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New examples of surfaces in H^3 with conformal normal Gauss map

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ABSTRACT

In this paper we give some examples of surfaces in H^3 with conformal normal Gauss map with respect to the second conformal structure and prove some global properties.

Key words: hyperbolic space, conformal normal Gauss map, ruled surface, translational surface.

It is well known that the Weierstrass representation formula has played an important role in studying minimal surfaces in R^3. To find a Weierstrass-type representation formula for simply connected immersed minimal surfaces in hyperbolic n-space, Kokubu (Kokubu 1997) considered the hyperbolic space as a Lie group G with a left invariant metric. Given a two-dimensional domain D and a map (φ) into the above Lie group, he considered the pullback (φ^i) of the Maurer-Cartan forms to D. Using the standard harmonic map equation for such maps, and by pulling back the Maurer-Cartan structure equations, he derived a complete integrability condition for the 1-forms (φ^i). Also, to assure that the associated harmonic map is a minimal surface, a conformality condition is imposed. The normal Gauss map into the hyperquadric is written down, which is the usual tangential Gauss map translated to the Lie algebra of G. On the other hand, Gálvez and Martínez (Gálvez and Martínez 2000) studied the properties of the Gauss map of a surface Σ immersed into the Euclidean 3-space R^3, particularly those related to the geometry of the immersion and the so-called second conformal structure of the surface, that is, the conformal structure on Σ induced by the second fundamental form. Motivated by their work, the author (Shi 2004) gave a Weierstrass representation formula for surfaces with prescribed normal Gauss map and Gauss curvature in H^3

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by using the second conformal structure on surfaces (see section 1 for the definition). From this, surfaces whose normal Gauss map are conformal maps have been found(see Theorem 1).

The propose of this paper is to classify locally the ruled surfaces with conformal normal Gauss map in H^3 and also give some new examples of complete properly embedded ruled surfaces. The rest of the paper is divided into three sections. The first one describes the definitions of the normal Gauss map in context of this paper and the second conformal structure on surfaces and states Theorem 1. The second section gets two local examples within the Euclidean ruled surface and in the last section some global properties of the ruled surfaces and translational surfaces are proven.

1 PRELIMINARIES

Take the upper half-space model of the hyperbolic 3-space $H^3 = \{(x_1, x_2, x_3) \in R^3 : x_3 > 0\}$ with the Riemannian metric $ds^2 = \frac{1}{x_3^2}(dx_1^2 + dx_2^2 + dx_3^2)$ and constant sectional curvature -1.

Let Σ be a connected 2-dimensional smooth surface and $x : \Sigma \to H^3$ be an immersion of Σ into H^3 with local coordinates u, v. The first and the second fundamental forms of the immersion are written, respectively, as $\mathrm{I} = Edu^2 + 2Fdudv + Gdv^2$ and $\mathrm{II} = Ldu^2 + 2Mdudv + Ndv^2$. The unit normal vector field of $x(\Sigma)$ in H^3 is written as

$$\vec{n} = x_3 e_{31}\frac{\partial}{\partial x_1} + x_3 e_{32}\frac{\partial}{\partial x_2} + x_3 e_{33}\frac{\partial}{\partial x_3},$$

where $e_{31}^2 + e_{32}^2 + e_{33}^2 = 1$. The Gauss equation is

$$K = -1 + \frac{LN - M^2}{EG - F^2}.$$

Identifying H^3 with the Lie group

$$H^3 = \left\{ \begin{pmatrix} 1 & 0 & 0 & \log x_3 \\ 0 & x_3 & 0 & x_1 \\ 0 & 0 & x_3 & x_2 \\ 0 & 0 & 0 & 1 \end{pmatrix} : (x_1, x_2, x_3) \in H^3 \right\},$$

the multiplication is defined as matrix multiplication and the unit element is $e = (0, 0, 1)$. The Riemannian metric is left$-$invariant and $X_1 = x_3\frac{\partial}{\partial x_1}$, $X_2 = x_3\frac{\partial}{\partial x_2}$, $X_3 = x_3\frac{\partial}{\partial x_3}$ are the left$-$invariant unit orthonormal vector fields. Now, the unit normal vector field of $x(\Sigma)$ can be written as $\vec{n} = e_{31}X_1 + e_{32}X_2 + e_{33}X_3$. Left translating $\vec{n}$ to $T_e(H^3)$, we obtain

$$\vec{\tilde{n}} = L_{x^{-1}*}(\vec{n}) = e_{31}\frac{\partial}{\partial x_1}(e) + e_{32}\frac{\partial}{\partial x_2}(e) + e_{33}\frac{\partial}{\partial x_3}(e) \in S^2(1) \subset T_e(H^3).$$

