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Rotational Solitary Wave Interactions over an Obstacle

M. V. FLAMARION

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ABSTRACT. In this work, we investigate the propagation of rotational solitary waves over a submerged obstacle in a vertically sheared shallow water channel with constant vorticity. In the weakly nonlinear, weakly dispersive regime the problem is formulated in the forced Korteweg-de Vries equation framework. The initial value problem for this equation is solved numerically using a Fourier pseudospectral method with an integrating factor. Solitary waves are taken as initial data and their interactions with an obstacle are analysed. We identify three types of regimes according to the intensity of the vorticity. A rotational solitary wave can bounce back and forth over the obstacle remaining trapped for large times, it can pass over the obstacle without reversing its direction or the wave can be blocked, i.e., it bounces back and forth above the obstacle until reaching a steady state. Such behaviour resembles the classical damped spring-mass system.

Keywords: gravity waves, solitary waves, KdV equation, shear flow.

1 INTRODUCTION

The forced Korteweg-de Vries (fKdV) equation has been used as a model to describe several problems in hydrodynamics for instance, problems related to the propagation of water waves over topographic obstacles, flow of water over rocks, [2, 16], ship wakes and ocean waves generated by storms [10].

For an irrotational flow of an incompressible fluid with constant density, it is well known that the flow over an obstacle is governed by two parameters, namely, the amplitude of the obstacle and the Froude number, which is defined as

$$F = \frac{U_0}{\sqrt{gh_0}},$$

where U_0 is the speed of the uniform flow, g is the acceleration of gravity and h_0 is the far field depth of a shallow water channel. The fKdV equation arises as a model to study nearly-critical flows ($F \approx 1$) for submerged obstacles with small amplitudes when compared to the average

UFRPE/Rural Federal University of Pernambuco, UACSA/Unidade Acadêmica do Cabo de Santo Agostinho, BR 101 Sul, 5225, 54503-900, Ponte dos Carvalhos, Cabo de Santo Agostinho, Pernambuco, Brazil – E-mail: marcelo.flamarion@ufrpe.br https://orcid.org/0000-0001-5637-7454

depth. A careful study on this model was first done by Wu and Wu [18] and later by several other authors [1, 5, 6, 9, 15, 19].

On the light of the fKdV framework, trapped waves have been extensively studied in the past few years. The terminology "trapped wave" is used to describe waves that remain trapped in a certain region of space, generally above an obstacle or in low-pressure regions. Grimshaw et al. [8] used the fKdV equation to investigate the interaction of a solitary wave with a external force of small amplitude asymptotically and numerically. They found regimes in which solitary waves remain partially or totally trapped at the external force. Lee and Whang [14] and Lee [13] considered a two-bumped obstacle and found solutions for the fKdV equation that remained trapped between the two obstacles for a certain period of time. Besides, the numerical stability of these trapped waves were analysed by disturbing their initial amplitudes as well as the obstacle heights. In the same spirit, Kim and Choi [12] verified that trapped waves have to cross a certain energy barrier in order to leave the region between the two bumps. Later, Flamarion and Ribeiro-Jr [7] studied the numerical stability of solitary trapped waves in a low pressure region with respect to perturbations of the amplitude of the initial data as well as the intensity of the pressure using the full Euler equations.

Considering a vertically sheared background flow in the presence of an even bottom, Johnson and Freeman [11] deduced a KdV-type equation, in which coefficients depend on the vorticity and the speed of the flow in the depth of the channel. More recently, Flamarion et al. [5] used asymptotic analysis for the full Euler equations to extend the KdV equation [11] to a rotational fKdV equation that can be used to model rotational flows with constant vorticity over obstacles. Then, they used the deduced rotational fKdV to validate their numerical methods to study rotational waves generated from rest by a current-topography interaction for the full Euler equations. Properties of this model were later reported in [4]. Although we use the rotational fKdV equation [5] in this article, our focus is on solitary wave interactions with a submerged obstacle not on the wave generation problem as done by Flamarion et al. [5]

