An Acad Bras Cienc (2023) 95(Suppl. 2): e20230345 DOI 10.1590/0001-3765202320230345
Anais da Academia Brasileira de Ciências | *Annals of the Brazilian Academy of Sciences*
Printed ISSN 0001-3765 I Online ISSN 1678-2690
www.scielo.br/aabc | www.fb.com/aabcjournal

MATHEMATICAL SCIENCES

Integral inequalities for closed linear Weingarten submanifolds in the product spaces

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Abstract: An integral inequality for closed linear Weingarten m-submanifolds with parallel normalized mean curvature vector field (*pnmc lw-submanifolds*) in the product spaces $M^n(c) \times \mathbb{R}$, $n > m \geq 4$, where $M^n(c)$ is a space form of constant sectional curvature $c \in \{-1, 1\}$, is proved. As an application is shown that the sharpness in this inequality is attained in the totally umbilical hypersurfaces, and in a certain family of standard product of the form $\mathbb{S}^1(\sqrt{1-r^2}) \times \mathbb{S}^{m-1}(r)$ with $0 < r < 1$ when $c = 1$. In the case where $c = -1$, is obtained an integral inequality whose sharpness is attained only in the totally umbilical hypersurfaces. When $m = 2$ and $m = 3$, an integral inequality is also obtained with equality happening in the totally umbilical hypersurfaces.

Key words: Closed pnmc lw-submanifolds, product spaces, totally umbilical hypersurfaces, standard product.

INTRODUCTION

Within the theory of isometric immersions, the characterization of closed submanifolds (compact with empty boundaries) with one of their constant curvatures using integral inequalities constitutes a classical research topic. Notable among these is Simons' integral inequality (see Simons 1968), which establishes a relationship between the squared norm of the second fundamental form and the dimension and codimension of the minimal submanifold in the unit sphere. It is worth highlighting that Simons' tool has proven effective not only in the study of minimal closed submanifolds in the sphere but also in the investigation of submanifolds with other constant curvatures, as well as in more general ambient spaces (see, for example, Chern et al. 1970, Lawson 1969, Ôtsuki 1970, dos Santos & da Silva 2021, and references therein).

In the context of hypersurfaces, Cheng & Yau (1977) investigated the rigidity of hypersurfaces with constant scalar curvature in a space form. They introduced a new second-order differential operator known as the square operator. Building upon Cheng-Yau's technique, Li (1996) studied the pinching problem concerning the square norm of the second fundamental form for complete hypersurfaces with constant scalar curvature. Later, Wei (2008) derived a Simons' type integral inequality for closed k-minimal rotational hypersurfaces immersed in $\mathbb{S}^{m+1}$, characterizing the equality through the standard product $\mathbb{S}^1(\sqrt{1-r^2}) \times \mathbb{S}^{m-1}(r)$. In higher codimension, Guo & Li (2013) extended the results of Li (1996) and showed that the only closed submanifolds with parallel normalized mean curvature (pnmc) in the unit sphere $\mathbb{S}^{m+p}$ with constant scalar curvature, and whose second fundamental form

2020 Mathematics Subject Classification: Primary 53C42; Secondary 53A10, 53C20.

satisfies appropriate boundedness, are the totally umbilical sphere $\mathbb{S}^m(r)$ and the standard product $\mathbb{S}^1(\sqrt{1-r^2}) \times \mathbb{S}^{m-1}(r)$.

Recently, Alías & Meléndez (2020) studied the rigidity of closed hypersurfaces with constant scalar curvature isometrically immersed in $\mathbb{S}^{m+1}$. In particular, they established a sharp Simons-type integral inequality for the squared norm of the traceless second fundamental form, with equality characterizing the totally umbilical hypersurfaces and the standard product $\mathbb{S}^1(\sqrt{1-r^2}) \times \mathbb{S}^{m-1}(r)$. More recently, by using the approach developed by Alías & Meléndez (2020), dos Santos & da Silva (2021) generalized the sharp Simons-type integral inequality of Alías & Meléndez (2020) for pnmc submanifolds immersed in the Riemannian product space $\mathbb{S}^n \times \mathbb{R}$ having constant second mean curvature. As an application, they showed that the sharpness in this inequality is attained in the totally umbilical hypersurfaces, and in a certain family of standard product of the form $\mathbb{S}^1(\sqrt{1-r^2}) \times \mathbb{S}^{m-1}(r) \subset \mathbb{S}^{m+1} \times \{t_0\} \hookrightarrow \mathbb{S}^n \times \mathbb{R}$, for some $t_0 \in \mathbb{R}$ with $n > m \geq 4$.

On the other hand, a natural extension of the submanifolds with constant second mean curvature is the linear Weingarten submanifolds. A submanifold is said to be linear Weingarten (here we will denote by lw-submanifolds) when the first and the second mean curvatures satisfy a certain linear relation. Here, we deal with m-dimensional closed pnmc lw-submanifolds immersed in a Riemannian product space $M^n(c) \times \mathbb{R}$, where $M^n(c)$ is a space form of constant sectional curvature $c = -1, 1$ with $n > m \geq 4$. In this setting, we extend the technique developed by the first two authors in dos Santos & da Silva (2021, 2022) in order to prove a sharp integral inequality for pnmc lw-submanifolds obtaining natural generalizations of the main results of Alías & Meléndez (2020) and dos Santos & da Silva (2021). Furthermore, we also obtain integral inequalities when $c = -1$, which is not contemplated in dos Santos & da Silva (2021).

This manuscript is organized as follows: In Section 1, we provide a brief review of fundamental concepts related to submanifolds immersed in a Riemannian product space $M^n(c) \times \mathbb{R}$. Subsequently, we establish a Simons' type formula for pnmc lw-submanifolds in $M^n(c) \times \mathbb{R}$ (see Proposition 1.2). In Section 2, we present auxiliary lemmas concerning pnmc lw-submanifolds in $M^n(c) \times \mathbb{R}$. Moving on to Section 3, we provide a lower estimate for a Cheng-Yau modified operator acting on the square norm of the traceless second fundamental form of such submanifolds (see Proposition 3.1). We then apply this result to establish our characterization theorems for closed pnmc lw-submanifolds in $M^n(c) \times \mathbb{R}$ with a constant angle between the normalized mean curvature and the unit vector field tangent to $\mathbb{R}$ (see Theorems 3.3 and 3.4). Finally, in the last section, we examine the cases of two and three dimensions (see Theorems 4.1 and 4.2).

1 - A SIMONS TYPE FORMULA FOR SUBMANIFOLDS IN $M^N(C) \times \mathbb{R}$

Along this manuscript, we will always deal with an m-dimensional connected submanifold Σ^m immersed in a Riemannian manifold $\overline{M}^{n+1}$ with $n \geq m$. We choose a local field of orthonormal frames $e_1, \dots, e_{n+1}$ in $\overline{M}^{n+1}$, with dual coframes $\omega_1, \dots, \omega_{n+1}$, such that, at each point of Σ^m, $e_1, \dots, e_m$ are tangent to Σ^m and $e_{m+1}, \dots, e_{n+1}$ are normal to Σ^m. We will use the following convention of indices:

$$1 \leq i, j, k, \dots \leq m \quad \text{and} \quad m+1 \leq \alpha, \beta, \gamma, \dots \leq n+1.$$

Now, restricting all the tensors to Σ^m, $\omega_\alpha = 0$ on Σ^m. Hence, $\sum_i \omega_{\alpha i} \wedge \omega_i = d\omega_\alpha = 0$ and as it is well known we get

$$\omega_{\alpha i} = \sum_j h_{ij}^\alpha \omega_j \quad \text{and} \quad h_{ij}^\alpha = h_{ji}^\alpha. \tag{1}$$

This gives

$$A = \sum_{\alpha,i,j} h_{ij}^\alpha \omega_i \otimes \omega_j e_\alpha, \quad h_{ij}^\alpha = \langle A_\alpha(e_i), e_j\rangle = \langle A(e_i, e_j), e_\alpha\rangle \tag{2}$$

with A denoting the second fundamental form of Σ^m in $\overline{M}^{n+1}$. The square length of the shape operator is

$$|A|^2 = \sum_\alpha |A_\alpha|^2 = \sum_{\alpha,i,j} (h_{ij}^\alpha)^2. \tag{3}$$

Furthermore, we define the mean curvature vector h and the mean curvature function H of Σ^m in $\overline{M}^{n+1}$, respectively by

$$h = \frac{1}{m}\sum_\alpha \operatorname{tr}(A_\alpha) e_\alpha \quad \text{and} \quad H = |h| = \frac{1}{m}\sqrt{\sum_\alpha \operatorname{tr}(A_\alpha)^2}, \tag{4}$$

where $\operatorname{tr}(A_\alpha) = \sum_i h_{ii}^\alpha$.