By the stereographic projection, we get the map $\Sigma \to C \bigcup \{\infty\}$,

$$g_1(x) = \frac{e_{31} + ie_{32}}{1 - e_{33}}, \vec{\tilde{n}} \in U_1 = S^2(1) \backslash \{N\},$$

$$g_2(x) = \frac{e_{31} - i e_{32}}{1 + e_{33}}, \vec{\tilde{n}} \in U_2 = S^2(1) \backslash \{S\}.$$

Call g_1(or g_2) the normal Gauss map of surface $x(\Sigma)$ (Kokubu 1997). On $U_1 \bigcap U_2$, $g_1 g_2 = 1$. In this paper, we only consider g_1 and write the normal Gauss map as $g : \Sigma \to C \bigcup \{\infty\}$. Then we have

$$e_{31} = \frac{g + \bar{g}}{1 + |g|^2}, e_{32} = -i \frac{g - \bar{g}}{1 + |g|^2}, e_{33} = \frac{|g|^2 - 1}{1 + |g|^2}.$$

Consider an immersion $x : \Sigma \to H^3$ with Gauss curvature $K > -1$. By the Gauss equation, we can choose a suitable orientation on $x(\Sigma)$ such that the second fundamental form II becomes a positive definite metric on Σ and induces a conformal structure on Σ, which is called the second conformal structure like (Klotz 1963). Σ will be considered as a Riemannian surface with the second conformal structure.

THEOREM 1 (Shi 2004). *Let Σ be a connected Riemannian surface and $x : \Sigma \to H^3$ be an immersion with Gauss curvature $K > -1$. Assume that the set of umbilics has no interior point. Then normal Gauss map $g : \Sigma \to C \bigcup \{\infty\}$ of $x(\Sigma)$ is conformal map if and only if the Gauss curvature K and the normal Gauss map g satisfy $K = -\frac{4|g|^2}{(1+|g|^2)^2} > -1$.*

The Weierstrass representation formulas for these surfaces can be found in (Shi 2004).

REMARK. For the Bryant's hyperbolic Gauss map of surface in H^3(see (Bryant 1987) for the definition), Gálvez, Martínez and Milán (Gálvez et al. 2000) proven that the hyperbolic Gauss map is conformal if and only if the surface is either flat or totally umbilic.

Locally, the graph $(u, v, f(u, v))$ satisfying $K = -\frac{4|g|^2}{(1+|g|^2)^2} > -1$ must satisfy the following fully nonlinear PDE of second order (Shi 2004),

$$f(f_{uu} f_{vv} - f_{uv}^2) + [(1 + f_v^2) f_{uu} - 2 f_u f_v f_{uv} + (1 + f_u^2) f_{vv}] = 0. \tag{1}$$

2 EXAMPLES

We consider the surfaces in H^3 as those ones in R^3. The simplest examples of surfaces with conformal normal Gauss map are the equidistant surfaces and horospheres $x_3 = const > 0$, i.e. ordinary Euclidean planes. They are totally umbilics with constant Gauss curvature and constant normal Gauss map.

The following theorem gives all the ruled surfaces with conformal normal Gauss map in H^3.

THEOREM 2. *Up to an isometric transformation*

$$(x_1, x_2, x_3) \to (x_1 \cos\theta - x_2 \sin\theta + a, x_1 \sin\theta + x_2 \cos\theta + b, x_3) \tag{2}$$

of H^3, every ruled surface with conformal normal Gauss map in H^3 is locally a part of one of the following,

(1) *equidistant surface with respect to a vertical hyperbolic plane,*

(2) *Horosphere* $x_3 = constant > 0$,

(3) $(u\cos v, c\cdot \sin v, u\sin v)$, *for a constant* $c \neq 0$,

(4) $(-c_2 \sin v + u\cos v, c_1 \cdot \sin v, c_2\cos v + u \sin v)$, *for constants* $c_1 \neq 0$ *and* $c_2 \neq 0$.

