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The hypersurfaces with conformal normal Gauss map in H^{n+1} and S_1^{n+1}

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ABSTRACT

In this paper, we introduce the fourth fundamental forms for hypersurfaces in H^{n+1} and space-like hypersurfaces in S_1^{n+1}, and discuss the conformality of the normal Gauss map of the hypersurfaces in H^{n+1} and S_1^{n+1}. Particularly, we discuss the surfaces with conformal normal Gauss map in H^3 and S_1^3, and prove a duality property. We give a Weierstrass representation formula for space-like surfaces in S_1^3 with conformal normal Gauss map. We also state the similar results for time-like surfaces in S_1^3. Some examples of surfaces in S_1^3 with conformal normal Gauss map are given and a fully nonlinear equation of Monge-Ampère type for the graphs in S_1^3 with conformal normal Gauss map is derived.

Key words: fourth fundamental form, conformal normal Gauss map, generalized Gauss map, duality property, de Sitter Gauss map, Monge-Ampère equation.

1 INTRODUCTION

It is well known that the classical Gauss map has played an important role in the study of the surface theory in R^3 and has been generalized to the submanifold of arbitrary dimension and codimension immersed into the space forms with constant sectional curvature (see Osserman 1980).

Particularly, for the n-dimensional submanifold $x: M \to V$ in space V with constant sectional curvature, Obata (Obata 1968) introduced the generalized Gauss map which assigns each point p of M to the totally geodesic n-subspace of V tangent to $x(M)$ at $x(p)$. He defined the third fundamental form of the submanifold in constant curvature space as the pullback of the metric of the set of all totally geodesic n-subspaces in V under the generalized Gauss map. He derived a relationship among the Ricci tensor of the immersed submanifold and the first, the second and the third fundamental forms of the immersion. Meanwhile, Lawson (Lawson 1970) discussed the generalized Gauss map of the immersed surfaces in S^3, and prove a duality property between the minimal surfaces in S^3 and their generalized Gauss map images.

Epstein (Epstein 1986) and Bryant (Bryant 1987) defined the hyperbolic Gauss map for surfaces in H^3, and Bryant (Bryant 1987) obtained a Weierstrass representation formula for constant mean curvature one

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surfaces with conformal hyperbolic Gauss map. Using the Weierstrass representation formula, Bryant also studied the properties of constant mean curvature one surfaces. Using the hyperbolic Gauss map, Gálvez and Martínez and Milán (Gálvez et al. 2000) studied the flat surfaces in H^3 with conformal hyperbolic Gauss map with respect to the second conformal structure on surfaces (see (Klotz 1963) for the definition), and obtained a Weierstrass representation formula for such surfaces.

Kokubu (Kokubu 1997) considered the n-dimensional hyperbolic space H^n as a Lie group G with a left-invariant metric, and defined the normal Gauss map of the surfaces which assigns each point of the surface to the tangent plane left translated to the Lie algebra of G. He also gave a Weierstrass representation formula for minimal surfaces in H^n. On the other hand, Gálvez and Martínez (Gálvez and Martínez 2000) studied the properties of the Gauss map of a surface Σ immersed into the Euclidean 3-space R^3 by using the second conformal structure on surface, and obtained a Weierstrass representation formula for surfaces with prescribed Gauss map. Motivated by their work, the author (Shi 2004) gave a Weierstrass representation formula for surfaces with prescribed normal Gauss map and Gauss curvature in H^3 by using the second conformal structure on surfaces. From this, the surfaces whose normal Gauss maps are conformal have been found, and the translational surfaces with conformal normal Gauss map locally are given. In (Shi 2006), the author classified locally the ruled surfaces with conformal normal Gauss map within the Euclidean ruled surfaces, and studied some global properties of the ruled surfaces and translational surfaces with conformal normal Gauss map.

Aiyama and Akutagawa (Aiyama and Akutagawa 2000) defined the normal Gauss map for space-like surfaces in the de Sitter 3-space S_1^3, and gave a Weierstrass representation formula for space-like surfaces in S_1^3 with prescribed mean curvature and normal Gauss map.