As it is well known, the basic equations of the submanifolds are the Gauss equation

$$R_{ijkl} = \overline{R}_{ijkl} + \sum_\beta (h_{ik}^\beta h_{jl}^\beta - h_{il}^\beta h_{jk}^\beta), \tag{5}$$

where $\overline{R}_{ijkl}$ and R_{ijkl} are the components of the curvature tensor of $\overline{M}^{n+1}$ and Σ^m, respectively, the Ricci equation

$$R_{\alpha\beta ij}^\perp = \overline{R}_{\alpha\beta ij} + \sum_k (h_{ik}^\alpha h_{kj}^\beta - h_{kj}^\alpha h_{ik}^\beta), \tag{6}$$

where $R_{\alpha\beta ij}^\perp$ are the components of the normal curvature tensor of Σ^m, and the Codazzi equation

$$h_{ijk}^\alpha - h_{ikj}^\alpha = -\overline{R}_{\alpha ijk}. \tag{7}$$

where h_{ijk}^α denote the first covariant derivatives of h_{ij}^α. Additionally,

$$|\nabla A|^2 = \sum_{\alpha,i,j,k} (h_{ijk}^\alpha)^2, \tag{8}$$

where ∇ denotes the covariant derivative of the second fundamental form A. In particular, we say that Σ^m is a parallel submanifold of $\overline{M}^{n+1}$ when $\nabla A = 0$ (see van der Veken & Vrancken 2008).

In this setting, the following Simons-type formula is well-known (see dos Santos & da Silva 2021, 2022):

Proposition 1.1. *Let Σ^m a submanifold immersed isometrically in a Riemannian manifold $\overline{M}^{n+1}$, $n \geq m$. Then, we have*

$$\begin{aligned}\frac{1}{2}\Delta|A|^2 = |\nabla A|^2 &+ \sum_{\alpha,i,j,k} h^\alpha_{ij}\left(h^\alpha_{kkij} - \overline{R}_{\alpha ikjk} - \overline{R}_{\alpha kkij}\right)\\ &+ \sum_{\alpha,\beta,i,j,k} h^\alpha_{ij}\left(-h^\beta_{kk}\overline{R}_{\alpha ij\beta} + 2h^\beta_{jk}\overline{R}_{\alpha\beta ki} - h^\beta_{ij}\overline{R}_{\alpha k\beta k} + 2h^\beta_{ki}\overline{R}_{\alpha\beta kj}\right)\\ &- \sum_{\alpha,\beta}\left(N(A_\alpha A_\beta - A_\beta A_\alpha) + [\mathrm{tr}(A_\alpha A_\beta)]^2 - \mathrm{tr}(A_\beta)\mathrm{tr}(A^2_\alpha A_\beta)\right)\\ &+ 2\sum_{\alpha,i,j,k,p} h^\alpha_{pj}\left(h^\alpha_{pk}\overline{R}_{pijk} + h^\alpha_{pj}\overline{R}_{pkik}\right),\end{aligned} \tag{9}$$

where $N(A) = \mathrm{tr}(AA^t)$ for all matrix $A = (a_{ij})$.

From now on, let us consider the case where the ambient space is a product space. Let $\overline{M}^{n+1} = M^n(c) \times \mathbb{R}$ be a product space, where $M^n(c)$ be a connected Riemannian manifold endowed with metric tensor $\langle\,,\rangle_M$ and of constant sectional curvature $c = -1, 1$ and $\mathbb{R}$ is the real line. Thus, the product space $M^n(c) \times \mathbb{R}$ is the differential manifold $M^n(c) \times \mathbb{R}$ endowed with the Riemannian metric

$$\langle v, w\rangle = \langle(\pi_M)_* v, (\pi_M)_* w\rangle_M + \langle(\pi_\mathbb{R})_* v, (\pi_\mathbb{R})_* w\rangle_\mathbb{R}, \tag{10}$$

with $(p,t) \in M^n(c) \times \mathbb{R}$ and $v, w \in T_{(p,t)}(M^n(c) \times \mathbb{R})$, where $\pi_\mathbb{R}$ and π_M denote the projections onto the corresponding factor. Associated with the product space, we know that, the vector field

$$\partial_t := (\partial/\partial_t)\big|_{(p,t)}, \quad (p,t) \in M^n(c) \times \mathbb{R} \tag{11}$$

is parallel and unitary, that is,

$$\overline{\nabla}\partial_t = 0 \quad \text{and} \quad \langle\partial_t, \partial_t\rangle = 1, \tag{12}$$

where $\overline{\nabla}$ is the Levi-Civita connection of the Riemannian metric of $M^n(c) \times \mathbb{R}$. Using the notations established in Fetcu & Rosenberg (2013), we write the decomposition

$$\partial_t = T + N \tag{13}$$

where $T := \partial_t^\top$ and $N := \partial_t^\perp$ denotes, respectively, the tangent and normal parts of the vector field ∂_t on the tangent and normal bundle of the submanifold Σ^m in $M^n(c) \times \mathbb{R}$. Moreover, from (12) and (13), we get the relation

$$1 = \langle\partial_t, \partial_t\rangle = |T|^2 + |N|^2. \tag{14}$$

It is clear that, if $T = 0$ then, ∂_t is normal to Σ^m and, hence Σ^m lies in $M^n(c)$.

Moreover, let us recall that the curvature tensor[1] of $M^n(c) \times \mathbb{R}$ satisfies, (see Daniel 2007),

$$\begin{aligned}\overline{R}(X,Y)Z = c(\langle X,Z\rangle Y - \langle Y,Z\rangle X) &+ c\langle Z,\partial_t\rangle(\langle Y,\partial_t\rangle X - \langle X,\partial_t\rangle Y)\\ &+ c(\langle Y,Z\rangle\langle X,\partial_t\rangle - \langle X,Z\rangle\langle Y,\partial_t\rangle)\partial_t,\end{aligned} \tag{15}$$

where $X, Y, Z \in \mathfrak{X}(M^n(c) \times \mathbb{R})$.

[1] We adopt for the $(1,3)$-curvature tensor the following definition of Chapter 3 of O'Neill (1983): $\overline{R}(X,Y)Z = \overline{\nabla}_{[X,Y]}Z - [\overline{\nabla}_X, \overline{\nabla}_Y]Z$.

In what follows, we will denote by ∇ and $\nabla^{\perp}$, respectively, the tangent and normal Levi-Civita connections along the tangent and normal bundle of Σ^m, a direct computation by (13) give us

$$\nabla_X T = A_N(X) \quad \text{and} \quad \nabla^{\perp}_X N = -A(T,X), \quad \text{for all} \quad X \in \mathfrak{X}(M), \tag{16}$$

where $A_N = \sum_\alpha \langle N, e_\alpha\rangle A_\alpha$ denotes the Weingarten operator in the N direction.

By this digression, our aim now is to get a Simons-type formula for a pnmc lw-submanifold Σ^m in $M^n(c)\times\mathbb{R}$. Firstly, since $M^n(c)\times\mathbb{R}$ locally symmetric, we have $\overline{R}_{\alpha ikjk} = \overline{R}_{\alpha kkij} = 0$. On the other hand, a direct computation from (15), gives $\overline{R}_{\alpha\beta kj} = 0$, for all α, β, j, k. Moreover,

$$\begin{aligned}\sum_{\alpha,\beta,i,j,k} h^\alpha_{ij}\left(h^\beta_{kk}\overline{R}_{\alpha ij\beta} + h^\beta_{ij}\overline{R}_{\alpha k\beta k}\right) = -cm^2H^2 + cm\langle A(T,T), h\rangle - cm|A_N|^2 \\ + cm^2\langle h, N\rangle^2 + c(m-|T|^2)|A|^2\end{aligned} \tag{17}$$

and

$$\begin{aligned}\sum_{\alpha,i,j,k,p} h^\alpha_{pj}\left(h^\alpha_{pk}\overline{R}_{pijk} + h^\alpha_{pj}\overline{R}_{pkik}\right) = -cm\sum_\alpha |A_\alpha(T)|^2 + c(m-|T|^2)|A|^2 \\ - cm^2H^2 + 2cm\langle A(T,T), h\rangle.\end{aligned} \tag{18}$$

Next, we will also consider the traceless second fundamental form

$$\phi = \sum_{\alpha,i,j}\phi^\alpha_{ij}\omega_i\otimes\omega_j e_\alpha, \quad \phi^\alpha_{ij} = \langle\phi_\alpha(e_i), e_j\rangle = h^\alpha_{ij} - \langle h, e_\alpha\rangle\delta_{ij}. \tag{19}$$

It is easy to check that each $\phi_\alpha = A_\alpha - \langle h, e_\alpha\rangle I$ is traceless and that

$$|\phi|^2 = \sum_\alpha |\phi_\alpha|^2 = \sum_{\alpha,i,j}(\phi^\alpha_{ij})^2 = |A|^2 - mH^2. \tag{20}$$

Observe that $|\phi|^2 = 0$ if and only if Σ^m is a totally umbilical submanifold of $M^n(c)\times\mathbb{R}$. Within this context, a standard computation give us

$$m|A_N|^2 = m|\phi_N|^2 + m^2\langle h, N\rangle^2 \tag{21}$$

and

$$\sum_\alpha |A_\alpha(T)|^2 = \sum_\alpha |\phi_\alpha(T)|^2 + 2\langle\phi_h(T), T\rangle + H^2|T|^2. \tag{22}$$