REMARK. We may check that, among all isometric transformations of H^3 (Korevaar et al. 1992), the horizontal Euclidean translations and rotations (2), the hyperbolic reflections with respect to a vertical hyperplane and the vertical hyperbolic translations

$$(x_1, x_2, x_3) \to (\lambda(x_1 - a), \lambda(x_2 - b), \lambda x_3), (\lambda > 0),$$

preserve the concept of the ruled surfaces and the conformality of the normal Gauss map of any surface with conformal normal Gauss map and the hyperbolic reflections

$$(x_1, x_2, x_3) \to (a, b, 0) + \frac{\lambda(x_1 - a, x_2 - b, x_3)}{(x_1-a)^2 + (x_2-b)^2 + x_3^2}, (\lambda > 0)$$

preserve the conformality of the normal Gauss map of all totally umbilics but totally geodesics.

PROOF. Generally, considered as surfaces in R^3, the ruled surfaces in H^3 can be represented as

$$x(u, v) = \alpha(v) + u\beta(v) : D \to H^3,$$

where $D(\subset R^2)$ is a parameter domain and $\alpha(v)$ and $\beta(v)$ are two vector value functions into R^3 corresponding to two curves in R^3.

First, we assume that β is locally nonconstant and without loss of generality assume that

$$\langle \beta(v), \beta(v)\rangle = 1, \langle \beta'(v), \beta'(v)\rangle = 1, \langle \alpha'(v), \beta'(v)\rangle = 0, \tag{3}$$

where $\langle \cdot, \cdot\rangle$ is the inner product in R^3. Write as $e = (0, 0, 1)$. We have $x_3 = \langle x, e\rangle$ and $\beta'' \neq 0$. The unit normal vector of the surface $x(u, v) : D \to H^3$ is given by

$$n = \frac{x_3 x_u \wedge x_v}{|x_u \wedge x_v|} = \frac{x_3(\beta \wedge (\alpha' + u\beta'))}{|\beta \wedge (\alpha' + u\beta')|},$$

where $X \wedge Y$ is the exterior product of the vectors X and Y in R^3 and $|\cdot|$ is the Euclidean norm in R^3. So

$$e_{33} = \frac{\langle \beta \wedge (\alpha' + u\beta'), e\rangle}{|\beta \wedge (\alpha' + u\beta')|}. \tag{4}$$

By a straight computation, we have,

$$\begin{aligned}
E &= \frac{1}{\langle x, e\rangle^2}, \quad F = \frac{\langle \alpha', \beta\rangle}{\langle x, e\rangle^2}, \quad G = \frac{\langle \alpha', \alpha'\rangle + u^2}{\langle x, e\rangle^2},\\
L &= \frac{\langle \beta \wedge (\alpha' + u\beta'), e\rangle}{|\beta \wedge (\alpha' + u\beta')|\langle x, e\rangle^2},\\
M &= \frac{\langle \beta', \beta \wedge (\alpha' + u\beta')\rangle}{|\beta \wedge (\alpha' + u\beta')|\langle x, e\rangle} + \frac{\langle \beta \wedge (\alpha' + u\beta'), e\rangle\langle \alpha', \beta\rangle}{|\beta \wedge (\alpha' + u\beta')|\langle x, e\rangle^2},\\
N &= \frac{\langle \alpha'' + u\beta'', \beta \wedge (\alpha' + u\beta')\rangle}{|\beta \wedge (\alpha' + u\beta')|\langle x, e\rangle} + \frac{\langle \beta \wedge (\alpha' + u\beta'), e\rangle(\langle \alpha', \alpha'\rangle + u^2)}{|\beta \wedge (\alpha' + u\beta')|\langle x, e\rangle^2}.
\end{aligned}$$

By the Gauss equation $K = -1 + \frac{LN-M^2}{EG-F^2}$ and (4), we know that

$$K = -\frac{4|g|^2}{(1+|g|^2)^2} = -1 + e_{33}^2 \tag{5}$$

is equivalent to

$$\begin{aligned} &\langle \beta \wedge (\alpha' + u\beta'), e\rangle \cdot \langle \alpha'' + u\beta'', \beta \wedge (\alpha' + u\beta')\rangle \\ &-\langle \alpha + u\beta, e\rangle \cdot \langle \beta', \beta \wedge \alpha'\rangle^2 \\ &= 2\langle \beta \wedge (\alpha' + u\beta'), e\rangle \cdot \langle \alpha', \beta\rangle \cdot \langle \beta', \beta \wedge \alpha'\rangle. \end{aligned} \tag{6}$$

