Anais da Academia Brasileira de Ciências (2003) 75(3): 271–278
(Annals of the Brazilian Academy of Sciences)
ISSN 0001-3765
www.scielo.br/aabc

Minimal Surfaces in Euclidean 3-Space and Their Mean Curvature 1 Cousins in Hyperbolic 3-Space

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Manuscript received on February 4, 2003; accepted for publication on June 4, 2003; presented by MANFREDO DO CARMO

ABSTRACT

We show that the Hopf differentials of a pair of isometric cousin surfaces, a minimal surface in euclidean 3-space and a constant mean curvature (CMC) one surface in the 3-dimensional hyperbolic space, with properly embedded annular ends, extend holomorphically to each end. Using this result, we derive conditions for when the pair must be a plane and a horosphere.

Key words: minimal surfaces, CMC 1 cousins, hyperbolic space.

INTRODUCTION

As there is a way to deform simply-connected CMC 1 surfaces in hyperbolic 3-space $\mathbb{H}^3$ to minimal surfaces in Euclidean 3-space $\mathbb{R}^3$ (Umehara and Yamada 1992), one might expect that there exist cousins in these two spaces that are not simply-connected. However, although there are now many known examples of minimal surfaces in $\mathbb{R}^3$ and also CMC 1 surfaces in $\mathbb{H}^3$ (see, for example, Bryant 1987, Rossman et al. 1997, 2001, Sá Earp and Toubiana 2001, Yu 2001, Umehara and Yamada 1993), and although non-simply-connected cousins pairs are easily found, such a pair of surfaces with embedded ends is yet to be found. Our purpose is to investigate whether such a pair can exist. Toward this goal, we apply recent results in Collin et al. 2001 about the behavior of embedded CMC 1 ends in $\mathbb{H}^3$ to give various conditions under which such a pair cannot exist.

RESULTS

Let $D \subset \mathbb{C}$ be a simply-connected domain in the complex plane. Fix a point $z_0 \in D$. Let g be a meromorphic function on D and ω a holomorphic 1-form on D such that ω has a zero of order $2k$

Mathematics Subject Classification: 53A10, 53C42.
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if and only if g has a pole of order k and so that ω has no other zeros. Set

$$\Phi_0(z) := \operatorname{Re} \int_{z_0}^{z} \left(1 - g^2, i(1+g^2), 2g\right) \omega \, .$$

Then $\Phi_0 : D \to \mathbb{R}^3$ is a minimal immersion with induced metric

$$\Phi_0^*(ds^2_{\mathbb{R}^3}) = \left(1 + |g|^2\right)^2 |\omega|^2 \, .$$

Furthermore, g is stereographic projection of the Gauss map of Φ_0. This is the Weierstrass representation.

On the other hand, for Weierstrass data (g, ω) on D, we can take $F : D \to SL(2, \mathbb{C})$ such that

$$F^{-1}dF = \begin{pmatrix} g & -g^2 \\ 1 & -g \end{pmatrix} \omega \, , \quad F(z_0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and set

$$\Phi_1(z) := F(z)\overline{F(z)}^t \, .$$

Then $\Phi_1 : D \to \{X\overline{X}^t \in \mathrm{Herm}(2); X \in SL(2, \mathbb{C})\} \cong \mathbb{H}^3$ is a CMC 1 immersion with induced metric $\Phi_1^*(ds^2_{\mathbb{H}^3}) = \Phi_0^*(ds^2_{\mathbb{R}^3})$, where $\mathbb{H}^3 = \mathbb{H}^3(-1)$ is the hyperbolic 3-space with sectional curvature -1. This is the Bryant representation (Bryant 1987, Umehara and Yamada 1993). F is unique up to the form $A \cdot F$, A a constant in $SL(2, \mathbb{C})$, so Φ_1 is unique up to rigid motions of $\mathbb{H}^3$ (see Umehara and Yamada 1993).

This shows that given data (g, ω) on D, we can locally construct a pair of isometric surfaces, a minimal surface $\Phi_0(D)$ in $\mathbb{R}^3$ and a CMC 1 surface $\Phi_1(D)$ in $\mathbb{H}^3$ (see Theorem 8 of Lawson 1970).

For both Φ_0 and Φ_1, the Hopf differential Q on D is defined by $Q = \omega dg$.

