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A new characterization of the Euclidean sphere

ABDÊNAGO A. BARROS[1], CÍCERO P. AQUINO[2] and JOSÉ N.V. GOMES[3]

[1]Departamento de Matemática, Universidade Federal do Ceará, Campus do Pici,
Bloco 914, 60455-760 Fortaleza, CE, Brazil
[2]Departamento de Matemática, Universidade Federal do Piauí, Campus Universitário
Ministro Petrônio Portella, 64049-550 Teresina, PI, Brazil
[3]Departamento de Matemática, Universidade Federal do Amazonas, Av. General
Rodrigo Octávio, 6200, 69080-900 Manaus, AM, Brazil

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ABSTRACT

In this paper, we obtain a new characterization of the Euclidean sphere as a compact Riemannian manifold with constant scalar curvature carrying a nontrivial conformal vector field which is also conformal Ricci vector field.

Key words: Conformal vector field, Scalar curvature, Euclidean sphere, Einstein manifold.

INTRODUCTION

In the middle of the last century many geometers tried to prove a conjecture concerning the Euclidean sphere as the unique compact orientable Riemannian manifold (M^n, g) admitting a metric of constant scalar curvature S and carrying a nontrivial conformal vector field X. Among them, we cite Bochner and Yano (1953), Goldberg and Kobayashi (1962), Lichnerowicz (1955), Nagano and Yano (1959), Obata (1962), Obata and Yano (1965, 1970) and Tashiro (1965). The attempts to prove this conjecture resulted into the rich literature which has currently been attracting a lot of attention in the mathematical community. We address the reader to the book of Yano (1970) for a summary of those results. Ejiri gave a counter example to this conjecture building metrics of constant scalar curvature on the warped product $\mathbb{S}^1 \times_f F^{n-1}$, where F^{n-1} is a compact Riemannian manifold of constant scalar curvature, while $\mathbb{S}^1$ stands for the Euclidean circle. In his example the conformal vector field is $X = f(d/dt)$, where d/dt is a unit vector field on $\mathbb{S}^1$ and f satisfies a certain ordinary differential equation, see Ejiri (1981) for details.

The primary concept involved in the study of this subject is of Lie derivatives. After all, what is the geometric meaning of the Lie derivative of a tensor T (or of a vector field Y) with respect to a vector field

Correspondence to: José Nazareno Vieira Gomes
E-mail: jnvgomes@pq.cnpq.br/jov217@lehigh.edu

X? This is a method that uses the flow of X to push values of T back to p and then differentiate. The result is called the Lie derivative $\mathscr{L}_X T$ of T with respect to X. A vector field X on Riemannian manifold (M^n, g) is called conformal if $\mathscr{L}_X g$ is a multiple of g. There are important applications of Lie derivatives in the study of how geometric objects such as Riemannian metrics, volume forms, and symplectic forms behave under flows. For instance, it is well-known that the Lie derivative of a vector field Y with respect to X is zero if and only if Y is invariant under the flow of X. It turns out that the Lie derivative of a tensor has exactly the same interpretation. For more details see the book of Lee (2003).

We highlight that Nagano and Yano (1959) have proved that the aforementioned conjecture is true when (M^n, g) is an Einstein manifold, i.e., the Ricci tensor of metric g satisfies $Ric = \frac{S}{n} g$. In this case, S is constant for dimensions $n \geq 3$. Thus if X is the conformal vector field with conformal factor ρ, that is, $\mathscr{L}_X g = 2\rho g$, we deduce $\mathscr{L}_X Ric = 2\rho\, Ric$. With this setting we define a *conformal Ricci vector field* on a Riemannian manifold (M^n, g) as a vector field X satisfying

$$\mathscr{L}_X Ric = 2\beta Ric, \tag{1}$$

for some smooth function $\beta : M^n \to \mathbb{R}$. In particular, on Einstein manifolds this concept is equivalent to the classical conformal vector field. With this additional condition the aforementioned conjecture is true. More precisely, we have the following theorem.