The purpose of this paper is to study the conformality of the normal Gauss map for hypersurfaces in H^{n+1} and space-like hypersurfaces in S_1^{n+1}, and to prove a duality property between the surfaces in H^3 and the space-like surfaces in S_1^3 with conformal normal Gauss map. The rest of this paper is organized as follows. In the second section, we describe the generalized definition of the normal Gauss map for hypersurfaces in H^{n+1} and space-like hypersurfaces in S_1^{n+1} (cf. Kokubu 1997, Aiyama and Akutagawa 2000). The third section introduces the fourth fundamental forms for hypersurfaces in H^{n+1} and S_1^{n+1}, and obtains a relation among the first, the second, the third and the fourth fundamental forms of the hypersurfaces. As an application, we discuss the conformality of the normal Gauss map for hypersurfaces in H^{n+1} and space-like hypersurfaces in S_1^{n+1}. By means of the generalized Gauss map of the surfaces in H^3 and S_1^3, the fourth one proves a duality property between the surfaces in H^3 and the space-like surfaces in S_1^3 with conformal normal Gauss map. The fifth one gives a Weierstrass representation formula for space-like surfaces in S_1^3 with conformal normal Gauss map, and the sixth one derives the fully nonlinear equation of Monge-Ampère type for space-like graphs in S_1^3 with conformal normal Gauss map and classifies locally the translational surfaces and the Euclidean ruled surfaces in S_1^3 with conformal normal Gauss map. In the last section, we state the similar results for time-like surfaces in S_1^3 with conformal normal Gauss map.

2 PRELIMINARIES

Take upper half-space models of hyperbolic space $H^{n+1}(-1)$ and de Sitter space $S_1^{n+1}(1)$

$$R_+^{n+1} = \left\{ \left(x_1, x_2, \ldots, x_{n+1}\right) \in R^{n+1} | x_{n+1} > 0 \right\}$$

with respectively Riemannian metric

$$ds^2 = \frac{1}{x_{n+1}^2}\left(dx_1^2 + dx_2^2 + \cdots + dx_{n+1}^2\right)$$

and Lorentz metric

$$ds^2 = \frac{1}{x_{n+1}^2}\left(dx_1^2 + dx_2^2 + \cdots + dx_n^2 - dx_{n+1}^2\right)$$

(see Aiyama and Akutagawa 2000 or section 4).

Let M be a n-dimensional Riemannian manifold and $x\colon M^n \to H^{n+1}$ (*resp.* $x\colon M^n \to S_1^{n+1}$) be an immersed hypersurface (*resp.* space-like hypersurface) with local coordinates $u_1, u_2, \ldots, u_n$. In this paper,we agree with the following ranges of indices: $1 \leq i, j, k, \ldots \leq n$ and $1 \leq A, B, C, \ldots \leq n+1$. The first and the second fundamental forms are given, respectively, by $\mathrm{I} = g_{ij}du_idu_j$ and $\mathrm{II} = h_{ij}du_idu_j$. The unit normal vector (*resp.* time-like unit normal vector) of $x(M)$ is

$$N = x_{n+1}\eta_1\frac{\partial}{\partial x_1} + x_{n+1}\eta_2\frac{\partial}{\partial x_2} + \cdots + x_{n+1}\eta_{n+1}\frac{\partial}{\partial x_{n+1}},$$

where $\eta_1^2 + \eta_2^2 + \cdots + \eta_{n+1}^2 = 1$ (*resp.* $\eta_1^2 + \eta_2^2 + \cdots + \eta_n^2 - \eta_{n+1}^2 = -1$).

We have the Weingarten formula

$$\frac{\partial \eta_A}{\partial u_k} = \frac{1}{x_{n+1}}\left(\eta_{n+1}\frac{\partial x_A}{\partial u_k} - g^{jl}h_{kl}\frac{\partial x_A}{\partial u_j}\right)\left(\textit{resp.}\ \frac{\partial \eta_A}{\partial u_k} = \frac{1}{x_{n+1}}\left(\eta_{n+1}\frac{\partial x_A}{\partial u_k} + g^{jl}h_{kl}\frac{\partial x_A}{\partial u_j}\right)\right).$$

