Assume that the hypotheses of the preceding theorem hold. Consider i_0 is the highest index with $|z_{i_0}| > 1$. Then $|z_{i_0+1}| \leq 1$. Hence one has the following estimates:

1. If $i_0 = n$, then $E_1 := |1 - |z_n||^n \leq \delta_n(1) \leq \delta_n(\rho_0)$.

2. If $|z_{i_0+1}| < 1$, then $E_2 := \min\{|1 - |z_{i_0}||^n, |1 - |z_{i_0+1}||^n\} \leq \delta_n(1) \leq \delta_n(\rho_0)$.

3. If $|z_{i_0+1}| = 1$, then $E_3 := |1 - \rho_1|z_{i_0}||^n \leq \delta_n(\rho_1) \leq \delta_n(\rho_0)$ with $\rho_1 = 2/(|z_{i_0}| + 1)$.

As a consequence of the previous remark, we may state the following corollary:

Corollary 15. *With the same assumptions of the preceding theorem, if*

$$\sum_{i=n+1}^{\infty} |a_i| < E,$$

where $E = E_1$, E_2 or E_3 are given as in the previous remark, then the null solution of (1.1) is unstable.

Example. Consider the sequence (a_n) given by

$$a_1 = 4, \quad a_2 = -4, \quad a_n = \frac{1}{2^{n-1}}, \quad n \geq 3.$$

(a_n) does not satisfy the assumptions of corollary 7. However the zeros of $p_2(z)$ are $z_1 = z_2 = 2$. Since

$$\sum_{k=3}^{\infty} |a_k| = \frac{1}{2} < |1-2|^2,$$

the null solution of (1.1) is unstable by the previous corollary.

RESUMO. In [4], S. Elaydi obteve uma caracterização da estabilidade da solução nula da equação a diferenças de Volterra

$$x_n = \sum_{i=0}^{n-1} a_{n-i} x_i, \quad n \geq 1,$$

localizando as raízes de sua equação característica

$$1 - \sum_{n=1}^{\infty} a_n z^n = 0.$$

A suposição de que $(a_n) \in \ell_1$ foi a única hipótese considerada para a validade daquela caracterização, que é uma condição insuficiente se o raio R de convergência da série de potência da equação anterior é igual a um. De fato, quando $R = 1$, esta caracterização entra em conflito com um resultado obtido por Erdös e colaboradores em [8]. Aqui, nós analisamos o caso $R = 1$ e mostramos que algumas partes daquela caracterização ainda se mantêm. Ainda, são apresentados estudos sobre a estabilidade para o caso $R < 1$. Finalmente, estudamos alguns resultados relativos a estabilidade via aproximações finitas.

Palavras-chave: equação a diferenças, estabilidade, convolução.

REFERENCES

[1]   K.S. Berenhaut & N.G. Vish. Equations of convolution type with monotone coefficients. *Journal of Difference Equations and Applications*, **17** (2011), 555–566.

[2]   S.K. Choi, Y.H. Goo, Y. Hoe & N.J. Koo. Asymptotic behavior of nonlinear Volterra difference systems. *Bull. Korean Math. Soc.*, **44** (2007), 177–184.

[3]   M.A. Crisci, V.B. Kolmanovskii, E. Russo & A. Vecchio. Boundedness of Discrete Volterra Equations. *Journal of Mathematical Analysis and Applications*, **211** (1997), 106–130.

[4]   S. Elaydi. Stability of Volterra difference equations of convolution type, Dynamical Systems. *Nankai Ser. Pure Appl. Math. Theoret. Phys.*, **4** (1993), 66–72.

[5]   S. Elaydi. An Introduction to Difference Equations, Undergraduate Texts in Mathematics, Springer Verlag, (2005).

[6]   S. Elaydi. Stability and asymptoticity of Volterra difference equations. A progress report. *J. Comp. Appl. Math.*, **228** (2009) 504–513.

[7]   S. Elaydi, E. Messina & A. Vecchio. A note on the asymptotic stability of linear Volterra difference equations of convolution type. *J. Difference Equ. Appl.*, **13** (2007), 1079–1084.

[8]   P. Erdös, W. Feller & H. Pollard. A property of power series with positive coefficients. *Bull. Amer. Math. Soc.*, **55** (1949), 201–204.

[9]   V.B. Kolmanovskii & N.P. Kosareva. Stability of Volterra Difference Equations. *Differential Equations*, **37** (2001), 1773–1782.

[10]   V.B. Kolmanovskii, E.C. Velasco & J.A.T. Muñoz. A survey: stability and boundedness of Volterra difference equations. *Nonlinear Analysis*, **53** (2003), 861–928.

[11]   E. Messina, Y. Muroya, E. Russo & A. Vecchio. Asymptotic Behavior of Solutions for Nonlinear Volterra Discrete Equations, Discrete Dynamics in Nature and Society (2008) Article ID 867623, 18 pp.

[12]   N. Van Minh. On the asymptotic behaviour of Volterra difference equations, *J. Difference Equ. Appl.*, **19** (2013), 1317–1330.

[13]   R. Nigmatulin. Asymptotic behavior of solutions of a nonlinear Volterra difference equation, Int. Electron. *J. Pure Appl. Math.*, **6** (2013), 123–125.

[14]   X.H. Tang & Z. Jiang. Asymptotic behavior of Volterra difference equation, *J. Difference Equ. Appl.*, **13** (2007), 25–40.