Theorem 1. *Let (M^n, g), $n \geq 3$, be a compact orientable Riemannian manifold with constant scalar curvature carrying a nontrivial conformal vector field X which is also a conformal Ricci vector field. Then M^n is isometric to a Euclidean sphere $\mathbb{S}^n(r)$. Moreover, up to a rescaling, the conformal factor ρ is given by*

$$\rho = \tau - \frac{h_v}{n}$$

and X is the gradient of the Hodge-de Rham function which in this case, up to a constant, is the function $\frac{1}{n}h_v$, where h_v is a height function on a unitary sphere $\mathbb{S}^n$ and τ is an appropriate constant.

We point out that Obata and Yano (1965) have obtained the same conclusion of the preceding theorem under the hypothesis $\mathscr{L}_X Ric = \alpha g$, for some smooth function α defined in M^n. Moreover, when M^n is an Einstein compact Riemannian manifold the result of Nagano and Yano (1959) is a consequence of Theorem 1. We also observe that the compactness of M^n in our result is an essential hypothesis. In fact, let us consider the hyperbolic space $\mathbb{H}^n$ as a hyperquadric of the Lorentz-Minkowski space $\mathbb{L}^{n+1}$ and v a nonzero fixed vector in $\mathbb{L}^{n+1}$. After a straightforward computation, it is easy to verify that the orthogonal projection $v^\top$ of v onto the tangent bundle $T\mathbb{H}^n$ provides a nontrivial conformal vector field on $\mathbb{H}^n$ for an appropriate choice of v. Consequently, since $\mathbb{H}^n$ is Einstein, it follows that $v^\top$ is also a nontrivial conformal Ricci vector field on $\mathbb{H}^n$.

PRELIMINARIES AND AUXILIARY RESULTS

To start with, we consider the Hilbert-Schmidt norm for tensors on a Riemannian manifold (M^n, g), i.e., the inner product $\langle T, S\rangle = \mathrm{tr}\big(TS^*\big)$. It is important to notice that for an orthonormal basis $\{e_1, \ldots, e_n\}$, we can use the natural identification of $(0,2)$-tensors with $(1,1)$-tensors, $T(e_i, e_j) = g(Te_i, e_j)$, to write

$$\langle T, S\rangle = \sum_{i,j} T_{ij} S_{ij} = \sum_i g(Te_i, Se_i).$$

We recall that the divergence of a $(1,r)$-tensor T on M^n is the $(0,r)$-tensor given by

$$(\mathrm{div}T)(v_1,\ldots,v_r)(p)=\mathrm{tr}\big(w\mapsto(\nabla_wT)(v_1,\ldots,v_r)(p)\big),$$

where $p\in M^n$ and $(v_1,\ldots,v_r)\in T_pM\times\ldots\times T_pM$.

Let X be a smooth vector field on M^n, and let φ be its flow. For any $p\in M^n$, if t is sufficiently close to zero, φ_t is a diffeomorphism from a neighborhood of p to a neighborhood of $\varphi_t(p)$, so φ_t^* pulls back tensors at $\varphi_t(p)$ to ones at p.

Given a $(0,r)$-tensor T on M^n, the Lie derivative of T with respect to X is the $(0,r)$-tensor $\mathscr{L}_XT$ given by

$$(\mathscr{L}_XT)_p=\frac{d}{dt}\Big|_{t=0}(\varphi_t^*T)_p=\lim_{t\to0}\frac{1}{t}\big[\varphi_t^*(T_{\varphi(t,p)})-T_p\big].$$

Fortunately, there is a simple formula for computing the Lie derivative without explicitly finding the flow. For any $(0,r)$-tensor T on M^n,

$$\begin{aligned}(\mathscr{L}_XT)(Y_1,\ldots,Y_r)&=X(T(Y_1,\ldots,Y_r))-T([X,Y_1],Y_2,\ldots,Y_r)-\ldots\\&\quad-T(Y_1,\ldots,Y_{r-1},[X,Y_r]),\end{aligned}$$

where $Y_1,\ldots,Y_r$ are any smooth vector fields on M^n, and $[X,Y_i]$ stands for the Lie bracket of X and Y_i.

In what follows we prove some lemmas and integral formulas which will be required later.