Now, let Σ^m be a *pnmc* submanifolds immersed in product space $M^n(c)\times\mathbb{R}$. This means that $H > 0$ and the normalized mean curvature vector field $\eta = h/H$ is parallel as a section of the normal bundle. In this setting, we will consider $\{e_{m+1}, \dots, e_{n+1}\}$ be a local orthonormal frame field in the normal bundle such that $e_{m+1} = \eta$. By this,

$$\mathrm{tr}(A_\eta) = mH \quad \text{and} \quad \mathrm{tr}(A_\alpha) = m\langle h, e_\alpha\rangle = 0, \quad \text{for all} \quad \alpha \geq m+2, \tag{23}$$

and by (19)

$$\phi^{m+1}_{ij} = h^{m+1}_{ij} - H\delta_{ij} \quad \text{and} \quad \phi^\alpha_{ij} = h^\alpha_{ij}, \quad \text{for all} \quad \alpha \geq m+2. \tag{24}$$

Since η parallel, the Ricci equation (6) guarantees that $A_\alpha A_\eta = A_\eta A_\alpha$ for all $\alpha \geq m+2$. Using this, (20) and (24),

$$\begin{aligned}\sum_{\alpha,\beta} \mathrm{tr}(A_\beta)\mathrm{tr}(A_\alpha^2 A_\beta) - \sum_{\alpha,\beta}\Big(N(A_\alpha A_\beta - A_\beta A_\alpha) + [\mathrm{tr}(A_\alpha A_\beta)]^2\Big) &= mH^2|\phi|^2 + mH\sum_\alpha \mathrm{tr}(\phi_\alpha^2\phi_\eta) \\ &\quad - \sum_{\alpha,\beta>m+1} N(\phi_\alpha\phi_\beta - \phi_\beta\phi_\alpha) - \sum_{\alpha,\beta}\Big[\mathrm{tr}(\phi_\alpha\phi_\beta)\Big]^2.\end{aligned} \tag{25}$$

Therefore, inserting (17), (18), (21), (22) and (25) in Proposition 1.1 we get

$$\begin{aligned}\frac{1}{2}\Delta|A|^2 &= |\nabla A|^2 + m\sum_{i,j} h_{ij}^{m+1}H_{ij} + cm|\phi_N|^2 - 2cm\sum_\alpha |\phi_\alpha(T)|^2 \\ &\quad + \Big(c(m-|T|^2) + mH^2\Big)|\phi|^2 - cm\langle\phi_h(T),T\rangle + mH\sum_\alpha \mathrm{tr}(\phi_\alpha^2\phi_\eta) \\ &\quad - \sum_{\alpha,\beta>m+1} N(\phi_\alpha\phi_\beta - \phi_\beta\phi_\alpha) - \sum_{\alpha,\beta}\Big[\mathrm{tr}(\phi_\alpha\phi_\beta)\Big]^2.\end{aligned} \tag{26}$$

According to Grosjean (2002) and Cao & Li (2007), we define the *r-th mean curvature function* H_r of an m-dimensional submanifold immersed in a Riemannian space, as follows: for any even integer $r \in \{0, 1, \dots, m-1\}$, the r-th are given by

$$\binom{n}{r}H_r := S_r = \frac{1}{r!}\sum_{\substack{i_1\dots i_r\\ j_1\dots j_r}} \delta_{j_1\dots j_r}^{i_1\dots i_r}\langle B_{i_1j_1}, B_{i_2j_2}\rangle\cdots\langle B_{i_{r-1}j_{r-1}}, B_{i_rj_r}\rangle, \tag{27}$$

where $\binom{n}{r}$ is the binomial coefficient, $\delta_{j_1\dots j_r}^{i_1\dots i_r}$ is the generalized Kronecker symbol and $B_{ij} = \sum_{\alpha,i,j} h_{ij}^\alpha e_\alpha$ with $\{e_{m+1}, \dots, e_{n+1}\}$ an orthonormal frame on the normal bundle. By convention, $H_0 = S_0 = 1$. For our study on submanifolds Σ^m in the product space $M^n(c)\times\mathbb{R}$, we will consider the *second mean curvature function* H_2, which is given by

$$m(m-1)H_2 = 2S_2 = m^2H^2 - |A|^2. \tag{28}$$

On the other hand, a natural extension of submanifolds having constant second mean curvature is the so-called linear Weingarten, in short, lw-submanifolds. A submanifold is said to be *linear Weingarten* red if its first and second mean curvatures are linearly related, that is,

$$H_2 = aH + b \tag{29}$$

for constants $a, b \in \mathbb{R}$. Observe that when $a = 0$, (29) reduces to H_2 constant.

For the study of the lw-submanifolds, we will consider the following Cheng-Yau's modified differential operator given by

$$L(u) = \sum_{i,j}\left[\left(mH - \frac{m-1}{2}a\right)\delta_{ij} - h_{ij}^{m+1}\right]u_{ij} = \left(mH - \frac{m-1}{2}a\right)\Delta u - \sum_{i,j} h_{ij}^{m+1}u_{ij}, \tag{30}$$

where u_{ij} stands for a component of the Hessian of $u \in C^2(M)$. From the tensorial point of view, (30) can be written as

$$L(u) = \mathrm{tr}(P\circ \mathrm{Hess}\, u), \tag{31}$$

with

$$P = \left(mH - \frac{m-1}{2}a\right)I - h^{m+1} \tag{32}$$

where I is the identity in the algebra of smooth vector fields on Σ^m and $h^{m+1} = (h_{ij}^{m+1})$ denotes the second fundamental form of Σ^m in the direction e_{m+1}. By (31), it is not difficult to see that

$$L(uv) = uL(v) + vL(u) + 2\langle P(\nabla u), \nabla v\rangle \tag{33}$$

for every $u, v \in C^2(M)$ and

$$L(f(u)) = f'(u)L(u) + f''(u)\langle P(\nabla u), \nabla u\rangle \tag{34}$$

for every smooth function $f : \mathbb{R} \to \mathbb{R}$.

Hence, taking $u = mH$ in (30), by (28) and (29), we obtain

$$\begin{aligned} L(mH) &= \sum_{i,j}\left[\left(mH - \frac{m-1}{2}a\right)\delta_{ij} - h_{ij}^{m+1}\right](mH)_{ij} \\ &= mH\Delta(mH) - \frac{m(m-1)}{2}\Delta(aH) - m\sum_{i,j} h_{ij}^{m+1}H_{ij} \\ &= \frac{1}{2}\Delta(m^2H^2 - m(m-1)H_2) - m^2|\nabla H|^2 - m\sum_{i,j} h_{ij}^{m+1}H_{ij} \\ &= \frac{1}{2}\Delta|A|^2 - m^2|\nabla H|^2 - m\sum_{i,j} h_{ij}^{m+1}H_{ij}. \end{aligned}$$

From all these results we have the following Simons-type formula for Cheng-Yau's modified operator acting on the mean curvature function of Σ^m in $M^n(c) \times \mathbb{R}$ which generalizes Proposition 2 of dos Santos & da Silva (2022):

Proposition 1.2. *If Σ^m is a pnmc lw-submanifold of $M^n(c) \times \mathbb{R}$, then we have*

$$\begin{aligned} L(mH) = |\nabla A|^2 - m^2|\nabla H|^2 + cm|\phi_N|^2 - 2cm\sum_{\alpha}|\phi_\alpha(T)|^2 \\ + \left(c(m - |T|^2) + mH^2\right)|\phi|^2 - cmH\langle\phi_\eta(T), T\rangle \\ + mH\sum_{\alpha}\mathrm{tr}(\phi_\alpha^2\phi_\eta) - \sum_{\alpha,\beta}\left(N(\phi_\alpha\phi_\beta - \phi_\beta\phi_\alpha) + \left[\mathrm{tr}(\phi_\alpha\phi_\beta)\right]^2\right). \end{aligned}$$

2 - KEY LEMMAS

In this section, we will present some necessary results for the proof of our results. The first ones are extensions of the Lemmas 1 and 2 of dos Santos & da Silva (2022) (see also Lemma 2.3 of dos Santos & da Silva (2021) and Lemmas 4.1 and 4.3 of dos Santos (2021)) to lw-submanifolds.

Lemma 2.1. *Let Σ^m be an lw-submanifold in the product space $M^n(c) \times \mathbb{R}$, such that $H_2 = aH + b$ with*

$$(m-1)a^2 + 4mb \geq 0. \tag{35}$$

Then

$$|\nabla A|^2 \geq m^2|\nabla H|^2. \tag{36}$$

Moreover, if the inequality (35) is strict and the equality occurs in (36), then Σ^m is an open piece of a parallel submanifold of $M^n(c) \times \mathbb{R}$.