In this work, we investigate numerically rotational solitary waves interactions with a submerged obstacle in a shallow water channel with a vertically sheared current using the rotational fKdV equation [5] as a model. Although there are many works on trapped waves for irrotational flows, to the best of our knowledge, there are no articles considering a sheared background flow. In our numerical simulations, we identify that when the vorticity is weak, rotational solitary waves tend to bounce back and forth above the obstacle remaining trapped for large times. Their amplitudes oscillate at each rebound and increase as time goes on. This indicates that all these waves may accumulate enough Kinect energy to pass over the obstacle at some point. Besides, when the vorticity is strong, rotational solitary waves can be blocked, i.e., these waves bounce back and forth above the obstacle a few times and then become stationary. Such behaviour resembles a damped spring-mass system, but here, the damping is due to the vorticity. Moreover, the numerical stability of these waves are investigated by disturbing the amplitude of the initial data.

This paper is organized as follows. In section 2 we present the mathematical formulation of the rotational fKdV equation. The numerical methods are presented in section 3, the numerical results in section 4 and the conclusion in section 5.

2 THE ROTATIONAL FORCED KORTEWEG-DE VRIES EQUATION

We consider a two-dimensional incompressible flow of an inviscid fluid with constant density in the presence of gravity force and a vertically sheared current with a submerged obstacle. In the weakly nonlinear, weakly dispersive regime Flamarion et al. [5] deduced the dimensionless fKdV equation

$$-2I_{31}\zeta_t - 2I_{31}f\zeta_x + 3I_{41}\zeta\zeta_x + J_1\zeta_{xxx} = h_x(x), \tag{2.1}$$

as a model to describe the flow over obstacles with small amplitudes. Here, we denote by $\zeta(x,t)$ the free-surface displacement over the undisturbed surface and $h(x)$ the submerged obstacle. The coefficients in (2.1) are defined as

$$I_{31} = \frac{\Omega + 2\gamma(\Omega)}{2\gamma(\Omega)^2(\Omega+\gamma(\Omega))^2},\ I_{41} = \frac{1}{3}\frac{\Omega^2 + 3\Omega\gamma(\Omega) + 3\gamma(\Omega)^2}{\gamma(\Omega)^3(\Omega+\gamma(\Omega))^3},\ J_1 = \frac{1}{3\gamma(\Omega)^3},$$

where

$$\gamma(\Omega) = -\frac{\Omega}{2} + \frac{\sqrt{\Omega^2+4}}{2}$$

and Ω is the vorticity parameter. The vertically sheared current with constant vorticity ($\omega = -\Omega$) is defined as

$$U(y) = \Omega y + \gamma(\Omega) + \varepsilon f,$$

where f is a constant and ε is a small positive parameter. The flow is called supercritical, subcritical or critical depending on whether $f > 0$, $f < 0$ or $f = 0$.

When the bottom is flat ($h_x = 0$), the fKdV equation (2.1) has solitary wave solutions given by

$$\zeta(x,t) = A\,\mathrm{sech}^2(k(x-ct)), \quad \text{where } c = f - \frac{2J_1}{I_{31}}k^2,\ A = \frac{4J_1}{I_{41}}k^2. \tag{2.2}$$

It is worth mentioning that when $f = 2J_1k^2/I_{31}$ the solitary wave solutions are stationary.

3 NUMERICAL METHODS

We solve the fKdV equation (2.1) through a Fourier pseudospectral method with an integrating factor. It solves the linear part of equation (2.1) exactly, which avoids numerical instabilities issues due to the dispersive term. The computational domain $[-L,L]$ is periodic with a uniform grid with N points and step $\Delta x = 2L/N$. Spatial derivatives are computed spectrally [17]. Furthermore, the time evolution is computed using the Runge-Kutta fourth-order method (RK4) with time step Δt. Flamarion et al. [5] considered a similar numerical method and tested its resolution with respect to N and Δx. They verified that solution is accurately captured using different grids. Therefore, for convenience we set the following parameters: $L = 1000$, $N = 2^{13}$, and $\Delta t = 0.01$.