Lemma 1. *For any symmetric $(0,2)$-tensor T on a Riemannian manifold (M^n,g) and $X\in\mathfrak{X}(M)$, holds*

$$\mathscr{L}_X|T|^2=2\langle\mathscr{L}_XT,T\rangle-2\langle T^2,\mathscr{L}_Xg\rangle.$$

In particular, if X is a conformal vector field with conformal factor ρ, then we have $\mathscr{L}_X|T|^2=2\langle\mathscr{L}_XT,T\rangle-4\rho|T|^2$.

Proof. Let $\{e_1,\ldots,e_n\}$ be a geodesic orthonormal frame at $p\in M^n$. Then we have

$$\begin{aligned}\mathscr{L}_X|T|^2&=\mathscr{L}_X\Big(\sum_{i,j}T_{ij}T_{ij}\Big)=2\sum_{i,j}T_{ij}X(T_{ij})\\&=2\sum_{i,j}T_{ij}\{(\mathscr{L}_XT)_{ij}+T([X,e_i],e_j)+T(e_i,[X,e_j])\}\\&=2\sum_{i,j}T_{ij}(\mathscr{L}_XT)_{ij}-2\sum_{i,j}T_{ij}\{T(\nabla_{e_i}X,e_j)+T(e_i,\nabla_{e_j}X)\}.\end{aligned}$$

Whence, we use that T is symmetric to deduce

$$\begin{aligned}\mathscr{L}_X|T|^2&=2\langle T,\mathscr{L}_XT\rangle-2\sum_{i,j}T_{ij}\{g(\nabla_{e_i}X,Te_j)+g(Te_i,\nabla_{e_j}X)\}\\&=2\langle T,\mathscr{L}_XT\rangle-\sum_j2g(\nabla_{Te_j}X,Te_j)-\sum_i2g(Te_i,\nabla_{Te_i}X)\\&=2\langle T,\mathscr{L}_XT\rangle-2\sum_i(\mathscr{L}_Xg)(Te_i,Te_i)\\&=2\langle T,\mathscr{L}_XT\rangle-2\sum_{i,j}T_{ij}^2(\mathscr{L}_Xg)_{ij},\end{aligned}$$

which completes the proof of the lemma.

Corollary 1. *Under the assumptions of Lemma 1 we have* $\mathscr{L}_X|T|^2 = 0$*, provided that* $\mathscr{L}_X g = 2\rho g$ *and* $\mathscr{L}_X T = 2\rho T$.

Another useful result is given by the following lemma.

Lemma 2. *Let* (M^n, g) *be a Riemannian manifold endowed with a symmetric* $(0,2)$*-tensor* T*. Then it holds*

$$X\langle T, g\rangle = \langle \mathscr{L}_X T, g\rangle - \langle T, \mathscr{L}_X g\rangle.$$

In particular, if $\mathscr{L}_X T = 2\beta T$*, then*

$$2\beta \operatorname{tr}(T) = \langle \nabla(\operatorname{tr}(T)), X\rangle + \langle T, \mathscr{L}_X g\rangle.$$

Proof. Let $\{e_1, \ldots, e_n\}$ be a geodesic orthonormal frame at a point $p \in M^n$. By the symmetry of T, we have

$$\begin{aligned} \langle \mathscr{L}_X T, g\rangle = \operatorname{tr}(\mathscr{L}_X T) &= \sum_i \big(X(T_{ii}) + 2T(\nabla_{e_i} X, e_i)\big) \\ &= X\langle T, g\rangle + 2\langle T, \nabla X\rangle \\ &= X\langle T, g\rangle + \langle T, \mathscr{L}_X g\rangle, \end{aligned}$$

that finishes the proof of the lemma.

Now we remind the reader that the traceless tensor of a symmetric $(0,2)$-tensor T on a Riemannian manifold $\big(M^n, g\big)$ is given by

$$\mathring{T} = T - \frac{\operatorname{tr}(T)}{n} g. \tag{2}$$

With this setting we prove the next corollary.