Expanding the above formula and noting the linearly independence of $1, u, u^2$ and u^3, we get

$$\langle \beta \wedge \beta', e\rangle \cdot \langle \beta'', \beta \wedge \beta'\rangle = 0, \tag{7}$$

$$\langle \beta \wedge \alpha', e\rangle \cdot \langle \beta'', \beta \wedge \beta'\rangle + \langle \beta \wedge \beta', e\rangle(\langle \alpha'', \beta \wedge \beta'\rangle + \langle \beta'', \beta \wedge \alpha'\rangle) = 0, \tag{8}$$

$$\begin{aligned} &\langle \beta \wedge \beta', e\rangle \cdot \langle \alpha'', \beta \wedge \alpha'\rangle + \langle \beta \wedge \alpha', e\rangle(\langle \alpha'', \beta \wedge \beta'\rangle + \langle \beta'', \beta \wedge \alpha'\rangle) \\ &-2\langle \alpha', \beta\rangle \cdot \langle \beta', \beta \wedge \alpha'\rangle \cdot \langle \beta \wedge \beta', e\rangle - \langle \beta, e\rangle \cdot \langle \beta', \beta \wedge \alpha'\rangle^2 = 0, \end{aligned} \tag{9}$$

$$\begin{aligned} &\langle \alpha, e\rangle \cdot \langle \beta', \beta \wedge \alpha'\rangle^2 + 2\langle \alpha', \beta\rangle \cdot \langle \beta', \beta \wedge \alpha'\rangle \cdot \langle \beta \wedge \alpha', e\rangle \\ &-\langle \beta \wedge \alpha', e\rangle \cdot \langle \alpha'', \beta \wedge \alpha'\rangle = 0 \end{aligned} \tag{10}$$

By (7), if $\langle \beta \wedge \beta', e\rangle \neq 0$ at a point p_0, then $\langle \beta'', \beta \wedge \beta'\rangle = 0$ in a neighbourhood U of p_0 and the curver $\beta(v)$ is a geodesic of S^2. Hence, $\langle \beta \wedge \beta', e\rangle$ is locally constant and so is globally constant.

CASE 1. If $\langle \beta \wedge \beta', e\rangle = 0$, then $\langle \beta \wedge \beta'', e\rangle = 0$. Hence β, β', β'' and e lie on same plane. We get $\langle \beta, \beta' \wedge \beta''\rangle = 0$. So the curve $\beta(v)$ is a plane curve in R^3, i.e. a unit circle, and the plane π on which β lies passes through the origin of R^3. By $\langle \beta \wedge \beta', e\rangle = 0$, we know π is a vertical plane on which x_3-axis lies. Revolving π around x_3-axis, i.e. making an isometric transformation $(x_1, x_2, x_3) \to (x_1 \cos\theta - x_2 \sin\theta, x_1 \sin\theta + x_2 \cos\theta, x_3)$, we may assume that π is the plane x_1ox_3. So $\beta(v) = (\cos v, 0, \sin v)$. Assume $\alpha(v) = (x_1(v), x_2(v), x_3(v))$, then equation systems (3) and (7)–(10) may be written as

$$x_1' \sin v - x_3' \cos v = 0, \tag{11}$$

$$-x_2'x_2'' \cos v = (x_2')^2 \sin v, \tag{12}$$

$$\begin{aligned} &x_3(x_2')^2 + 2(x_1' \cos v + x_3' \sin v)(x_2')^2 \cos v \\ &= x_2' \cos v(x_2'x_3'' \cos v + x_1'x_2'' \sin v - x_1''x_2' \sin v - x_2''x_3' \cos v). \end{aligned} \tag{13}$$

Solving (11)(12)(13), we get

$$\alpha(v) = (-c_2 \sin v + c_3, c_1 \sin v + c_4, c_2 \cos v),$$

where $c_1 \neq 0$, c_2,c_3 and c_4 are constants. Making a translation $(x_1, x_2, x_3) \to (x_1+c_3, x_2+c_4, x_3)$, we get (3) and (4) of Theorem 2.