Identitying H^{n+1} and S_1^{n+1} with Lie group (Kokubu 1997)

$$G = \left\{\begin{pmatrix} 1 & 0 & \cdots & 0 & \log x_{n+1} \\ 0 & x_{n+1} & \cdots & 0 & x_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & x_{n+1} & x_n \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} : (x_1, x_2, \ldots, x_{n+1}) \in R_+^{n+1}\right\},$$

the multiplication is defined as matrix multiplication and the identity is $e = (0, 0, \ldots, 0, 1)$. The Riemannian metric of H^{n+1} and the Lorentz metric of S_1^{n+1} are left-invariant, and

$$\widetilde{X}_1 = x_{n+1}\frac{\partial}{\partial x_1},\ \widetilde{X}_2 = x_{n+1}\frac{\partial}{\partial x_2}, \ldots,\ \widetilde{X}_{n+1} = x_{n+1}\frac{\partial}{\partial x_{n+1}}$$

are the left-invariant unit orthonormal vector fields. Now, the unit normal vector (*resp.* time-like unit normal vector) field of $x(M)$ can be written as $N = \eta_1\widetilde{X}_1 + \eta_2\widetilde{X}_2 + \cdots + \eta_{n+1}\widetilde{X}_{n+1}$. Left translating N to $T_e(R_+^{n+1})$, we obtain

$$\widetilde{N}\colon M \to S^n(1) \subset T_e\left(R_+^{n+1}\right)\left(\textit{resp.}\ \widetilde{N}\colon M \to H^n(-1) \subset T_e\left(R_+^{n+1}\right)\right),$$

$$\widetilde{N} = L_{x^{-1}*}(N) = \eta_1\frac{\partial}{\partial x_1}(e) + \eta_2\frac{\partial}{\partial x_2}(e) + \cdots + \eta_{n+1}\frac{\partial}{\partial x_{n+1}}(e).$$

Call $\widetilde{N}$ the normal Gauss map of the immersed hypersurface $x: M \to H^{n+1}$ (*resp.* space-like hypersurface $x: M \to S_1^{n+1}$) (Kokubu 1997, Aiyama and Akutagawa 2000).

3 THE FOURTH FUNDAMENTAL FORM

DEFINITION. Let M be a n-dimensional Riemannian manifold. Call $\mathrm{IV} = \langle d\widetilde{N}, d\widetilde{N}\rangle$ the fourth fundamental form of the immersed hypersurface $x: M \to H^{n+1}$ (*resp.* space-like hypersurface $x: M \to S_1^{n+1}$), where the scalar product $\langle \cdot, \cdot \rangle$ is induced by the Euclidean metric of R^{n+1} (*resp.* the Lorentz-Minkowski metric of L^{n+1}).

THEOREM 3.1. *Let M be a n-dimensional Riemannian manifold with Ricci tensor Ric. Let $x: M \to H^{n+1}$ (resp. $x: M \to S_1^{n+1}$) be an immersed hypersurface (resp. space-like hypersurface) with mean curvature $H = \frac{1}{n}tr(\mathrm{II})$. Then*

$$\mathrm{IV} \;=\; \eta_{n+1}^2 \mathrm{I} - 2\eta_{n+1}\mathrm{II} + \mathrm{III} \tag{3.1}$$

$$\big(\textit{resp. } \mathrm{IV} \;=\; \eta_{n+1}^2 \mathrm{I} + 2\eta_{n+1}\mathrm{II} + \mathrm{III}\big), \tag{3.2}$$

where $\mathrm{III} = nH\mathrm{II} - (n-1)\mathrm{I} - Ric$ *(resp.* $\mathrm{III} = nH\mathrm{II} - (n-1)\mathrm{I} + Ric$*) is Obata's third fundamental form of* $x(M)$ (see Obata 1968).

PROOF. At first we prove the Theorem for H^{n+1}. Choose the normal coordinates $u_1, u_2, \ldots, u_n$ near $p \in M$. By the Weingarten formula, we get

$$\begin{aligned} \mathrm{IV} \;&=\; \langle d\widetilde{N}, d\widetilde{N}\rangle = \frac{\partial \eta_A}{\partial u_i}\frac{\partial \eta_A}{\partial u_j} du_i du_j \\ &=\; \frac{1}{x_{n+1}^2}\left(\eta_{n+1}\frac{\partial x_A}{\partial u_i} - h_{ik}\frac{\partial x_A}{\partial u_k}\right)\left(\eta_{n+1}\frac{\partial x_A}{\partial u_j} - h_{jl}\frac{\partial x_A}{\partial u_l}\right) du_i du_j \\ &=\; \big(\eta_{n+1}^2\delta_{ij} - 2\eta_{n+1}h_{ij} + h_{ik}h_{jk}\big) du_i du_j. \end{aligned} \tag{3.3}$$

$\mathrm{III} = h_{ik}h_{jk}du_i du_j$ is the third fundamental form (Obata 1968) and by the Gauss equation, $\mathrm{III} = nH\mathrm{II} - (n-1)\mathrm{I} - Ric$. (3.1) is proved.