Corollary 2. *Let* (M^n, g) *be a Riemannian manifold endowed with a symmetric* $(0,2)$*-tensor* T *such that* $\mathscr{L}_X T = 2\beta T$*, with* $\mathscr{L}_X g = 2\rho g$*. Then we have:*

1. $2\big(\beta - \rho\big)\operatorname{tr}(T) = \langle \nabla(\operatorname{tr}(T)), X\rangle$.
2. *If* $\operatorname{tr}(T)$ *is a non null constant, then* $\beta = \rho$ *and* $\mathscr{L}_X \mathring{T} = 2\rho \mathring{T}$.

Proof. Since $\mathscr{L}_X g = 2\rho g$ we have immediately the first item. Now, if $\operatorname{tr}(T)$ is a non zero constant, then we have $\beta = \rho$, which implies $\mathscr{L}_X \mathring{T} = \mathscr{L}_X T - \frac{\operatorname{tr}(T)}{n}\mathscr{L}_X g = 2\rho \mathring{T}$, finishing the proof of the corollary.

Next, we apply the previous results to the Ricci tensor. First of all, given a conformal vector field X on a Riemannian manifold M^n such that $\mathscr{L}_X g = 2\rho g$, we have the next well-known formulae, see e.g. Obata and Yano (1970).

$$\mathscr{L}_X Ric = -(n-2)\nabla^2 \rho - (\Delta\rho) g, \tag{3}$$

$$\mathscr{L}_X S = -2(n-1)\Delta\rho - 2S\rho, \tag{4}$$

and

$$\mathscr{L}_X G = -(n-2)\Big(\nabla^2 \rho - \frac{1}{n}(\Delta\rho) g\Big), \tag{5}$$

where $G = Ric - \frac{S}{n} g = \mathring{Ric}$.

We claim that if M^n is compact with constant scalar curvature S and ρ is not constant, then equation (4) allows us to infer that S is positive. Indeed, since ρ is not constant $\frac{S}{n-1}$ belongs to the spectrum of the Laplacian of M^n. Therefore, we deduce the next lemma.

Lemma 3. *Let (M^n, g) be a compact Riemannian manifold with constant scalar curvature such that $\mathscr{L}_X Ric = 2\beta\, Ric$ and $\mathscr{L}_X g = 2\rho g$. Then we have $\beta = \rho$ and $\mathscr{L}_X |G|^2 = 0$.*

Proof. Since $\mathscr{L}_X Ric = 2\beta\, Ric$ and $S = \mathrm{tr}(Ric)$ is a positive constant we get by Corollary 2 that $\beta = \rho$ and $\mathscr{L}_X G = 2\rho G$. Therefore, applying Corollary 1 we have $\mathscr{L}_X |G|^2 = 0$, which completes the proof of the lemma.

Taking into account the second contracted Bianchi identity: $\mathrm{div}(Ric) = \frac{1}{2} dS$ and using the identity $\mathrm{div}(\frac{S}{n} g) = \frac{1}{n} dS$ we obtain the following relation

$$\mathrm{div}(G) = \frac{n-2}{2n} dS.$$

Therefore, we can write

$$\rho \mathrm{div}(G)(\nabla\rho) = \frac{n-2}{4n} \langle \nabla S, \nabla \rho^2 \rangle.$$

The next equation is well-known. For details of a more general case see for example Lemma 1 in Barros and Gomes (2013).

$$\mathrm{div}(\rho G(\nabla\rho)) = \rho \mathrm{div}(G)(\nabla\rho) + \rho \langle \nabla^2 \rho, G \rangle + G(\nabla\rho, \nabla\rho). \tag{6}$$

Since $\langle G, g \rangle = 0$, from (5) we have

$$\langle \mathscr{L}_X G, G \rangle = -(n-2) \langle \nabla^2 \rho, G \rangle.$$

Applying Lemma 1 to this identity we obtain

$$\langle \nabla^2 \rho, G \rangle = -\frac{1}{n-2} \Big(\frac{1}{2} \mathscr{L}_X |G|^2 + 2\rho |G|^2 \Big). \tag{7}$$