CASE 2. If $\langle \beta \wedge \beta', e\rangle \neq 0$, then $\langle \beta'', \beta \wedge \beta'\rangle \equiv 0$. We also get that the curve β is a plane curve in R^3 and the plane π on which the curve β lies passes through the origin of R^3 but is not totally geodesic. Similar to case 1, making a rotation transformation around x_3-axis, we may assume that x_1-axis lies on π. Now, $\beta(v)$is a unit circle on π. $\beta(v) = (\cos v, \sin v \cos\theta, \sin v \sin\theta)$, where constant θ is the angle between plane x_1ox_2 and π. By (8), we get $\langle \alpha'', \beta \wedge \beta'\rangle = 0$. Multiplying (10) by $\langle \beta \wedge \beta', e\rangle$ and using (9), we get

$$\langle \alpha, e\rangle \cdot \langle \beta \wedge \beta', e\rangle \cdot \langle \beta', \beta \wedge \alpha'\rangle^2 = \langle \beta \wedge \alpha', e\rangle \cdot \langle \beta, e\rangle \cdot \langle \beta', \beta \wedge \alpha'\rangle^2.$$

Then, according to whether $\langle \beta', \beta \wedge \alpha'\rangle = 0$ or not, we again get two cases from equation systems (3) and (7)−(10),

CASE (i),

$$\langle \alpha', \beta'\rangle = 0, \langle \beta \wedge \beta', e\rangle \neq 0, \langle \alpha'', \beta \wedge \beta'\rangle = 0,$$
$$\langle \beta', \beta \wedge \alpha'\rangle = 0, \langle \alpha'', \beta \wedge \alpha'\rangle = 0,$$

and

CASE (ii),

$$\langle \alpha', \beta'\rangle = 0, \langle \beta \wedge \beta', e\rangle \neq 0, \langle \alpha'', \beta \wedge \beta'\rangle = 0, \langle \beta', \beta \wedge \alpha'\rangle \neq 0,$$
$$\langle \alpha, e\rangle \cdot \langle \beta \wedge \beta', e\rangle = \langle \beta, e\rangle \cdot \langle \beta \wedge \alpha', e\rangle,$$
$$\langle \beta \wedge \beta', e\rangle \cdot \langle \alpha'', \beta \wedge \alpha'\rangle - 2\langle \alpha', \beta\rangle \cdot \langle \beta', \beta \wedge \alpha'\rangle \cdot \langle \beta \wedge \beta', e\rangle$$
$$= \langle \beta, e\rangle \cdot \langle \beta', \beta \wedge \alpha'\rangle^2.$$

In the case (i), taking derivative on $\langle \alpha'', \beta\wedge\beta'\rangle = 0$, we get $\langle \alpha''', \beta\wedge\beta'\rangle = 0$. So $\langle \alpha'\wedge\alpha'', \alpha'''\rangle = 0$, i.e. the torsion of the curve $\alpha(v)$ is zero. We infer that the curve $\alpha(v)$ is a plane curve and the plane on which the curve α lies parallels to π. Hence x satisfies either (1) or (2) of Theorem 2. As for the case (ii), similar to the proof in the case 1, we know that this is a contradictory system of equations.

Next, when β is constant, we have $\beta \wedge e \neq 0$. Or else, by proposition 34 in chapter 7 of (Spivak 1979), we have $K \equiv -1$. A contradiction. We may assume

$$\langle \alpha', \beta\rangle = 0. \tag{14}$$

Thus $\alpha(v)$ is a plane curve and (6) becomes

$$\langle \beta \wedge \alpha', e\rangle \cdot \langle \alpha'', \beta \wedge \alpha'\rangle = 0. \tag{15}$$

We have $\langle \beta \wedge \alpha', e\rangle \neq 0$. Or else, $\alpha(v)$ is a plane curve and lies on a plane parallel to the one spanned by β and e. Then $x : D \to H^3$ is totally geodesic plane H^2 and $K \equiv -1$. This is contradictory to $K > -1$. So $\langle \alpha'', \beta \wedge \alpha'\rangle = 0$. By (14) , we know that $\alpha(v)$ is a straight line in R^3 and $x : D \to H^3$ is either a equidistant surface or a horosphere $x_3 = constant > 0$. We have proved Theorem 2.