Proof. Inserting $H_2 = aH + b$ in (28) we have

$$m^2H^2 = |A|^2 + m(m-1)(aH+b). \tag{37}$$

By taking the derivative in (37),

$$2|A|\nabla|A| = \left(2m^2H - m(m-1)a\right)\nabla H \tag{38}$$

and consequently

$$4|A|^2|\nabla|A||^2 = \left(2m^2H - m(m-1)a\right)^2|\nabla H|^2. \tag{39}$$

It is not difficult to check that

$$\left(2m^2H - m(m-1)a\right)^2 = 4m^2|A|^2 + m^2(m-1)(4mb + (m-1)a^2). \tag{40}$$

Thus by using (35),

$$4|A|^2|\nabla|A||^2 = \left[4m^2|A|^2 + m^2(m-1)(4mb + (m-1)a^2)\right]|\nabla H|^2 \geq 4m^2|A|^2|\nabla H|^2. \tag{41}$$

Now, from Kato's inequality

$$|\nabla|A||^2 \leq |\nabla A|^2 \tag{42}$$

we obtain

$$m^2|A|^2|\nabla H|^2 \leq |A|^2|\nabla|A||^2 \leq |A|^2|\nabla A|^2. \tag{43}$$

Therefore, we have either

$$|A|^2 = 0 \quad \text{and} \quad m^2|\nabla H|^2 = |\nabla A|^2 = 0 \tag{44}$$

or

$$|\nabla A|^2 \geq m^2|\nabla H|^2. \tag{45}$$

If the inequality (35) is strict, from (41) we get

$$\left(2m^2H - m(m-1)a\right)^2 > 4m^2|A|^2. \tag{46}$$

Now, let us assume in addition that the equality holds in (36) on Σ^m. In this case, we wish to show that H is constant on Σ^m. Suppose, by contradiction, that it does not occur. Consequently, there exists a point $p \in \Sigma^m$ such that $|\nabla H(p)| > 0$. So, one deduces from (39) that

$$4|A|^2(p)|\nabla A|^2(p) > 4m^2|A|^2(p)|\nabla H(p)|^2 \tag{47}$$

and, since $|\nabla A|^2(p) = m^2|\nabla H(p)|^2 > 0$, we arrive at a contradiction. Hence, in this case, we conclude that H must be constant on Σ^m. □

Lemma 2.2. *Let Σ^m be a pnmc lw-submanifold in the product space $M^n(c) \times \mathbb{R}$, such that $H_2 = aH + b$ with $b \geq 0$. Then the operator P defined in (32) is positive semidefinite. In the case where $b > 0$, we have that P is positive definite.*

Proof. Let us consider $\{e_1, \dots, e_m\}$ an orthonormal frame on Σ^m such that $h_{ij}^{m+1} = \lambda_i^{m+1}\delta_{ij}$. Since $b \geq 0$, from (37), we have

$$m^2H^2 = |A|^2 + m(m-1)(aH+b) \geq (\lambda_i^{m+1})^2 + m(m-1)aH, \tag{48}$$

for each principal curvature λ_i^{m+1} of Σ^m, $i = 1, \dots, m$.

On the other hand, with a straightforward computation, we verify that

$$\begin{aligned}(\lambda_i^{m+1})^2 \leq m^2H^2 - m(m-1)aH &= \left(mH - \frac{m-1}{2}a\right)^2 - \frac{(m-1)^2}{4}a^2 \\ &\leq \left(mH - \frac{m-1}{2}a\right)^2.\end{aligned} \tag{49}$$

Now, we claim that $mH - \frac{m-1}{2}a \geq 0$. For this, let us consider two cases. When $a \leq 0$, our assertion is immediate. Otherwise, if $a > 0$, from (37) we see that

$$mH(mH-(m-1)a) = |A|^2 + m(m-1)b > 0, \tag{50}$$

since Σ^m is a pnmc submanifold. Thus, $mH - (m-1)a > 0$ and consequently, $mH - \frac{m-1}{2}a \geq 0$ as claimed.

So, from (49) we obtain

$$-mH + \frac{m-1}{2}a \leq \lambda_i^{m+1} \leq mH - \frac{m-1}{2}a, \qquad i = 1, \dots, m, \tag{51}$$

and hence, for each $i \in \{1, \dots, m\}$

$$0 \leq mH - \frac{m-1}{2}a - \lambda_i \leq 2mH - (m-1)a. \tag{52}$$

Since $mH - \frac{m-1}{2}a - \lambda_i$ are the eigenvalues of P, follows that P is positive semidefinite. Similarly if $b > 0$. □

Given a unit normal vector field $\xi \in X(\Sigma)^\perp$, we say that a submanifold Σ^m of $M^n(c) \times \mathbb{R}$ has *constant ξ-angle* if the angle between ξ and ∂_t is constant, that is, the function $\langle \xi, \partial_t \rangle$ is constant along of Σ^m. We should notice that constant η-angle submanifolds, where $\eta = h/H$, corresponds to a natural extension of hypersurfaces with constant angle in a product space, which was widely studied by Dillen and many other authors (see, for instance, Dillen et al. 2007, Dillen & Munteanu 2009, Navarro et al. 2016, Nistor 2017). By using this context, the next result is a suitable adaptation of Lemma 2.1 of dos Santos & da Silva (2021) which assures that the integral of the L operator acting on any nonnegative function is equal to zero.

Lemma 2.3. *Let Σ^m be a closed pnmc lw-submanifold in $M^n(c) \times \mathbb{R}$ such that $H_2 = aH + b$. If Σ^m has constant η-angle, then this angle is always zero. Moreover*

$$\int_\Sigma L(u)d\Sigma = 0, \tag{53}$$

for all nonnegative functions $u \in C^2(\Sigma)$.

Proof. By a standard tensorial computation, it is not difficult to see that

$$L(u) = \operatorname{div}(P(\nabla u)) - \langle \operatorname{div}(P), \nabla u \rangle, \tag{54}$$

for every $u \in C^2(M)$, where

$$\mathrm{div}(P) = \sum_i (\nabla_{e_i} P)e_i = \mathrm{tr}(\nabla P), \tag{55}$$

with ∇P is defined as

$$\nabla P(X, Y) = (\nabla_X P)Y = \nabla_X P(Y) - P(\nabla_X Y), \quad X, Y \in TM. \tag{56}$$

From this and (32), we write

$$\nabla P(e_i, e_i) = m\langle \nabla H, e_i\rangle e_i - \nabla h^{m+1}(e_i, e_i), \tag{57}$$

where $\{e_1, \dots, e_{n+1}\}$ an orthonormal frame on $M^n \times \mathbb{R}$ adapted to Σ^m, that is, $\{e_1, \dots, e_m\}$ are tangent to Σ^m and choose $e_{m+1} = \eta$. By Codazzi equation (7),

$$\begin{aligned}\langle \nabla h^{m+1}(e_i, e_i), X\rangle &= \langle (\nabla_{e_i} h^{m+1})e_i, X\rangle = \langle e_i, (\nabla_{e_i} h^{m+1})X\rangle \\ &= \langle (\nabla_X h^{m+1})e_i, e_i\rangle - \sum_j \langle X, e_j\rangle \overline{R}_{(m+1)iji},\end{aligned} \tag{58}$$

for all $X \in TM$. By using (15), a direct computation give us

$$\sum_j \langle X, e_j\rangle \overline{R}_{(m+1)iji} = c\langle \eta, \partial_t\rangle \big(\langle e_i, T\rangle\langle X, e_i\rangle - \langle T, X\rangle\langle e_i, e_i\rangle\big). \tag{59}$$

Hence,

$$\mathrm{div}(P) = m\nabla H - m\nabla H - c(m-1)\langle \eta, \partial_t\rangle T = -c(m-1)\langle \eta, \partial_t\rangle T. \tag{60}$$

On the other hand, we take the vector field $X = uT$. Computing its divergence, we obtain

$$\mathrm{div}(X) = u\,\mathrm{div}(T) + T(u) = u\mathrm{div}(T) + \langle T, \nabla u\rangle. \tag{61}$$

By (16), $\mathrm{div}(T) = m\langle h, \partial_t\rangle$. So,

$$\mathrm{div}(X) = um\langle h, \partial_t\rangle + \langle T, \nabla u\rangle. \tag{62}$$

Since Σ^m has constant η-angle, we get

$$\mathrm{div}(\langle \eta, \partial_t\rangle X) = um\langle \eta, \partial_t\rangle\langle h, \partial_t\rangle + \langle \eta, \partial_t\rangle\langle T, \nabla u\rangle. \tag{63}$$

Therefore, as $\langle h, \partial_t\rangle = H\langle \eta, \partial_t\rangle$, from (54), (62) and (63),

$$\mathrm{div}\,(P(\nabla u)) = L(u) - (m-1)\mathrm{div}(\langle \eta, \partial_t\rangle X) + m(m-1)uH\langle \eta, \partial_t\rangle^2. \tag{64}$$

Taking into account Stokes' Theorem,

$$\int_\Sigma L(u)d\Sigma = -cm(m-1)\langle \eta, \partial_t\rangle^2 \int_\Sigma uHd\Sigma. \tag{65}$$

Finally, let us choose u a positive constant function. Since $H > 0$, from (65) we must have $\langle \eta, \partial_t\rangle = 0$. Therefore, inserting this in (65) we obtain the result. □

The following two results are fundamental to our study and can be found in Li & Li (1992) and Santos (1994), respectively.