Comparing (6) and (7) we infer

$$\mathrm{div}(\rho G(\nabla\rho)) = \frac{n-2}{4n} \langle \nabla S, \nabla \rho^2 \rangle - \frac{1}{n-2} \Big(\frac{\rho}{2} \mathscr{L}_X |G|^2 + 2\rho^2 |G|^2 \Big) + G(\nabla\rho, \nabla\rho). \tag{8}$$

In what follows we assume that (M, g) is an orientable Riemannian manifold. If M is not orientable, we take the orientable double covering $\tilde{M}$ of M and induce, in the natural manner, the Riemannian metric $\tilde{g}$ on $\tilde{M}$. Then (M, g) and $(\tilde{M}, \tilde{g})$ have the same local geometry.

As a direct consequence from (8) and Stokes' Theorem we obtain the following lemma.

Lemma 4. *Let (M^n, g) be a compact orientable Riemannian manifold endowed with a conformal vector field X of conformal factor ρ, then*

$$\int_M G(\nabla\rho, \nabla\rho) dM = \frac{1}{n-2} \int_M \Big(\frac{\rho}{2} \mathscr{L}_X |G|^2 + 2\rho^2 |G|^2 \Big) dM + \frac{n-2}{4n} \int_M \rho^2 \Delta S dM.$$

Before stating our next result we note that $|\nabla^2 \rho - \frac{\Delta\rho}{n} g|^2 = |\nabla^2 \rho|^2 - \frac{1}{n} (\Delta\rho)^2$. Accordingly, the Bochner formula becomes

$$\begin{aligned} \frac{1}{2} \Delta |\nabla\rho|^2 &= G(\nabla\rho, \nabla\rho) + \frac{S}{n} |\nabla\rho|^2 + |\nabla^2 \rho - \frac{\Delta\rho}{n} g|^2 + \frac{1}{n} (\Delta\rho)^2 \\ &\quad - \langle \nabla\rho, \nabla(\frac{S\rho}{n-1} + \frac{\mathscr{L}_X S}{2(n-1)}) \rangle, \end{aligned}$$

where in the last term we use equation (4). Then,

$$\begin{aligned}\frac{1}{2}\Delta|\nabla\rho|^2 &= G(\nabla\rho,\nabla\rho)+|\nabla^2\rho-\frac{\Delta\rho}{n}g|^2-\frac{S\big(|\nabla\rho|^2+\rho\Delta\rho\big)}{n(n-1)}\\ &\quad -\frac{1}{2n(n-1)}\Big((\mathscr{L}_XS)\Delta\rho+n\langle\nabla S,\nabla\rho^2\rangle+n\langle\nabla\rho,\nabla(\mathscr{L}_XS)\rangle\Big).\end{aligned}$$

By integration,

$$\int_M\Big(G(\nabla\rho,\nabla\rho)+|\nabla^2\rho-\frac{\Delta\rho}{n}g|^2+\frac{1}{2n}(\rho^2\Delta S+(\mathscr{L}_XS)\Delta\rho)\Big)dM=0. \tag{9}$$

In the notation of (2) we have $|\mathring{\nabla}^2\rho|^2=|\nabla^2\rho-\frac{\Delta\rho}{n}g|^2$. Comparing Lemma 4 with equation (9) we get the lemma.

Lemma 5. *Let (M^n,g) be a compact orientable Riemannian manifold endowed with a conformal vector field X of conformal factor ρ, then*

1. $\int_M\Big(\frac{\rho}{n-2}\big(\frac{1}{2}\mathscr{L}_X|G|^2+2\rho|G|^2\big)+|\mathring{\nabla}^2\rho|^2+\frac{S}{2}\mathrm{div}(\rho\nabla\rho)+\frac{\mathscr{L}_XS}{2n}\Delta\rho\Big)dM=0.$
2. $\int_M\Big(\frac{\rho}{n-2}\langle\mathscr{L}_XG,G\rangle+|\mathring{\nabla}^2\rho|^2+\frac{S}{2}\mathrm{div}(\rho\nabla\rho)+\frac{\mathscr{L}_XS}{2n}\Delta\rho\Big)dM=0.$

Proof. First assertion is a direct combination of Lemma 4 and equation (9), while the second one follows from Lemma 1 and the first assertion. Indeed, from this latter lemma we have $\frac{1}{2}\mathscr{L}_X|G|^2+2\rho|G|^2=\langle\mathscr{L}_XG,G\rangle$, which completes our proof.