Next, similar to the above proof, for S_1^{n+1}, we have

$$\mathrm{IV} \;=\; \big(\eta_{n+1}^2\delta_{ij} + 2\eta_{n+1}h_{ij} + h_{ik}h_{jk}\big) du_i du_j. \tag{3.4}$$

Similar to the proof of (3.1), we can prove (3.2).

Next, we consider the applications of these formulas (3.1)–(3.4). In the following of this paper, that the normal Gauss map is conformal means that the fourth fundamental form is proportional to the second fundamental form, i.e. $\mathrm{IV} = \rho\mathrm{II}$ for some smooth function ρ on M (Shi 2004, 2006).

THEOREM 3.2. *Let M be a n-dimensional Riemannian manifold and $x: M \to H^{n+1}$ (resp. $x: M \to S_1^{n+1}$) be an immersed hypersurface (resp. space-like hypersurface) without umbilics. Then the normal Gauss map of $x(M)$ is conformal if and only if at each point of M, there exists exactly two distinct principal curvatures and the sectional curvature $R(X \wedge Y) = -1 + \eta_{n+1}^2$ (resp. $R(X \wedge Y) = 1 - \eta_{n+1}^2$), where the vectors X and Y belong to different principal direction spaces.*

PROOF. The case of H^{n+1}. For any point $p \in M$, let $\{e_1, e_2, \dots, e_n\}$ be a local frame field so that (h_{ij}) is diagonalized at this point, i.e. $h_{ij}(p) = \lambda_i \delta_{ij}$. By IV $= \rho$II and (3.3), we get, for $i = 1, 2, \dots, n$, that

$$\eta_{n+1}^2 - 2\eta_{n+1}\lambda_i + \lambda_i^2 = \rho\lambda_i, \tag{3.5}$$

i.e.

$$\lambda_i^2 - (\rho + 2\eta_{n+1})\lambda_i + \eta_{n+1}^2 = 0. \tag{3.6}$$

Because $x(M)$ has no umbilics, the equation (3.6) with respect to λ_i has exactly two distinct solutions λ and μ and $\lambda\mu = \eta_{n+1}^2$. By the Gauss equation, one may prove $R(X \wedge Y) = -1 + \lambda\mu = -1 + \eta_{n+1}^2$.

Conversely, choose the local tangent frame $\{e_1, e_2, \dots, e_n\}$ and the dual frame $\{\omega_1, \omega_2, \dots, \omega_n\}$ near p, such that $h_{ij} = 0, i \neq j$ and $h_{11} = h_{22} = \cdots = h_{rr} = \lambda \neq \mu = h_{r+1r+1} = \cdots = h_{nn}$ at p. Then $\eta_{n+1}^2 = \lambda\mu$. By (3.3),

$$\begin{aligned}
\mathrm{IV} &= \left(\eta_{n+1}^2 - 2\eta_{n+1}\lambda + \lambda^2\right)\left(\omega_1^2 + \cdots + \omega_r^2\right) \\
&\quad + \left(\eta_{n+1}^2 - 2\eta_{n+1}\mu + \mu^2\right)\left(\omega_{r+1}^2 + \cdots + \omega_n^2\right) \\
&= \left(\mu - 2\eta_{n+1} + \lambda\right) \lambda \left(\omega_1^2 + \cdots + \omega_r^2\right) \\
&\quad + \left(\lambda - 2\eta_{n+1} + \mu\right) \mu \left(\omega_{r+1}^2 + \cdots + \omega_n^2\right) \\
&= \left(\lambda - 2\eta_{n+1} + \mu\right)\mathrm{II}.
\end{aligned}$$

The sufficiency has been proved for H^{n+1}. Similarly, we can prove Theorem 3.2 for S_1^{n+1}.

REMARK. By (3.5), we know that the normal Gauss map of all totally umbilical hypersurfaces except totally geodesic hyperspheres in H^{n+1} are conformal. Similarly, for space-like hypersurfaces in S_1^{n+1}, since $\eta_{n+1} \neq 0$, the normal Gauss maps of all totally umbilical space-like hypersurfaces except totally geodesic space-like hypersurfaces are conformal.