Lemma 2.4. *Let $B_1, \ldots, B_p$, where $p \geq 2$, be symmetric $m \times m$ matrices. Then*

$$\sum_{\alpha,\beta=1}^{p} \left(N(B_\alpha B_\beta - B_\beta B_\alpha) + [\mathrm{tr}(B_\alpha B_\beta)]^2 \right) \leq \frac{3}{2} \left(\sum_{\alpha=1}^{p} N(B_\alpha) \right)^2 .$$

Lemma 2.5. *Let $B, C : \mathbb{R}^m \to \mathbb{R}^m$ be symmetric linear maps that $[B, C] = 0$ and $\mathrm{tr}(B) = \mathrm{tr}(C) = 0$, then*

$$-\frac{m-2}{\sqrt{m(m-1)}} |B|^2 |C| \leq \mathrm{tr}(B^2 C) \leq \frac{m-2}{\sqrt{m(m-1)}} |B|^2 |C|.$$

Moreover, the equality holds if and only if $(m-1)$ of the eigenvalues x_i of B and corresponding eigenvalues y_i of C satisfy

$$|x_i| = \sqrt{\frac{N(B)}{m(m-1)}}, \; x_i y_i \geq 0 \quad \textit{and} \quad y_i = \sqrt{\frac{N(C)}{m(m-1)}} \left(\textit{resp.} -\sqrt{\frac{N(C)}{m(m-1)}} \right).$$

We will conclude this section by quoting the following codimension reduction result for submanifolds in the product space $M^n(c) \times \mathbb{R}$, see Lemma 1.6 of Mendonça & Tojeiro (2013).

Lemma 2.6. *Let Σ^m be a submanifold of $M^n(c) \times \mathbb{R}$ and let N be the normal vector field defined by* (13). *Assume that $L := N_1 + \mathrm{span}\{N\}$ is a subbundle of $T\Sigma^\perp$ with rank $q < n - m + 1$ and that $\nabla^\perp N_1 \subset L$, where N_1 denotes the first normal subspace of Σ^m. Then the codimension of Σ^m reduces to q, that is, Σ^m is contained in a totally geodesic submanifold $M^{m+q-1}(c) \times \mathbb{R}$ of $M^n(c) \times \mathbb{R}$.*

3 - MAIN RESULTS

In our first result, we obtain a suitable lower estimate for the operator L applied on the squared norm of the traceless operator of a lw-submanifold, which will be also essential to the proofs of our main results.

Proposition 3.1. *Let Σ^m be a pnmc lw-submanifold in a product space $M^n(c) \times \mathbb{R}$, $n > m \geq 4$, such that $H_2 = aH + b$ with $a, b \geq 0$. Then*

$$L(|\phi|^2) \geq -2(m-1) \left(|\phi|^2 \varphi_{a,b,c}(|\phi|, |T|) - Q_c \right) \sqrt{\frac{|\phi|^2}{m(m-1)} + b + \frac{a^2}{4}}, \tag{66}$$

where

$$Q_c = cm|\phi_N|^2 - 2cm \sum_\alpha |\phi_\alpha(T)|^2 - cmH\langle \phi_\eta(T), T \rangle \tag{67}$$

and

$$\begin{aligned} \varphi_{a,b,c}(x,y) = \frac{m-2}{m-1} x^2 + cy^2 - m \left(a - \frac{m-2}{\sqrt{m(m-1)}} x \right) \sqrt{\frac{x^2}{m(m-1)} + b + \frac{a^2}{4}} \\ + \frac{m(m-2)a}{2\sqrt{m(m-1)}} x - m \left(\frac{a^2}{2} + b + c \right). \end{aligned} \tag{68}$$

In particular, if $b > 0$ and equality holds in (66)*, then Σ^m is a part of a parallel submanifold in $M^n(c) \times \mathbb{R}$ with two distinct principal curvatures, one of which is simple.*

Proof. From Cauchy Schwarz's inequality and Lemma 2.5, we get

$$\sum_{\alpha,\beta}[\mathrm{tr}(\phi_\alpha\phi_\beta)]^2 \leq |\phi_\eta|^4 + 2|\phi_\eta|^2\left(|\phi|^2 - |\phi_\eta|^2\right) + \sum_{\alpha,\beta>m+1}[\mathrm{tr}(\phi_\alpha\phi_\beta)]^2, \tag{69}$$

and

$$mH\sum_{\alpha}\mathrm{tr}(\phi_\alpha^2\phi_\eta) \geq -\frac{m(m-2)}{\sqrt{m(m-1)}}H|\phi|^2|\phi_\eta|. \tag{70}$$

Besides these, by Lemma 2.4, we also can estimate

$$\sum_{\alpha,\beta>m+1}\left(N(\phi_\alpha\phi_\beta - \phi_\beta\phi_\alpha) + [\mathrm{tr}(\phi_\alpha\phi_\beta)]^2\right) \leq \frac{3}{2}\left(\sum_{\alpha>m+1}|\phi_\alpha|^2\right)^2 = \frac{3}{2}\left(|\phi|^2 - |\phi_\eta|^2\right)^2. \tag{71}$$

Then, inequalities (69), (70) and (71), becomes in

$$\begin{aligned} mH\sum_{\alpha}\mathrm{tr}(\phi_\alpha^2\phi_\eta) - \sum_{\alpha,\beta\neq m+1}N(\phi_\alpha\phi_\beta - \phi_\beta\phi_\alpha) - \sum_{\alpha,\beta}[\mathrm{tr}(\phi_\alpha\phi_\beta)]^2 \\ \geq -\frac{m(m-2)}{\sqrt{m(m-1)}}H|\phi|^2|\phi_\eta| - |\phi_\eta|^4 \\ -2|\phi_\eta|^2\left(|\phi|^2 - |\phi_\eta|^2\right) - \frac{3}{2}\left(|\phi|^2 - |\phi_\eta|^2\right)^2. \end{aligned} \tag{72}$$

After some standard computations, we can express (72) as follows:

$$\begin{aligned} mH^2|\phi|^2 - \frac{m(m-2)}{\sqrt{m(m-1)}}H|\phi|^2|\phi_\eta| - \frac{3}{2}|\phi|^4 + |\phi|^2|\phi_\eta|^2 - \frac{1}{2}|\phi_\eta|^4 \\ = (|\phi| - |\phi_\eta|)\left(\frac{m(m-2)}{\sqrt{m(m-1)}}H|\phi|^2 - \frac{1}{2}(|\phi| - |\phi_\eta|)(|\phi| + |\phi_\eta|)^2\right) \\ + |\phi|^2\left(-|\phi|^2 - \frac{m(m-2)}{\sqrt{m(m-1)}}H|\phi| + mH^2\right). \end{aligned} \tag{73}$$

Hence, by replacing (73) into Proposition 1.2,

$$\begin{aligned} L(mH) \geq |\nabla A|^2 - m^2|\nabla H|^2 + cm|\phi_N|^2 - 2cm\sum_{\alpha}|\phi_\alpha(T)|^2 - cmH\langle\phi_\eta(T), T\rangle \\ + (|\phi| - |\phi_\eta|)\left(\frac{m(m-2)}{\sqrt{m(m-1)}}H|\phi|^2 - \frac{1}{2}(|\phi| - |\phi_\eta|)(|\phi| + |\phi_\eta|)^2\right) \\ + |\phi|^2\left(-|\phi|^2 - \frac{m(m-2)}{\sqrt{m(m-1)}}H|\phi| + c(m - |T|^2) + mH^2\right). \end{aligned} \tag{74}$$

On the other hand, from (20) and (37), we write

$$H^2 = \frac{1}{m(m-1)}|\phi|^2 + aH + b, \tag{75}$$

and since $a, b \geq 0$ we obtain

$$H \geq \frac{1}{\sqrt{m(m-1)}}|\phi|. \tag{76}$$

Moreover, the following inequality is well known (see Equation 3.5 of Guo & Li 2013)

$$(|\phi| - |\phi_\eta|)(|\phi| + |\phi_\eta|)^2 \leq \frac{32}{27}|\phi|^3. \tag{77}$$

Thus, from (76) and (77) we conclude that

$$\frac{m(m-2)}{\sqrt{m(m-1)}}H|\phi|^2 - \frac{1}{2}(|\phi| - |\phi_\eta|)(|\phi| + |\phi_\eta|)^2 \geq \left(\frac{m-2}{m-1} - \frac{16}{27}\right)|\phi|^3. \tag{78}$$

Assuming that $m \geq 4$,

$$\frac{m-2}{m-1} - \frac{16}{27} > 0. \tag{79}$$

Therefore, from (78) and (79), (74) becomes

$$\begin{aligned} L(mH) \geq\ & |\nabla A|^2 - m^2|\nabla H|^2 + cm|\phi_N|^2 - 2cm\sum_\alpha |\phi_\alpha(T)|^2 - cmH\langle \phi_\eta(T), T\rangle \\ & + |\phi|^2\left(-|\phi|^2 - \frac{m(m-2)}{\sqrt{m(m-1)}}H|\phi| + c(m - |T|^2) + mH^2\right). \end{aligned} \tag{80}$$

On the other hand, from (50) we have $mH - (m-1)a > 0$. Since $m \geq 4$, it follows that $H - \frac{a}{2} \geq \frac{1}{m}(mH - (m-1)a) > 0$. Consequently, by making a direct computation, (75) can be written as follows:

$$H - \frac{a}{2} = \sqrt{\frac{|\phi|^2}{m(m-1)} + b + \frac{a^2}{4}}. \tag{81}$$