DEFINITION 1. *Let M be a Riemann surface and $\Phi_0 : M \to \mathbb{R}^3$ a conformal minimal immersion. Then a CMC 1 immersion $\Phi_1 : M \to \mathbb{H}^3$ is a* cousin surface *of Φ_0 if*

$$\Phi_1^*\left(ds^2_{\mathbb{H}^3}\right) = \Phi_0^*\left(ds^2_{\mathbb{R}^3}\right)$$

holds. We refer to any such pair of surfaces Φ_0 and Φ_1 as cousins.

The following lemma is immediately obtained from §177 of Nitsche 1989:

LEMMA 2. *Let (g, ω) be the Weierstrass data of a simply-connected CMC 1 surface $\Phi_1 : D \to \mathbb{H}^3$. Then any cousin minimal surface Φ_0 in $\mathbb{R}^3$ can be represented (up to a rigid motion) by the Weierstrass data $(g, e^{i\theta}\omega)$ for some $\theta \in [0, \pi)$.*

Recall that a surface has *finite topology* if it is homeomorphic to a compact Riemann surface $\overline{M}$ with a finite number of points $\{p_1, \ldots, p_k\}$ removed, which we write as $M = \overline{M} \setminus \{p_1, \ldots, p_k\}$. We have the following proposition, which follows directly from results in Collin et al. 2001 and Sá Earp and Toubiana 2001.

PROPOSITION 3. *Let* $M = \overline{M} \setminus \{p_1, \dots, p_k\}$ *be a Riemann surface of finite topology, and let* $\Phi_1 : M \to \mathbb{H}^3$ *be a conformal CMC* 1 *immersion with properly embedded annular ends. Let* $\Phi_0 : M \to \mathbb{R}^3$ *be a minimal immersion with embedded ends, and assume that* Φ_1 *and* Φ_0 *are cousins. Then the Hopf differentials of* Φ_1 *and* Φ_0 *are holomorphic on* $\overline{M}$.

REMARK 4. By Theorem 10 of Collin et al. 2001, all properly embedded annular CMC 1 ends in $\mathbb{H}^3$ are conformal to a punctured disk, thus the assumption that Φ_1 is conformal is not actually a restriction on the possible choices of Φ_1. Because Φ_0 and Φ_1 are cousins, $\Phi_0 : M \to \mathbb{R}^3$ is also conformal.

PROOF OF PROPOSITION 3. Let $\varphi_1 : \Delta_\varepsilon^* \to \mathbb{H}^3$ be an arbitrary end of Φ_1, where $\Delta_\varepsilon^* = \{z \in \mathbb{C}; 0 < |z| < \varepsilon\}$ for some $\varepsilon > 0$. As noted in Remark 4, we may assume that φ_1 is conformal. Let $\varphi_0 : \Delta_\varepsilon^* \to \mathbb{R}^3$ be the corresponding minimal end. By Theorem 10 of Collin et al. 2001, φ_1 has finite total curvature and is regular. Then by Umehara and Yamada 1993, we can take the Weierstrass data associated with φ_1 in the following form:

$$g(z) = z^\mu \hat{g}(z)\,, \quad \hat{g}(0) \neq 0\,, \qquad \omega = z^\nu \hat{w}(z)dz\,, \quad \hat{w}(0) \neq 0\,,$$

where $\hat{g}$, $\hat{w}$ are nonzero holomorphic functions on $\Delta_\varepsilon = \{z \in \mathbb{C}; |z| < \varepsilon\}$, and $\mu,\ \nu \in \mathbb{R}$, $\mu > 0$, $\nu \leq -1$, $\mu + \nu \in \mathbb{Z}$, $\mu + \nu \geq -1$.

By Lemma 2, there exists a $\theta \in [0, \pi)$ such that $(g, e^{i\theta}\omega)$ is the Weierstrass data associated with φ_0. Because g is stereographic projection of the Gauss map of φ_0, g is well-defined on Δ_ε^*, so $\mu \in \mathbb{N}$ and hence $-\nu \in \mathbb{N}$.

The first and second coordinates of φ_0 are

$$\mathrm{Re} \int_{z_0}^{z} (1 - g^2) e^{i\theta}\omega\,, \quad -\mathrm{Im} \int_{z_0}^{z} (1 + g^2) e^{i\theta}\omega\,,$$

and φ_0 is asymptotic to a catenoid or planar end, by Schoen 1983. Also $g(0) = 0$, and the limiting normal of the end φ_0 must be vertical. Therefore, ν must be -2 for the end to be embedded, and $\hat{w}'(0)$ must be 0 for the end φ_0 to be well-defined on Δ_ε^*.