For H^3 and S_1^3, by Theorem 3.2, we immediately get

THEOREM 3.3. *Let M be a 2-dimensional Riemannian manifold and $x\colon M \to H^3$ (resp. $x\colon M \to S_1^3$) be an immersed surface (resp. space-like surface) without umbilics. Then the normal Gauss map of $x(M)$ is conformal if and only if the Gauss curvature $K = -1 + \eta_3^2$ (resp. $K = 1 - \eta_3^2$).*

REMARK. In (Shi 2004, 2006), we assume that the second fundamental form is positive definite and induces the conformal structure on the surfaces in H^3. Here, the assumption with respect to the positive definite second fundamental form is dropped.

THEOREM 3.4. *Let M be a n-dimensional Einstein manifold and $x\colon M \to H^{n+1}$ (resp. $x\colon M \to S_1^{n+1}$) be an immersed hypersurface (resp. space-like hypersurface) with non-degenerate second fundamental form and without umbilics. If the normal Gauss map of $x(M)$ is conformal map, i.e. IV $= \rho$II, then $n = 2$ and $\rho = 2(H - \eta_3)$ (resp. $\rho = 2(H + \eta_3)$).*

PROOF. We only prove the Theorem for H^{n+1}. M is an Einstein manifold, so $Ric = \frac{S}{n}I$, where S is the scalar curvature of M. (3.1) becomes

$$\left(\eta_{n+1}^2 - (n-1) - \frac{S}{n}\right)\mathrm{I} + (nH - 2\eta_{n+1} - \rho)\mathrm{II} = 0.$$

Because $x(M)$ has no umbilics, we have

$$nH = 2\eta_{n+1} + \rho.$$

By Theorem 3.2 and its proof, we assume that $\lambda_1 = \cdots = \lambda_r = \lambda \neq \mu = \lambda_{r+1} = \cdots = \lambda_n$, then

$$r\lambda + (n-r)\mu = 2\eta_{n+1} + \rho.$$

By (3.6),

$$\lambda + \mu = 2\eta_{n+1} + \rho.$$

So $(r-1)\lambda + (n-r-1)\mu = 0$. By Theorem 3.2, λ and μ have same signature. So $r = 1$ and $n = 2$. Hence $\rho = 2H - 2\eta_3$.

4 A DUALITY FOR THE SURFACES IN H^3 AND S_1^3 WITH CONFORMAL NORMAL GAUSS MAPS

Let L^4 be the Minkowski 4-space with canonical coordinates X_0, X_1, X_2, X_3 and Lorentz-Minkowski scalar product $-X_0^2 + X_1^2 + X_2^2 + X_3^2$. The Minkowski model of H^3 is given by

$$H^3 = \left\{(X_0, X_1, X_2, X_3)\middle| - X_0^2 + X_1^2 + X_2^2 + X_3^2 = -1, X_0 > 0\right\}$$

and is identified with the upper half-space model R_+^3 of H^3 by

$$(x_1, x_2, x_3) = \left(\frac{X_1}{X_0 - X_3}, \frac{X_2}{X_0 - X_3}, \frac{1}{X_0 - X_3}\right).$$

Accordingly, the space-like normal vector of the surface in the Minkowski model of H^3 is

$$N = N_0\frac{\partial}{\partial X_0} + N_1\frac{\partial}{\partial X_1} + N_2\frac{\partial}{\partial X_2} + N_3\frac{\partial}{\partial X_3},$$

where

$$\begin{aligned} N_0 &= \frac{X_1}{X_0 - X_3}\eta_1 + \frac{X_2}{X_0 - X_3}\eta_2 + \frac{1 - X_0(X_0 - X_3)}{X_0 - X_3}\eta_3, \\ N_1 &= \eta_1 - X_1\eta_3, \ N_2 = \eta_2 - X_2\eta_3, \\ N_3 &= \frac{X_1}{X_0 - X_3}\eta_1 + \frac{X_2}{X_0 - X_3}\eta_2 + \frac{1 - X_3(X_0 - X_3)}{X_0 - X_3}\eta_3. \end{aligned}$$