We are in the right position to prove our main result.

PROOF OF THEOREM 1

Firstly, we observe that we are supposing that there exists a vector field X on M^n such that $\mathscr{L}_Xg=2\rho g$ and $\mathscr{L}_XRic=2\beta\, Ric$, for some smooth functions ρ and β on M^n, where ρ is non-constant. Consequently, from Lemma 3 we obtain $\beta=\rho$ and $\mathscr{L}_X|G|^2=0$. Secondly, from item (1) of Lemma 5 and equation (4) we get

$$\int_M\Big(\frac{2}{n-2}\rho^2|G|^2+|\nabla^2\rho+\frac{S\rho}{n(n-1)}g|^2\Big)dM=0.$$

Taking into account that ρ is non-constant the preceding identity allows us to achieve $G=0$ and $\nabla^2\rho=-\frac{S}{n(n-1)}\rho g$. Therefore, we can apply a classical result due to Obata (1962), for instance, to conclude that M^n is isometric to a Euclidean sphere $\mathbb{S}^n(r)$. Moreover, $\nabla^2\rho=\frac{\Delta\rho}{n}g$ and $\Delta(\Delta\rho)+\frac{S}{n-1}\Delta\rho=0$ (see equation (4)). Rescaling the metric we can assume that $S=n(n-1)$. Then we conclude that $\Delta\rho$ is the first eigenvalue of the unitary sphere $\mathbb{S}^n$. Whence, there exists a fixed vector $v\in\mathbb{S}^n$ such that $\Delta\rho=h_v=-\frac{1}{n}\Delta h_v$. Thus we have $\Delta(\rho+\frac{1}{n}h_v)=0$, which gives $\rho=\tau-\frac{1}{n}h_v$. Setting $u=-\rho$ we obtain

$$\mathscr{L}_{\nabla u}g=2\nabla^2u=-2\nabla^2\rho=2\rho g=\mathscr{L}_Xg.$$

It is also true that $\mathscr{L}_{\nabla u}Ric=2\rho Ric$. Besides, by Hodge-de Rham decomposition theorem we can write $X=Y+\nabla\ell$, for some vector field Y with $\mathrm{div}Y=0$ and ℓ is the Hodge-de Rham function. So, $\Delta u=\mathrm{div}X=\Delta\ell$ which implies $u-\ell$ is constant. Note that this is sufficient to complete our proof.

A MORE GENERAL CASE

We notice that $\mathscr{L}_X G = -(n-2)\mathring{\nabla}^2\rho$ and $\mathscr{L}_X G = \mathscr{L}_X Ric - \frac{1}{n}\mathscr{L}_X(Sg)$ give

$$\begin{aligned}
\mathscr{L}_X Ric &= \frac{1}{n}\mathscr{L}_X(Sg) - (n-2)\mathring{\nabla}^2\rho \\
&= 2\rho\big(\frac{S}{n}g\big) + \frac{1}{n}(\mathscr{L}_X S)g - (n-2)\mathring{\nabla}^2\rho \\
&= 2\rho\big(Ric - G\big) + \frac{1}{n}(\mathscr{L}_X S)g - (n-2)\mathring{\nabla}^2\rho \\
&= 2\rho Ric + \frac{1}{n}(\mathscr{L}_X S)g + T,
\end{aligned}$$

where $T = -2\rho G - (n-2)\mathring{\nabla}^2\rho$. Therefore, we deduce

$$\mathscr{L}_X Ric = 2\rho\, Ric + \frac{1}{n}(\mathscr{L}_X S)g + T,$$

where $\mathrm{tr}(T) = 0$.