By using this and (75) we can write

$$\begin{aligned} & -|\phi|^2 - \frac{m(m-2)}{\sqrt{m(m-1)}}H|\phi| + c(m - |T|^2) + mH^2 \\ & = -\frac{m-2}{m-1}|\phi|^2 - \frac{m(m-2)}{\sqrt{m(m-1)}}|\phi|\left(\sqrt{\frac{|\phi|^2}{m(m-1)} + b + \frac{a^2}{4}} + \frac{a}{2}\right) \\ & \quad + ma\sqrt{\frac{|\phi|^2}{m(m-1)} + b + \frac{a^2}{4}} + m\left(\frac{a^2}{2} + b\right) + c(m - |T|^2) \\ & = -|\phi|^2\varphi_{a,b,c}(|\phi|, |T|), \end{aligned} \tag{82}$$

where $\varphi_{a,b,c}$ is a real function defined in (68). Since $b \geq 0$, Lemma 2.1 assures that

$$|\nabla A|^2 - m^2|\nabla H|^2 \geq 0. \tag{83}$$

Therefore, inserting (83) and (82) into (80), we obtain.

$$L(mH) \geq cm|\phi_N|^2 - cmH\langle \phi_\eta(T), T\rangle - 2mc\sum_\alpha |\phi_\alpha(T)|^2 - |\phi|^2\varphi_{a,b,c}(|\phi|, |T|), \tag{84}$$

where $\varphi_{a,b,c}$ is defined in (68).

Now, Lemma 2.2 guarantees that the operator P is positive definite since $b \geq 0$. So, by (33) and (75), we can write

$$\frac{1}{m-1}L(|\phi|^2) = 2HL(mH) + 2m\langle P(\nabla H), \nabla H\rangle - aL(mH) \geq 2\left(H - \frac{a}{2}\right)L(mH). \tag{85}$$

Hence, by inserting (84) in (85), we get (66).

Finally, if equality holds in (66), considering that $b > 0$ and P is positive definite, we can deduce from (85) that H is constant. Moreover, (83) must also be satisfied as an equality. Since we already established that H is constant, this implies $\nabla A = 0$, indicating that the second fundamental form is parallel. Additionally, in order to achieve equality in Lemma 2.5, (70) must also be an equality. Consequently, we conclude that Σ^m is a parallel submanifold of $M^n(c) \times \mathbb{R}$ with exactly two distinct principal curvatures, one of which is simple. □

Remark 3.2. Since the mean curvature vector field is normalized, it follows that $H > 0$. By using (75),

$$|\phi|^2 = m(m-1)(H^2 - aH - b). \tag{86}$$

If $a = b = 0$ and there exists a point $p \in \Sigma^m$ such that $|\phi|(p) = 0$, then H must vanish, which contradicts the fact that $H > 0$. Therefore, we conclude that $|\phi|$, a, and b cannot vanish simultaneously.

Now, we are ready to give proof of our first result.

Theorem 3.3. *Let Σ^m be a closed pnmc lw-submanifold in $\mathbb{S}^n \times \mathbb{R}$, $n > m \geq 4$, such that $H_2 = aH + b$ with $a, b \geq 0$. If Σ^m has constant η-angle, then*

$$\int_\Sigma |\phi|^{p+2} F_{a,b}(|\phi|, |T|) d\Sigma \geq 0, \tag{87}$$

for every real number $p > 2$, where $F_{a,b}$ is the real function given by

$$\begin{aligned} F_{a,b}(x,y) = \frac{m-2}{m-1}x^2 + (2m+1)y^2 - m\left(a - \frac{m-2}{\sqrt{m(m-1)}}x\right)\sqrt{\frac{x^2}{m(m-1)} + b + \frac{a^2}{4}} \\ + \frac{m(m-2)a}{2\sqrt{m(m-1)}}x - m\left(\frac{a^2}{2} + b + 1\right). \end{aligned} \tag{88}$$

Moreover, if $b > 0$ the equality holds in (87) if and only if:

(i) either Σ^m is a totally umbilical hypersurface in $\mathbb{S}^{m+1} \times \{t_0\} \hookrightarrow \mathbb{S}^n \times \mathbb{R}$ for some $t_0 \in \mathbb{R}$;

(ii) or $|\phi|^2 = \gamma(m,a,b)$, where $\gamma(m,a,b)$ is a positive constant depending only on m, a, b and Σ^m is isometric to a standard product $\mathbb{S}^1(\sqrt{1-r^2}) \times \mathbb{S}^{m-1}(r) \subset \mathbb{S}^{m+1} \times \{t_0\} \hookrightarrow \mathbb{S}^n \times \mathbb{R}$ for some $t_0 \in \mathbb{R}$, with $r = \sqrt{(m-1)/m(H_2+1)} > 0$.

Proof. Firstly, let us take $c = 1$ in Proposition 3.1. By using Cauchy-Schwarz inequality, we get

$$-2m\sum_\alpha |\phi_\alpha(T)|^2 \geq -2m\sum_\alpha |\phi_\alpha|^2|T|^2 = -2m|\phi|^2|T|^2. \tag{89}$$

On the other hand, since If Σ^m has constant η-angle and $\eta = e_{m+1}$ is parallel, by (16),

$$0 = X\langle \eta, \partial_t\rangle = \langle \nabla_X^\perp \eta, N\rangle + \langle \eta, \nabla_X^\perp N\rangle = -\langle A(T,X), \eta\rangle = -\langle A_\eta(T), X\rangle, \tag{90}$$

for all $X \in X(M)$. So, from (24), $\phi_\eta(T) = -HT$. Thus, from (89),

$$\begin{aligned} Q_1 &= m|\phi_N|^2 - 2m\sum_\alpha |\phi_\alpha(T)|^2 - mH\langle \phi_\eta(T), T\rangle \\ &\geq m|\phi_N|^2 + mH^2|T|^2 - 2m|\phi|^2|T|^2 \geq -2m|\phi|^2|T|^2, \end{aligned} \tag{91}$$

with equality holding if and only if $\phi_N = T = 0$. Thus, inserting (91) in (66), we obtain

$$L(|\phi|^2) \geq -2(m-1)|\phi|^2 F_{a,b}(|\phi|, |T|)\sqrt{\frac{|\phi|^2}{m(m-1)} + b + \frac{a^2}{4}}, \tag{92}$$

where $F_{a,b}(x, y)$ is given in (88).

From now on, for simplicity, we will denote $u = |\phi|^2$. So, (85) can be rewritten as follows

$$L(u) \geq -\sqrt{\frac{m-1}{m}} u F_{a,b}(\sqrt{u}, |T|)\sqrt{4u + m(m-1)(a^2+4b)}. \tag{93}$$

Taking into account that $u \geq 0$ and $b \geq 0$, from Remark 3.2 and (93) we get

$$u^{\frac{p+2}{2}} F_{a,b}\left(\sqrt{u}, |T|\right) \geq -\sqrt{\frac{m}{m-1}} \frac{u^{\frac{p}{2}}}{\sqrt{4u + m(m-1)(a^2+4b)}} L(u), \tag{94}$$

for every real number p. By closedness of Σ^m, we can integrate both sides of (94) in order to obtain

$$\int_\Sigma u^{\frac{p+2}{2}} F_{a,b}\left(\sqrt{u}, |T|\right) d\Sigma \geq -\sqrt{\frac{m}{m-1}} \int_\Sigma \frac{u^{\frac{p}{2}}}{\sqrt{4u + m(m-1)(a^2+4b)}} L(u) d\Sigma. \tag{95}$$

Now, we will define the function

$$f(t) = \int_{t_0}^{t} g(s) ds, \tag{96}$$

where $g(s)$ is given by

$$g(s) = \frac{s^{p/2}}{\sqrt{4s + m(m-1)(a^2+4b)}}, \quad s \geq 0. \tag{97}$$

Since $p > 2$, $b \geq 0$ and g is a smooth function, we have that f is well defined (see Remark 3.2) and $f \geq 0$. Hence, taking into the integral, from (34) and Lemma 2.3, we have

$$0 = \int_\Sigma L(f(u)) d\Sigma = \int_\Sigma f'(u) L(u) d\Sigma + \int_\Sigma f''(u)\langle P(\nabla u), \nabla u\rangle d\Sigma, \tag{98}$$

that is,

$$-\int_\Sigma f'(u) L(u) d\Sigma = \int_\Sigma f''(u)\langle P(\nabla u), \nabla u\rangle d\Sigma. \tag{99}$$

Taking the first and second derivatives of (96), we have

$$f'(t) = \frac{t^{p/2}}{\sqrt{4t + m(m-1)(a^2+4b)}} \geq 0 \tag{100}$$

and

$$f''(t) = \frac{4(p-1)t^{p/2} + pm(m-1)(a^2+4b)t^{\frac{p-2}{2}}}{2\left(4t + m(m-1)(a^2+4b)\right)^{3/2}} \geq 0. \tag{101}$$

Lemma 2.2 assures that the operator P is positive semidefinite, using (99), (100) and (101) in (95), we can estimate

$$\int_\Sigma u^{\frac{p+2}{2}} F_{a,b}\left(\sqrt{u}, |T|\right) d\Sigma \geq \sqrt{\frac{m}{m-1}} \int_\Sigma f''(u)\langle P(\nabla u), \nabla u\rangle d\Sigma \geq 0. \tag{102}$$

Therefore, we conclude

$$\int_{\Sigma} u^{\frac{p+2}{2}} F_{a,b}\left(\sqrt{u}, |T|\right) d\Sigma \geq 0. \tag{103}$$

This proves inequality (87).