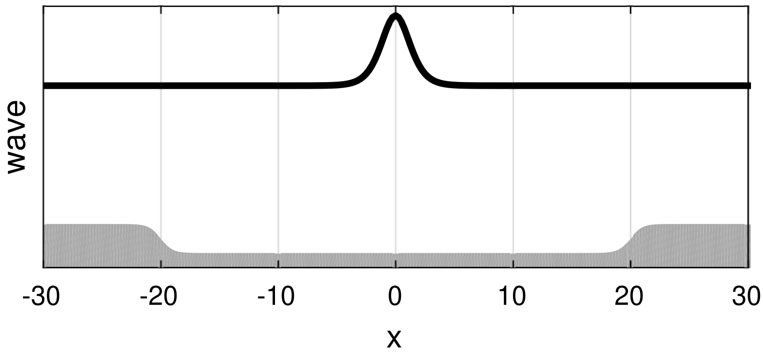

Figure 1: Sketch of the physical problem.

The initial condition of equation (2.1) is taken as a solitary wave defined in equation (2.2) and the submerged obstacle is modelled by the localised function

$$h(x) = \delta(\tanh(x-\beta) - \tanh(x+\beta)). \tag{3.1}$$

Since h decays to zero as $|x| \to \infty$, for large values of L, we can approximate the boundary conditions by periodic conditions. A sketch of the physical problem at t = 0 is depicted in Figure 1. Unless mentioned otherwise, we fix $\delta = 10^{-3}$ and $\beta = 20$ in all following simulations.

In this paper we do not attempt an exhaustive study of trapped waves, instead, we present a few examples and highlight their main properties. For this reason, we limit ourselves to consider solitary waves as in equation (2.2) with amplitude $A = 0.5$. However, the results presented here can be easily extended for solitary waves with different amplitudes.

4 NUMERICAL RESULTS

In the absence of a variable bottom topography, the fKdV equation (2.1) has solitary wave solutions (2.2), and steady solutions can be obtained by taking $f = 2J_1k^2/I_{31}$. Notice that if a solitary wave is above the obstacle (see Figure 1), steady waves are no longer given by the formula (2.2). Although the amplitude of the obstacle is small, it still affects the solitary wave speed. In order to find solitary waves that bounce back and forth above the obstacle, we disturb f slightly by taking

$$f = 2J_1k^2/I_{31} + 0.01,$$

which implies that $c = 0.01$ is the solitary-wave speed when the bottom is flat.

Figure 2 (top left) displays the evolution of a trapped solitary wave above the obstacle (3.1). The solitary wave is initially set with its crest located at $x = 0$. This wave starts moving downstream until it reaches the shallower region, and then, it is reflected back and moves upstream. This dynamic is repeated for large times (see Figure 2 (bottom)). Details of how the amplitude changes as a function of the position of the solitary-wave crest is depicted in Figure 2 (top right). It is