Locally, the ruled surfaces (3) and (4) in Theorem 2 can be represented as a graph $(u, v, f(u, v))$.

COROLLARY 1. $f(u, v) = \pm\dfrac{c_1c_2 + uv}{\sqrt{c_1^2 - v^2}}$ *is a solution of equation* (1)*, where* $c_1 \neq 0$ *and* c_2 *are constants.*

3 GLOBAL PROPERTIES

The equidistant surfaces and horosphere $x_3 = const > 0$ are simply connected, complete, totally umbilics and properly embedded surfaces. Generally, we have the following global properties for the ruled surfaces obtained in theorem 2.

THEOREM 3. *The following are simply connected, complete and properly embedded surfaces in* H^3*,*

(1) $x(u, v) = (u\cos v, c \cdot \sin v, u\sin v) : D_1 = \{(u, v)|0 < u < +\infty, 0 < v < \pi\} \to H^3$*, and*

(2) $x(u, v) = (-c_2\sin v + u\cos v, c_1\sin v, c_2\cos v + u\sin v)$:

$$D_2 = \left\{(u, v)| -\infty < u < +\infty, \arctan\frac{u}{c_2} - \frac{\pi}{2} < v < \arctan\frac{u}{c_2} + \frac{\pi}{2}\right\} \to H^3,$$

where $c > 0$*,* $c_1 > 0$ *and* $c_2 > 0$ *are constants.*

PROOF.

(1) The map $x : D_1 \to H^3$ is one-to-one. For any compact subset S in H^3, it is easy to know that the relative closed set $x^{-1}(S)(\subset\subset D_1)$, as the subset of R^2, is bounded and closed set of R^2. Hence $x^{-1}(S)$ is a compact set. So, $x : D_1 \to H^3$ is a proper map and $x(D_1)$ is a complete surface in H^3. On the basis of this, we can infer that $x : D_1 \to r(D_1)$ is closed. So x maps D_1 homeomorphically onto its image with the induced topology and $x : D_1 \to H^3$ is embedded.

(2) By a parameter transformation $u = \bar{u}$ and $v = \bar{v} + \arctan\frac{\bar{u}}{c_2} - \frac{\pi}{2}$, D_2 is diffeomorphic to $D_2' = \{(\bar{u}, \bar{v})| -\infty < \bar{u} < +\infty, 0 < \bar{v} < \pi\}$. Then $x : D_2 \to H^3$ becomes $x : D_2' \to H^3$ with

$$x(\bar{u}, \bar{v}) = \left(\sqrt{\bar{u}^2 + c_2^2}\cos\bar{v}, \frac{c_1(\bar{u}\sin\bar{v} - c_2\cos\bar{v})}{\sqrt{\bar{u}^2 + c_2^2}}, \sqrt{\bar{u}^2 + c_2^2}\sin\bar{v}\right).$$

Similar to the proof of (1), we can prove (2).

We have obtained the translational surfaces with conformal normal Gauss map in nonparameter form (Shi 2004),

$$f(u, v) = \sqrt{a^2 - u^2} \pm \sqrt{b^2 - v^2}$$

satisfying (1). Now, the parameter form of these translational surfaces is locally given by $x(u, v) = (a \cos u, b \cos v, a \sin u + b \sin v)$.

THEOREM 4.

(1) *The map* $x(u, v) = (\cos u, \cos v, \sin u + \sin v) : D_1 \to H^3$ *is a simply connected, complete and properly embedded surface, where* D_1 *is a simply connected open domain of* R^2 *enclosed by four straight lines* $v = u \pm \pi$, $v = -u + 2\pi$ *and* $v = -u$.

(2) *For* $0 < a < b$, *the image of the map* $x(u, v) = (a \cos u, b \cos v, a \sin u + b \sin v) : D_2 \to H^3$ *is a complete and properly embedded surface diffeomorphic to* $S^1 \times R$, *where*

$$D_2 = \left\{ (u, v) | -\infty < u < +\infty, -\arcsin\left(\frac{a}{b}\sin u\right) < v < \pi + \arcsin\left(\frac{a}{b}\sin u\right) \right\}.$$

PROOF.