We assume that the equality holds in (103) and $b > 0$. By (102), we get

$$\int_{\Sigma} f''(u)\langle P(\nabla u), \nabla u\rangle d\Sigma = 0, \tag{104}$$

where

$$f''(u) = \frac{4(p-1)u^{p/2} + pm(m-1)(a^2+4b)u^{\frac{p-2}{2}}}{2\left(4u + m(m-1)(a^2+4b)\right)^{3/2}} \geq 0, \tag{105}$$

with equality holding if and only if $p > 2$ and $u = 0$. Since $b > 0$, from Lemma 2.2, P is positive definite, consequently

$$\langle P(\nabla u), \nabla u\rangle \geq 0 \tag{106}$$

with equality if and only if $\nabla u = 0$. Therefore, it follows from (104) that:

$$f''(u)\langle P(\nabla u), \nabla u\rangle = 0 \quad \text{on} \quad \Sigma^m, \tag{107}$$

which implies that the function $u = |\phi|^2$ must be constant, either $u \equiv 0$ or $u \equiv u_0 > 0$.

If $|\phi|^2 = u \equiv 0$, then Σ^m is a totally umbilical submanifold. Hence, by (91) we get $T = 0$. Otherwise, if $|\phi|^2 = u \equiv u_0 > 0$. From this, the equality in (103) implies in

$$\int_{\Sigma} F_{a,b}(|\phi|, |T|) d\Sigma = 0. \tag{108}$$

Hence, from (91) we also must have $\phi_N = 0$ and $T = 0$ in the non-totally umbilical case. Thus, by this, (88) can be written as follows, $F_{a,b} = const.$ and follows by (108) that $F_{a,b} = 0$. Consequently, we must have that $|\phi|^2 = \gamma(m, a, b)$, where $\gamma(m, a, b)$ is the only positive root of $\varphi_{a,b,1}$. From this, we are able to see that all inequalities obtained along of the proof become equalities. In particular, the equality holds in (70) and (83). So, from Lemmas 2.1 and 2.5 we must have that Σ^m is a parallel submanifold in $\mathbb{S}^n$ with two distinct principal curvatures one of which is simple. Besides this, also occurs the equality in (80), which implies $|\phi| = |\phi_\eta|$. In both cases, $|\phi| = 0$ or $|\phi| = |\phi_\eta|$, we can always get that $\phi_\alpha = 0$ for all $\alpha > m+1$. By using this, since $n > m$, if $n = m+1$, then $\Sigma^m \hookrightarrow \mathbb{S}^{m+1} \times \mathbb{R}$ and as $T = 0$, we obtain that Σ^m is a hypersurface of $\mathbb{S}^{m+1} \times \{t_0\}$ for some $t_0 \in \mathbb{R}$. So, let us assume then $n > m+1$. Once $A_\alpha = 0$ for all $\alpha \geq m+2$, we observe that the first normal subspace

$$N_1 = \{\xi \in TM^\perp ; A_\xi = 0\}^\perp = \mathrm{span}\{\eta\}, \tag{109}$$

has dimension 1 and $\nabla^\perp N_1 \subset L = N_1 + \mathrm{span}\{N\}$. Since η is orthogonal to ∂_t we have that $\mathrm{rank}(L) = q = 2$. Finally, we observe that the condition $n - m + 1 > 2 = q$ is satisfied. Therefore we can apply Lemma 2.6 in order to obtain that Σ^m lies in a totally geodesic submanifold $\mathbb{S}^{m+1} \times \mathbb{R}$ of $\mathbb{S}^n \times \mathbb{R}$. So, we can conclude that Σ^m is an isoparametric hypersurface in $\mathbb{S}^{m+1} \times \{t_0\}$, for some $t_0 \in \mathbb{R}$, with at most two distinct principal curvatures. Therefore, we can use Theorem 1.1 of dos Santos & da Silva 2021 (see also Theorem 1 of Alías et al. 2012) in order to conclude that Σ^m must be isometric to the following standard product $\mathbb{S}^1(\sqrt{1-r^2}) \times \mathbb{S}^{m-1}(r)$ with $r = \sqrt{(m-2)/m(H_2+1)}$. □

In the case $c = -1$, we have:

Theorem 3.4. *Let Σ^m be a closed pnmc lw-submanifold in $\mathbb{H}^n \times \mathbb{R}$, $n > m \geq 4$, such that $H_2 = aH + b$ with $a, b \geq 0$. If Σ^m has constant η-angle, then*

$$\int_\Sigma |\phi|^{p+2} G_{a,b}(|\phi|, |T|) d\Sigma \geq 0, \tag{110}$$

for every real number $p > 2$, where $G_{a,b}$ is given by

$$\begin{aligned} G_{a,b}(x,y) = \frac{m-2}{m-1}x^2 - (m+1)y^2 - m\left(a - \frac{m-2}{\sqrt{m(m-1)}}x\right)\sqrt{\frac{x^2}{m(m-1)} + b + \frac{a^2}{4}} \\ + \frac{m(m-2)a}{2\sqrt{m(m-1)}}x - m\left(\frac{a^2}{2} + b - 2\right). \end{aligned} \tag{111}$$

Moreover, if $b > 0$ the equality holds in (110) if and only if Σ^m is a totally umbilical hypersurface in $\mathbb{H}^{m+1} \times \{t_0\} \hookrightarrow \mathbb{H}^n \times \mathbb{R}$ for some $t_0 \in \mathbb{R}$.

Proof. Let us consider a local orthonormal frame field $\{e_{m+1}, \ldots, e_{n+1}\}$ in the normal bundle such that $e_{m+1} = \eta$. Then, from (19), it is easy to see that

$$\phi_N = \sum_{\alpha=m+1}^{n+1} \langle N, e_\alpha \rangle \phi_\alpha. \tag{112}$$

From Cauchy-Schwarz's inequality and Hilbert-Schmidt's norm definition, we have

$$\begin{aligned} |\phi_N|^2 &= \sum_{\alpha,i} \langle N, e_\alpha \rangle^2 \langle \phi_\alpha(e_i), \phi_\alpha(e_i) \rangle \\ &\leq \sum_{\alpha,i} |N|^2 |e_\alpha|^2 \langle \phi_\alpha(e_i), \phi_\alpha(e_i) \rangle = |N|^2 |\phi|^2. \end{aligned} \tag{113}$$

Hence, from (14), (90) and (113),

$$\begin{aligned} Q_{-1} &= -m|\phi_N|^2 + mH\langle \phi_\eta(T), T\rangle + 2m\sum_\alpha |\phi_\alpha(T)|^2 \\ &\geq -m|N|^2|\phi|^2 + mH\langle \phi_\eta(T), T\rangle + 2m|\phi_\eta(T)|^2 + 2m\sum_{\alpha > m+1}|\phi_\alpha(T)|^2 \\ &\geq -m(1 - |T|^2)|\phi|^2 + mH\langle \phi_\eta(T), T\rangle + 2m|\phi_\eta(T)|^2 \\ &= -m(1 - |T|^2)|\phi|^2 - mH^2|T|^2 + 2mH^2|T|^2 \\ &= -m(1 - |T|^2)|\phi|^2 + mH^2|T|^2 \geq -m(1 - |T|^2)|\phi|^2, \end{aligned} \tag{114}$$

with equality holding if and only if $T = 0$. Thus, inserting (114) in (66),

$$L(|\phi|^2) \geq -2(m-1)|\phi|^2 G_{a,b}(|\phi|, |T|)\sqrt{\frac{|\phi|^2}{m(m-1)} + b + \frac{a^2}{4}}, \tag{115}$$

where $G_{a,b}(x,y)$ is given in (111).