Lemma 2.4 of Sá Earp and Toubiana 2001 showed that $0 \neq \hat{g}(0)\hat{w}(0) = (1 - \mu^2)/4\mu$. So μ cannot be 1 because $\hat{g}(0) \neq 0$ and $\hat{w}(0) \neq 0$. Furthermore, Lemma 2.9 of Sá Earp and Toubiana 2001 showed that

$$\hat{w}'(0) = \begin{cases} 2\hat{w}(0)^2\hat{g}(0) & \text{if } \mu = 2\,, \\ 0 & \text{if } \mu \geq 3\,. \end{cases}$$

So μ cannot be 2. Therefore $\mu \geq 3$.

Thus the Hopf differentials ωdg and $e^{i\theta}\omega dg$ have order $\mu + \nu - 1 \geq 0$ at $z = 0$. Hence they are holomorphic at each end, as well as on M itself. □

An end $\varphi_0 : \Delta_\varepsilon^* \to \mathbb{R}^3$ (resp. $\varphi_1 : \Delta_\varepsilon^* \to \mathbb{H}^3$) is said to be a *planar end* (resp. *horosphere end*) if $\mu + \nu \geq 0$. So we have the following corollary:

COROLLARY 5. *Hypotheses being as in Proposition 3, then Φ_0 has only planar ends and Φ_1 has only horosphere ends.*

COROLLARY 6. *Let $M = \overline{M} \setminus \{p_1, \ldots, p_k\}$ be a Riemann surface of finite topology so that $\overline{M}$ has genus zero. Let $\Phi_0 : M \to \mathbb{R}^3$, $\Phi_1 : M \to \mathbb{H}^3$ be properly immersed cousin surfaces with embedded ends. Then Φ_0 is a plane and Φ_1 is a horosphere.*

PROOF. Since there exists no nonzero holomorphic 2-differential on the sphere $\mathbb{C} \cup \{\infty\}$, the Hopf differential is identically zero. So both $\Phi_0(M)$ and $\Phi_1(M)$ are totally umbilic. Therefore Φ_0 is a plane and Φ_1 is a horosphere. □

COROLLARY 7. *Let $M = \overline{M} \setminus \{p_1, \ldots, p_k\}$ be a Riemann surface of finite topology so that $\overline{M}$ has genus γ. Let $\Phi_0 : M \to \mathbb{R}^3$, $\Phi_1 : M \to \mathbb{H}^3$ be properly immersed cousin surfaces with embedded ends, and suppose they have total curvature more than -16π. Then Φ_0 is a plane and Φ_1 is a horosphere.*

PROOF. Lopez 1992 showed that any minimal surface with total curvature -4π or -8π has a non-holomorphic Hopf differential Q on $\overline{M}$. Thus the only possibility (other than a plane) is that $\Phi_0 : M \to \mathbb{R}^3$ is a properly immersed minimal surface with embedded planar ends and total curvature -12π. By Theorem 4 of Jorge and Meeks 1983, each end of Φ_0 is embedded if and only if

$$\int_M K\,dA = -4\pi(k + \gamma - 1) \tag{1}$$

holds, where K and dA are the Gaussian curvature and the area element of Φ_0. So $k+\gamma = 4$. Since any complete minimal surface with finite total curvature and one embedded end is a plane, and since the only complete minimal surface in $\mathbb{R}^3$ with finite total curvature and two embedded ends is the catenoid (Schoen 1983), $\Phi_0(M)$ is a torus with three embedded planar ends. But Theorem 26 of Kusner and Schmitt 1992 showed that such a surface does not exist, completing the proof. □

COROLLARY 8. *Let $M = \overline{M} \setminus \{p_1, \ldots, p_k\}$ be a Riemann surface of finite topology so that $\overline{M}$ has genus one. Let $\Phi_0 : M \to \mathbb{R}^3$, $\Phi_1 : M \to \mathbb{H}^3$ be properly immersed cousin surfaces with embedded ends. Then Φ_0 and Φ_1 each have at least 4 ends.*

PROOF. By Theorem 4 of Jorge and Meeks 1983 again, the right hand side of (1) is $-4k\pi$. So $k \geq 4$, by Corollary 7. □

REMARK 9. Theorem 3 of Miyaoka and Sato 1994 found examples of complete minimal surfaces of genus one with four embedded ends, but they all contain non-planar ends.