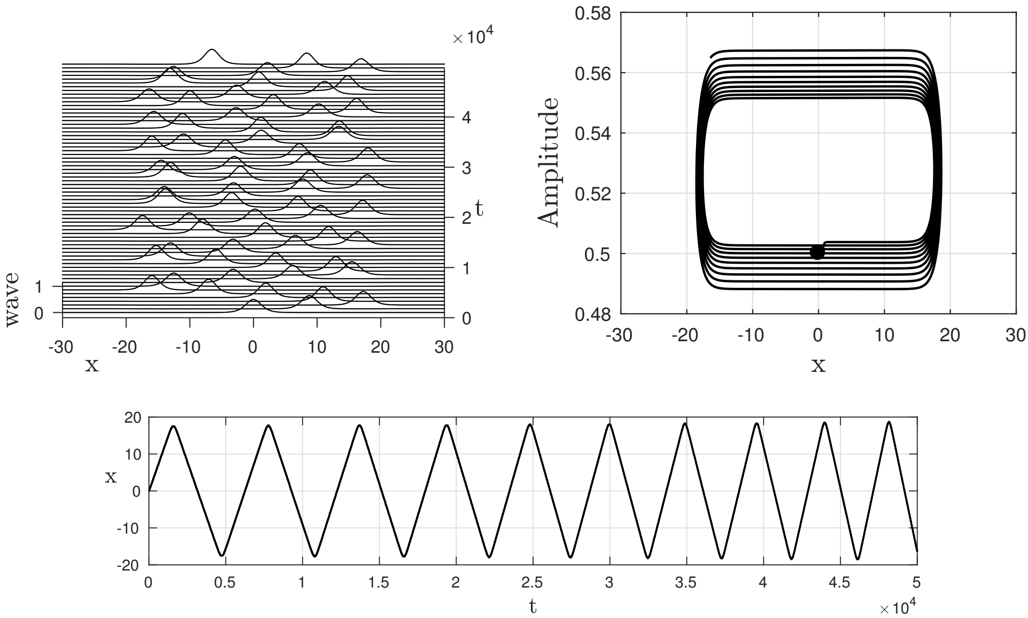

Figure 2: Top (left): trapped solitary wave over the obstacle. Top (right): amplitude of the trapped wave as a function of its crests position. Bottom: the crest position of the solitary wave as a function of time. Parameters: $A = 0.5$, $f = 0.26$ and $\Omega = 0$.

interesting to observe that when the wave bounces back and forth where the bottom is nearly flat ($-10 < x < 10$) its amplitude remains unchanged. In addition, the amplitude of this wave oscillates increasing over time which leads us to conclude that this wave might overcome the obstacle at some point. These results agree qualitatively with the ones reported on the works of Grimshaw et al. [8] and in the recent work of Ermakov and Stepanyants [3].

Similar regimes were found for $\Omega \in [0,2)$. Furthermore, the trapping mechanism turn out to be robust with respect to perturbations of the initial data, in the sense that for small perturbations of the amplitude of the initial data, the wave solution still remains trapped above the obstacle.

Besides the solitary trapped wave displayed in Figure 2, we also find regimes in which the wave is blocked. In this case, the solitary wave bounces back and forth above the obstacle until it reaches a steady state. Blocked waves only occur when Ω is negative. In this case, the solitary waves propagate downstream reaching the shallower region ($x = 20$) and then are reflected back. However, differently from the previous case, these waves are reflected back before reaching the shallower region ($x = -20$). They move back and forth above the obstacle, but their speed decreases until they reach a steady state approaching a wave-limit. A particular case of this dynamic is depicted in Figure 3 (top left) and its amplitude as a function of its crest position in Figure 3 (top right). In the amplitude vs. crest position space, the equation (2.1) can be interpreted as a dynamical system in which the amplitude of wave-limit and its crest position is a stable spiral point. Somehow, the vorticity acts similar to the manner in which a damping acts in a spring-mass system (see Figure 3 (bottom)). Moreover, if we disturb the initial amplitude of the solitary wave, this wave still approaches the very same wave-limit, which shows that the blocking mechanism is robust with respect to small perturbations of the amplitude of the initial data.