At this point the proof follows as the one of Theorem 3.3 until reaching inequality (110). If the equality holds, from (104) we have

$$\langle P(\nabla|\phi|^2), \nabla|\phi|^2\rangle = 0, \tag{116}$$

since

$$f''(|\phi|) = \frac{4(p-1)|\phi|^p + m(m-1)(4b+a^2)p|\phi|^{p-2}}{2\,(4|\phi|^2 + m(m-1)(4b+a^2))^{3/2}} > 0,$$

where it was used that $p > 2$ and $b > 0$. Hence, being P positive definite, from (116) it follows that $|\phi|$ is constant along of M^n. If $|\phi| = 0$, then M^n is a totally umbilical, and as before, Σ^m hypersurface in $\mathbb{H}^{m+1} \hookrightarrow \mathbb{H}^n \times \{t_0\}$, for some $t_0 \in \mathbb{R}$. Otherwise, $|\phi|$ is a positive constant. So, from the equality (110),

$$\int_\Sigma G_{a,b}(|\phi|, |T|)d\Sigma = 0. \tag{117}$$

Therefore, reasoning as in the last part of Theorem 3.3, we must have that Σ^m is an isoparametric hypersurface of $\mathbb{H}^{m+1} \times \{t_0\} \hookrightarrow \mathbb{H}^n \times \mathbb{R}$, for some $t_0 \in \mathbb{R}$. So, taking into account Theorem 2 of Alías et al. (2012), we conclude that M^n should be isometric to a hyperbolic cylinder $\mathbb{H}^1(-\sqrt{1+r^2}) \times \mathbb{S}^{n-1}(r)$, which is not closed manifold. Therefore, the equality holds in (110) if, and only if, Σ^m is a totally umbilical hypersurface in $\mathbb{H}^{m+1}$. □

Remark 3.5. Let us recall that a submanifold Σ^m of $M^n(c) \times \mathbb{R}$ is said to be a *vertical cylinder* over M^{m-1} if $\Sigma^m = \pi_M^{-1}(M^{m-1})$ where M^{m-1} is a submanifold of $M^n(c)$. It is not difficult to check that Σ^m is a non-minimal parallel vertical cylinder in $M^n(c) \times \mathbb{R}$ if, and only if, M^{m-1} is a non-minimal parallel submanifold in $M^n(c)$. Moreover, its mean curvature vector field h is given by $h = \frac{m-1}{m}h_0$, where h_0 denotes the mean curvature vector field of M^{m-1}. Hence, Σ^m is a pnmc lw-submanifold of $M^n(c) \times \mathbb{R}$ having constant η-angle and that is not lies in a slice provided vertical cylinders are characterized by the fact that ∂_t is always tangent to Σ^m (see Fetcu & Rosenberg 2013). Therefore, we conclude that the hypothesis of the submanifold to be closed in Theorems 3.3 and 3.4 is, indeed, necessary.

4 - FURTHER RESULTS FOR $M = 2$ AND $M = 3$

We should notice that when $m = 2$ and $m = 3$, the integral inequalities obtained in Theorems 3.3 and 3.4 holds. To see this, it is sufficient to do a rereading on the first inequality of (72). In fact, from (72),

$$\begin{aligned} mH\sum_\alpha \mathrm{tr}(\phi_\alpha^2\phi_m) - \sum_{\alpha,\beta\neq m+1} N(\phi_\alpha\phi_\beta - \phi_\beta\phi_\alpha) - \sum_\alpha[\mathrm{tr}(\phi_\alpha\phi_\beta)]^2 \\ \geq -\frac{m(m-2)}{\sqrt{m(m-1)}}H|\phi|^2|\phi_\eta| - \frac{1}{2}|\phi_\eta|^4 + |\phi|^2|\phi_\eta|^2 - \frac{3}{2}|\phi|^4. \end{aligned} \tag{118}$$

A straightforward computation, gives

$$|\phi|^2|\phi_\eta|^2 \geq -\frac{1}{2}|\phi_\eta|^4 - \frac{1}{2}|\phi|^4, \tag{119}$$

with equality holding if and only if $|\phi| = |\phi_\eta| = 0$, that is, if and only if Σ^m is totally umbilical submanifold. Besides this,

$$|\phi|^2 = |\phi_\eta|^2 + \sum_{\alpha > m+1}|\phi_\alpha|^2 \geq |\phi_\eta|^2, \tag{120}$$

with equality holding if and only if $|\phi_\alpha| = 0$ for $\alpha > m+1$. Hence, inserting (119) and (120) in (118), we get

$$mH\sum_{\alpha}\mathrm{tr}(\phi_\alpha^2\phi_\eta) - \sum_{\alpha,\beta\neq m+1}N(\phi_\alpha\phi_\beta - \phi_\beta\phi_\alpha) - \sum_{\alpha}[\mathrm{tr}(\phi_\alpha\phi_\beta)]^2 \geq -|\phi|^2\left(\frac{m(m-2)}{\sqrt{m(m-1)}}H|\phi| + 3|\phi|^2\right).$$

By using this in Proposition 1.2,

$$\begin{aligned} L(mH) \geq\ & |\nabla A|^2 - m^2|\nabla H|^2 + cm|\phi_N|^2 - 2cm\sum_{\alpha}|\phi_\alpha(T)|^2 - cmH\langle\phi_\eta(T), T\rangle \\ & + \left(c(m - |T|^2) + mH^2 - \frac{m(m-2)}{\sqrt{m(m-1)}}H|\phi| - 3|\phi|^2\right)|\phi|^2. \end{aligned} \tag{121}$$

Hence, by replacing (75) and (81) in (121), we have

$$L(mH) \geq |\nabla A|^2 - m^2|\nabla H|^2 + Q_c - |\phi|^2\overline{\varphi}_{a,b,c}(|\phi|, |T|), \tag{122}$$

where

$$\overline{\varphi}_{a,b,c}(x,y) = 2x^2 + \varphi_{a,b,c}(x,y) \tag{123}$$

with Q_c and $\varphi_{a,b,c}(x,y)$ defined in (67) and (68), respectively.

Therefore, since $b \geq 0$, we can apply Lemma 2.1 together with inequality (85) in (122) in order to obtain:

$$L(|\phi|^2) \geq -2(m-1)\left(|\phi|^2\overline{\varphi}_{a,b,c}(|\phi|, |T|) - Q_c\right)\sqrt{\frac{|\phi|^2}{m(m-1)} + b + \frac{a^2}{4}}. \tag{124}$$

By this previous digression, we obtain:

Theorem 4.1. *Let Σ^m be a closed pnmc lw-submanifold in $\mathbb{S}^n \times \mathbb{R}$, $n > m$, such that $H_2 = aH + b$ with $a, b \geq 0$. If Σ^m has constant η-angle, then*

$$\int_\Sigma |\phi|^{p+2}\overline{F}_{a,b}(|\phi|, |T|)d\Sigma \geq 0, \tag{125}$$

for every real number $p > 2$, where $\overline{F}_{a,b}$ is the real function given by

$$\overline{F}_{a,b}(x,y) = 2x^2 + F_{a,b}(x,y),$$

with $F_{a,b}(x,y)$ defined in (88). *Moreover, the equality holds in* (125) *if and only if Σ^m is a totally umbilical hypersurface in $\mathbb{S}^{m+1} \times \{t_0\} \hookrightarrow \mathbb{S}^n \times \mathbb{R}$ for some $t_0 \in \mathbb{R}$.*

Proof. The proof follows the same steps as the proof of Theorem 3.3 until we reach inequality (103), changing the function $\varphi_{a,b,c}$ by $\overline{\varphi}_{a,b,c}$ along of the computations. If the equality in (125) holds, then also occurs equality in (119) and hence, Σ^m is a totally umbilical. Besides this, the equality also occurs in (91), from where we conclude that $T = 0$. Therefore, Σ^m is a totally umbilical hypersurface in $\mathbb{S}^{m+1} \times \{t_0\} \hookrightarrow \mathbb{S}^n \times \mathbb{R}$ for some $t_0 \in \mathbb{R}$. □

Following the same steps of the proof of Theorems 3.4 and 4.1, we have:

Theorem 4.2. *Let Σ^m be a closed pnmc lw-submanifold in $\mathbb{H}^n \times \mathbb{R}$, $n > m$, such that $H_2 = aH + b$ with $a, b \geq 0$. If Σ^m has constant η-angle, then*

$$\int_\Sigma |\phi|^{p+2}\overline{G}_{a,b}(|\phi|, |T|)d\Sigma \geq 0, \tag{126}$$

for every real number $p > 2$, where $\overline{G}_{a,b}$ is given by

$$\overline{G}_{a,b}(x, y) = 2x^2 + G_{a,b}(x, y),$$

with $G_{a,b}(x, y)$ defined in (111). *Moreover, the equality holds in* (126) *if and only if Σ^m is a totally umbilical hypersurface in $\mathbb{H}^{m+1} \times \{t_0\} \hookrightarrow \mathbb{H}^n \times \mathbb{R}$ for some $t_0 \in \mathbb{R}$.*

Remark 4.3. The approach developed here is not effective to find parallel submanifolds with two distinct principal curvatures as in Proposition 3.1, because of this, we use it only in the cases where $m = 2$ and $m = 3$. So, following Remark 3.2 of Guo & Li (2013), it is an interesting question is to know if Proposition 3.1 holds or not for $m = 2$ and $m = 3$.

Acknowledgments

The authors would like to express their thanks to the referees for reading the manuscript in great detail and for the valuable suggestions and comments that helped to improve the paper. The first author is partially supported by Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq), Brazil, grants 431976/2018-0 and 311124/2021-6 and Propesqi (UFPE).

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How to cite

DOS SANTOS FR, DA SILVA SF & DE SOUSA AF. 2023. Integral inequalities for closed linear Weingarten submanifolds in the product spaces. An Acad Bras Cienc 95: e20230345. DOI 10.1590/0001-3765202320230345.

Manuscript received on April 1, 2023;
accepted for publication on May 24, 2023

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