REMARK 10. Costa 1993 and Kusner and Schmitt 1992 found examples of complete minimal surfaces of genus one with four embedded planar ends. But none of them satisfies the condition that the Hopf differential extends holomorphically to the ends.

Defining annular ends to be those which are homeomorphic to punctured disks, Theorem 12 of Collin et al. 2001 showed that each end of a properly embedded non-totally-umbilic CMC 1 surface $\Phi_1 : M \to \mathbb{H}^3$ with annular ends is asymptotic to an end of a CMC 1 catenoid. In particular, such a surface does not have horosphere ends. We saw in the proof of Proposition 3, in conjunction with Remark 4, that any single embedded annular end asymptotic to a CMC 1 catenoid in $\mathbb{H}^3$ cannot have a corresponding minimal cousin in $\mathbb{R}^3$ with an embedded end. Hence, Φ_1 does not have a cousin $\Phi_0 : M \to \mathbb{R}^3$ with embedded ends. So we have the following corollary, in which we do not need to assume that M has finite topology, since finite topology was not assumed in Theorem 12 of Collin et al. 2001:

Corollary 11. *Let M be a Riemann surface. Let $\Phi_1 : M \to \mathbb{H}^3$ be a conformal CMC 1 proper embedding with annular ends, and let $\Phi_0 : M \to \mathbb{R}^3$ be a minimal surface with embedded ends. Assume that Φ_1 and Φ_0 are cousins. Then Φ_0 is a plane and Φ_1 is a horosphere.*

Remark 12. Regarding Corollary 11:

(i) If the assumption that Φ_1 is embedded is removed, then the pair of cousin surfaces given by the Weierstrass data

$$(g, \omega) = \left(z, \frac{n^2-1}{4} z^{-2} dz\right) \quad \text{on} \quad M = \mathbb{C} \setminus \{0\}, \quad n \in \mathbb{N} \setminus \{1\}$$

is a counterexample. In fact, each end of Φ_1 in this example is an n-fold cover of an embedded end, and Φ_0 is an embedding.

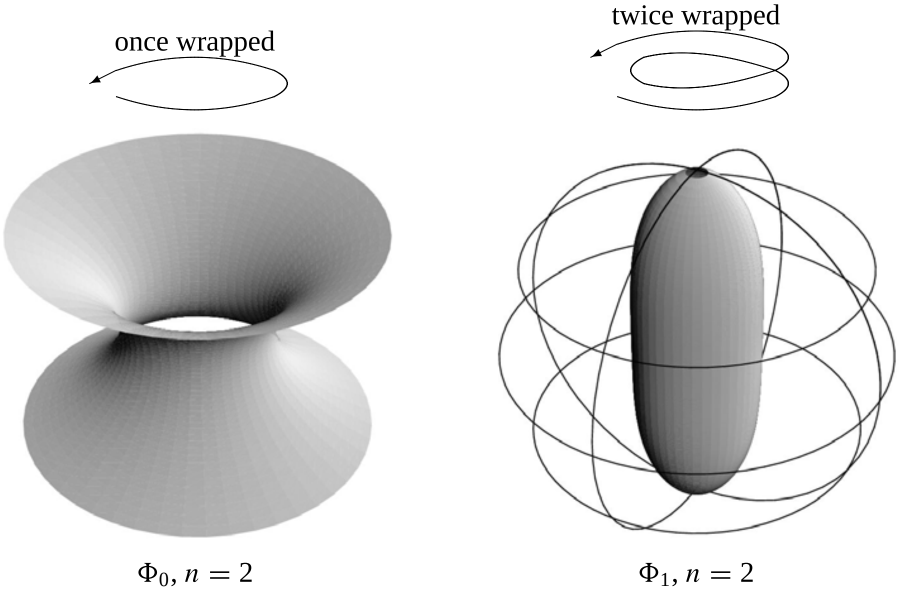

Fig. 1 – The surfaces Φ_0 and Φ_1 in (i) of Remark 12.

(ii) If the assumption that Φ_1 is embedded is replaced with the weaker assumption that only the ends are embedded, then any possible counterexamples can not satisfy the conditions of Corollaries 6 or 7 or 8.