(1) By a parameter transformation $u = \bar{u} + \bar{v}$ and $v = \bar{u} - \bar{v}$, D_1 is diffeomorphic to $D_1' = \left\{ (\bar{u}, \bar{v}) | 0 < \bar{u} < \pi, -\frac{\pi}{2} < \bar{v} < \frac{\pi}{2} \right\} \subset R^2$. Then $x : D_1 \to H^3$ becomes $x : D_1' \to H^3$ with

$$x(\bar{u}, \bar{v}) = (\cos(\bar{u} + \bar{v}), \cos(\bar{u} - \bar{v}), 2 \sin \bar{u} \cos \bar{v}) : D_1' \to H^3.$$

Similar to the proof of (1) in Theorem 3, we can prove (1).

(2) It is easy to know that image $x(D_2)$ is diffeomorphic to $S^1 \times R$. For any divergent curve $\alpha : [0, 1) \to D_2$, when $t \to 1-$, $\alpha(t)$ tends to either the boundary curve, $v = -\arcsin\left(\frac{a}{b}\sin u\right)$ and $v = \pi + \arcsin\left(\frac{a}{b}\sin u\right)$ or ∞. For the former, if the length of $x(\alpha(t))$ is finite, then there exist a compact set S in H^3 containing completely the curve $x(\alpha(t))$. However, when restricting $x(D_2)$ on S, there exists a positive constant ε_0 such that $a \sin u + b \sin v \geq \varepsilon_0$. We may assume $\varepsilon_0 < min\{a, b - a\}$. So $\alpha(t)$ must satisfy $-\arcsin\left(\frac{a}{b}\sin u - \frac{\varepsilon_0}{b}\right) \leq v \leq \pi + \arcsin\left(\frac{a}{b}\sin u - \frac{\varepsilon_0}{b}\right)$. So $\alpha(t)$ does not tend to the boundary of D_2, which is contradictory that $\alpha(t)$ tends to the boundary of D_2. So the length of $x(\alpha(t))$ is infinite. For the latter, as $t \to 1-$, $u(t) \to \infty$. The first fundamental form of $x : D_2 \to H^3$ satisfies

$$ds^2 = \frac{a^2 du^2 + 2ab \cos u \cos v du dv + b^2 dv^2}{(a \sin u + b \sin v)^2} \geq \frac{a^2 \sin^2 u du^2}{(a \sin u + b)^2}.$$

So the length of $x(\alpha(t))$

$$L \geq \int_0^1 \frac{a |\sin u(t)| |u'(t)|}{a \sin u(t) + b} dt.$$

Noting that $\int_0^{2\pi} \frac{a|\sin u|}{a\sin u+b}du > 0$, we have $L = +\infty$. In a word, the length of $x(\alpha(t))$ is always infinite. Hence the image $x(D_2)$ is complete.

Next, we prove that the image $x(D_2)$ is a proper surface. When restricting $x(D_2)$ on any compact set S in H^3 mentioned above, the first fundamental form of $x : D_2 \to H^3$ satisfies $ds^2 \leq \frac{C}{\varepsilon_0^2}(du^2 + dv^2)$, where C is constant and depend only on a and b. So, we infer that the restriction of S to $x(D_2)$ is bounded closed subset of $x(D_2)$ and hence is compact subset of $x(D_2)$. We complete the proof.

REMARK. The curves in Theorem 3, corresponding respectively to $v = \frac{\pi}{2}$ on $x(D_1)$ and to $v = \pm\frac{\pi}{2}$ on $x(D_2)$ and the curves in Theorem 4, corresponding to $u = 0$,. $u = \pi$, $v = 0$ and $v = \pi$ on $x(D_1)$ and $x(D_2)$ are all geodesics of H^3 follow which $K = -1$ and accordingly the second fundamental forms are degenerate. Furthermore, every such a geodesic is mapped to a point by the normal Gauss map.

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RESUMO

Neste trabalho, damos alguns exemplos de superfícies no espaço hiperbólico de dimensão três com aplicação de Gauss conforme relativamente à segunda estrutura conforme e provamos algumas propriedades globais.

Palavras-chave: espaço hiperbólico, aplicação normal de Gauss conforme, superfície regrada, superfície de translação.

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