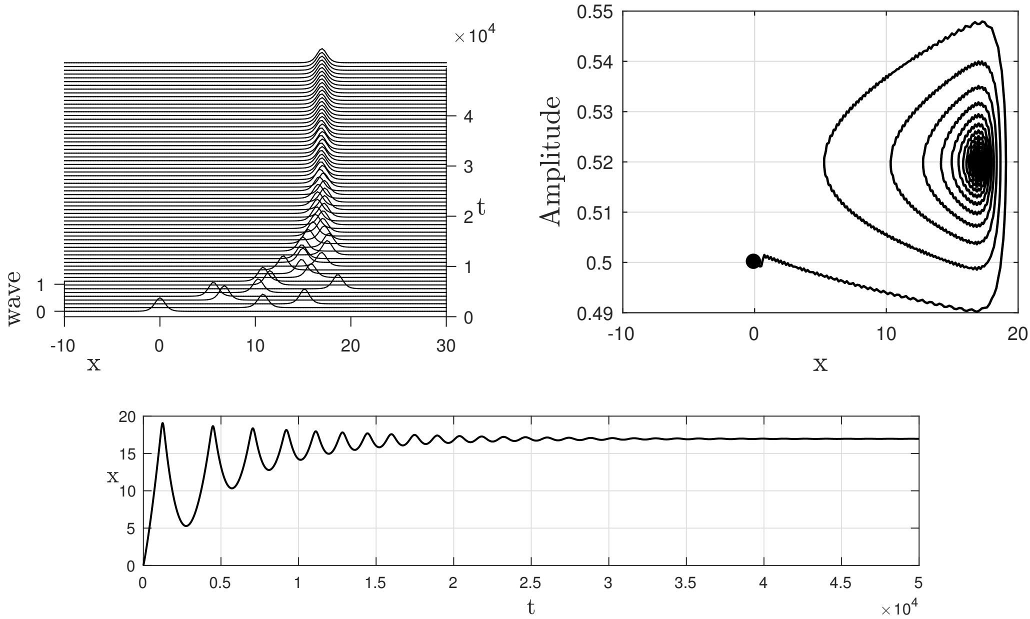

Figure 3: Top (left): trapped solitary wave over the obstacle. Top (right): amplitude of the trapped wave as a function of its crests position. Bottom: the crest position of the solitary wave as a function of time. Parameters: $A = 0.5$, $f = 0.3377$ and $\Omega = -1.3$.

We point out that for each value of Ω fixed in the interval $(-1.7, 0)$, the rotational solitary wave approaches a wave-limit at large times. Although, these wave-limits are close to each other, they are not the same. For each value of $\Omega \in (-1.7, 0)$ we have a different wave-limit. Figure 4 shows the crest position of rotational solitary waves as a function of time for different values of Ω. As we can see, the solution reaches an equilibrium faster when the vorticity is stronger. In addition, when it becomes larger ($\Omega < -1.7$) rotational waves are no longer blocked. These waves seem to have enough energy to pass over the obstacle without reversing their direction. We call them passage waves.

The results presented here help us understand some important features of trapped rotational solitary waves over a submerged obstacle. Although the fKdV is a reduced model, its solutions agree well with solutions of the full model when the obstacle has small amplitude and the channel is shallow [5]. Thus, we conjecture that the results present here still hold for the full Euler equations.

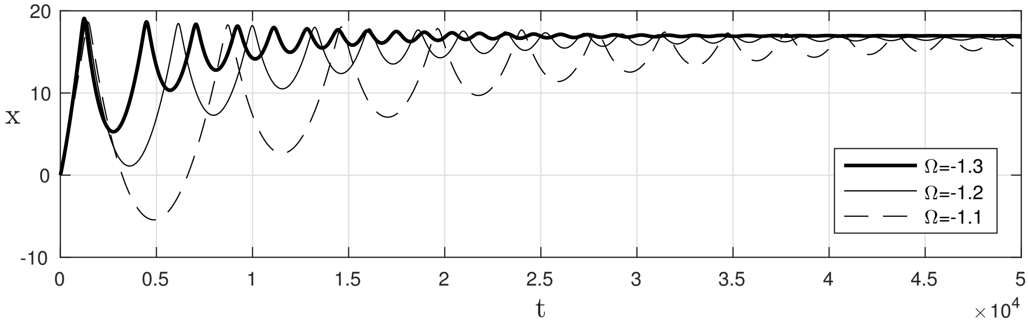

Figure 4: The crest position of solitary waves as a function of time for different values of Ω. Parameters: $A = 0.5$, $f = 0.3377$.

5 CONCLUSIONS

In this paper, we have investigated rotational solitary wave interactions with a submerged obstacle in a vertically sheared channel. We have considered the weakly nonlinear, weakly dispersive regime which allowed us to formulate the problem in rotational fKdV framework. Numerically, we investigated how the sheared flow affects the interaction between solitary waves and an obstacle. These interactions were classified into thee types according to the intensity of the vorticity, namely, trapped waves, blocked waves and passage waves. Moreover, we found that the vorticity acts as a damping in the solitary-wave crest position vs. time space, which resembles a damped spring-mass system. This study is the first step in understanding the dynamic of rotational solitary waves over submerged obstacles, with further investigations to be pursued in the future.

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