(iii) If the weaker assumption in (ii) is used, and the assumption that the ends of Φ_0 are embedded is removed, then the pair of cousin surfaces given by the Weierstrass data

$$(g, \omega) = \left(z^n, \frac{1-n^2}{4n} z^{-1-n} dz \right) \quad \text{on} \quad M = \mathbb{C} \setminus \{0\}, \quad n \in \mathbb{N} \setminus \{1\}$$

is a counterexample to the corollary. In fact, in this example, each end of Φ_1 is embedded, and each end of Φ_0 is an n-fold cover of an embedded end.

(iv) If the assumption that Φ_1 is embedded is kept, but the assumption that each end of Φ_0 is embedded is removed, then the author does not know of any counterexamples to the corollary.

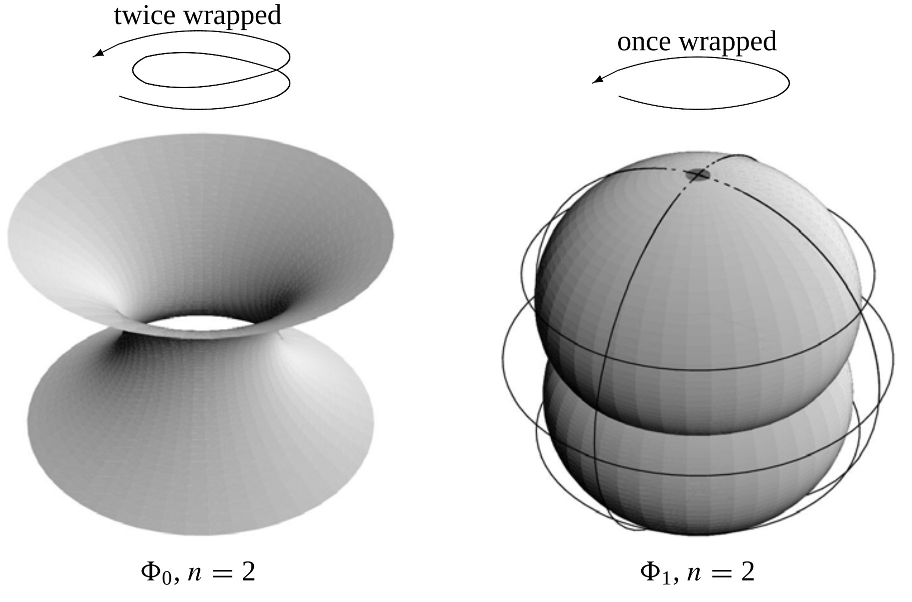

Fig. 2 – The surfaces Φ_0 and Φ_1 in (iii) of Remark 12.

REMARK 13. Theorem 3.3 of Choi et al. 1990 showed that a properly embedded minimal surface in $\mathbb{R}^3$ which has more than one end is minimally rigid. Corollary 3.4 of Umehara and Yamada 1992 showed that if cousin surfaces $\tilde{\Phi}_c : \Delta_\varepsilon \to \mathbb{H}^3(-c^2)$ $(c > 0)$ associated with a minimal surface $\Phi_0 \circ \rho : \Delta_\varepsilon \to \mathbb{R}^3$ are well-defined on Δ_ε^* for all c, then all of the surfaces in the associate family of Φ_0 are well-defined on Δ_ε^*, where

$$\rho : \Delta_\varepsilon \ni z \mapsto \varepsilon e^{(z-\varepsilon)/(z+\varepsilon)} \in \Delta_\varepsilon^*$$

is the projection. However, this cannot lead us to another proof of Corollary 11, because we only assume that the Φ_c is well-defined when $c = 0, 1$. Furthermore, we allow M to have positive genus, so we are not considering well-definedness merely on domains which are simply-connected or homeomorphic to Δ_ε^*.

ACKNOWLEDGMENTS

The author thanks Professors Wayne Rossman and Hitoshi Furuhata for their assistance with this work.

RESUMO

Mostramos que as diferenciais de Hopf de um par de superfícies primas, a saber, uma superfície mínima em um espaço euclideano de dimensão 3 e uma superfície de curvatura média constante (CMC) um em um espaço hiperbólico de dimensão 3, se estendem holomorficamente em cada fim. Usando este resultado, obtemos condições para que o par seja um plano e uma horosfera.

Palavras-chave: superfícies mínimas, prismas de CMC 1, espaços hiperbólicos.

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