Let us suppose that $\mathscr{L}_X Ric = 2\beta\, Ric + T$ and $\mathscr{L}_X g = 2\rho g$, where T is a $(0,2)$-tensor on M^n. By using (3) and (4) we deduce

$$\mathrm{tr}(T) = -2(n-1)\Delta\rho - 2\beta S$$

and

$$\mathscr{L}_X S = \mathrm{tr}(T) + 2(\beta-\rho)S.$$

In particular, if S is a non null constant and $\rho \neq 0$, we have

$$\mathrm{tr}(T) = 0 \quad \text{if and only if} \quad \beta = \rho. \tag{10}$$

On the other hand, $\mathscr{L}_X G = \mathscr{L}_X Ric - \frac{1}{n}\mathscr{L}_X(Sg)$ gives

$$\mathscr{L}_X G = 2\beta G + T - \frac{\mathrm{tr}(T)}{n}g \tag{11}$$

and

$$\mathscr{L}_X |G|^2 = 4(\beta-\rho)|G|^2 + 2\langle T, G\rangle. \tag{12}$$

Lemma 6. *Let (M^n, g), $n \geq 3$, be a compact orientable Riemannian manifold endowed with a conformal vector field X whose conformal factor is ρ. If $\mathscr{L}_X Ric = 2\beta\, Ric + T$, then the following integral formula holds:*

$$\int_M \Big(\frac{1}{n-2}\big(2\beta\rho|G|^2 + \rho\langle T, G\rangle\big) + |\nabla^2\rho - \frac{\Delta\rho}{n}g|^2 + \frac{S}{2}\mathrm{div}(\rho\nabla\rho) + \frac{\mathscr{L}_X S}{2n}\Delta\rho\Big)dM = 0.$$

Proof. First we notice that from (12) we obtain

$$\frac{\rho}{2}\mathscr{L}_X|G|^2 + 2\rho^2|G|^2 = 2\beta\rho|G|^2 + \rho\langle T, G\rangle. \tag{13}$$

Therefore, using (13) in the first assertion of Lemma 5, we have

$$\int_M \Big(\frac{1}{n-2}\big(2\beta\rho|G|^2 + \rho\langle T, G\rangle\big) + |\nabla^2\rho - \frac{\Delta\rho}{n}g|^2 + \frac{S}{2}\mathrm{div}(\rho\nabla\rho) + \frac{\mathscr{L}_X S}{2n}\Delta\rho\Big)dM = 0$$

which completes the proof of the lemma.

Proceeding we use this lemma to obtain the following theorem.

Theorem 2. *Let (M^n, g), $n \geq 3$, be a compact orientable Riemannian manifold with constant scalar curvature S. Suppose that there exists a nontrivial conformal vector field X on M^n such that $\mathscr{L}_X g = 2\rho g$ and $\mathscr{L}_X Ric = 2\beta\, Ric + T$. If $\mathrm{tr}(T) = 0$ and $\int_M \Big(2\rho^2|G|^2 + \rho\langle G, T\rangle\Big) dM \geq 0$, then M^n is isometric to a Euclidean sphere.*

Moreover, up to a rescaling, the conformal factor ρ is given by $\rho = \tau - \frac{h_v}{n}$ and X is the gradient of the Hodge-de Rham function which in this case, up to a constant, is the function $\frac{1}{n}h_v$, where h_v is a height function on a unitary sphere $\mathbb{S}^n$ and τ is an appropriate constant.

Proof. It follows from (10) that $\beta = \rho$. Now we may use Lemma 6 to deduce that $|\nabla^2\rho - \frac{\Delta\rho}{n}g|^2 = 0$. But, from (4) we have $\Delta\rho = -\frac{S}{n-1}\rho$. Therefore, we deduce $\nabla^2\rho = -\frac{S}{n(n-1)}\rho g$, from which we may apply Obata's Theorem, consult Obata (1962), to conclude that M^n is isometric to a Euclidean sphere. Moreover, from (11) we have that T is null tensor. So, the latter claim is proved following the same steps given in the proof of Theorem 1.

Remark 1. *Notice that if $T = \lambda\rho G$, with $(2+\lambda) \geq 0$, then the conditions of the previous theorem are verified. In particular, for $\lambda = -2$, we have $\mathscr{L}_X Ric = 2\rho\frac{S}{n}g$, which allows us to obtain the result due to Obata and Yano (1965).*

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