We get

$$\eta_3 = \frac{N_0 - N_3}{X_3 - X_0}. \tag{4.1}$$

The Minkowski model of the de Sitter 3-space is defined as

$$S_1^3 = \{(X_0, X_1, X_2, X_3) | -X_0^2 + X_1^2 + X_2^2 + X_3^2 = 1\} \simeq S^2 \times R$$

and can be divided into three components as follows (cf. Aiyama and Akutagawa 2000),

$$\begin{aligned} S_- &= \{X \in S_1^3 \mid X_0 - X_3 < 0\} \simeq R^3, \\ S_0 &= \{X \in S_1^3 \mid X_0 - X_3 = 0\} \simeq S^1 \times R, \\ S_+ &= \{X \in S_1^3 \mid X_0 - X_3 > 0\} \simeq R^3. \end{aligned}$$

Identify S_- and S_+ with upper half-space model R_+^3 of the de Sitter 3-space by (cf. Aiyama and Akutagawa 2000)

$$(x_1, x_2, x_3) = \left(\frac{X_1}{|X_0 - X_3|}, \frac{X_2}{|X_0 - X_3|}, \frac{1}{|X_0 - X_3|} \right).$$

For space-like surface $X \colon M \to S_1^3$, let $U_- = X^{-1}(S_-)$ and $U_+ = X^{-1}(S_+)$, then $U_- \cup U_+$ is the open dense subset of M. On $U_- \cup U_+$, the time-like unit normal vector is

$$N = N_0 \frac{\partial}{\partial X_0} + N_1 \frac{\partial}{\partial X_1} + N_2 \frac{\partial}{\partial X_2} + N_3 \frac{\partial}{\partial X_3},$$

where

$$\begin{aligned} N_0 &= \frac{X_1}{X_0 - X_3}\eta_1 + \frac{X_2}{X_0 - X_3}\eta_2 - \frac{1 + X_0(X_0 - X_3)}{X_0 - X_3}\eta_3, \\ N_1 &= \eta_1 - X_1\eta_3, \; N_2 = \eta_2 - X_2\eta_3, \\ N_3 &= \frac{X_1}{X_0 - X_3}\eta_1 + \frac{X_2}{X_0 - X_3}\eta_2 - \frac{1 + X_3(X_0 - X_3)}{X_0 - X_3}\eta_3. \end{aligned}$$

We get

$$\eta_3 = \frac{N_0 - N_3}{X_3 - X_0}. \tag{4.2}$$

REMARK. In (Aiyama and Akutagawa 2000), the normal Gauss map of the space-like surface $X \colon M \to S_1^3$ is defined globally on M. Because of the density of U_- and U_+ in M, in this paper, we may consider that the normal Gauss map of the space-like surface $X \colon M \to S_1^3$ is defined on U_- and U_+.

Let $X \colon M \to H^3$ (*resp.* $X \colon M \to S_1^3$) be an immersed surface (*resp.* space-like surface). Parallel translating the space-like (*resp.* time-like) unit normal vector N to the origin of L^4, one gets the map $N \colon M \to S_1^3$ (*resp.* $N \colon M \to H^3$) which is usually called generalized Gauss map of $X \colon M \to H^3$ (*resp.* $X \colon M \to S_1^3$). The generalized Gauss map image can be considered as the surface in S_1^3 (*resp.* H^3).

THEOREM 4.1. (Kokubu 2005, Prop. 3.5). (1) *Let $X \colon M \to H^3$ be a 2-dimensional immersed surface. Then its generalized Gauss map $N \colon M \to S_1^3$ is a branched space-like immersion into S_1^3 with branch points where $K = -1$. And, when $K \neq -1$, the curvature of $N \colon M \to S_1^3$ is $K^* = \dfrac{K}{K+1}$ and the volume element is $dV_N = |K+1| dV_X$.*

(2) *Let* $X: M \to S_1^3$ *be a 2-dimensional space-like immersed surface. Then its generalized Gauss map* $N: M \to H^3$ *is a branched immersion into* H^3 *with branch points where* $K = 1$*. And, when* $K \neq 1$*, the curvature of* $N: M \to H^3$ *is* $K^* = \dfrac{K}{1-K}$ *and the volume element is* $dV_N = |1-K|dV_X$.

PROOF. In the context of this paper, we prove (2). For any $p \in M$, let $\{e_0, e_1, e_2, e_3\}$ be the orthonormal frame near p, such that $e_3 = X, e_0 = N$. Let $\{\omega_0, \omega_1, \omega_2, \omega_3\}$ be the dual frame. The connection 1-forms is $\omega_\alpha^\beta, \alpha, \beta = 0, 1, 2, 3$. The coefficients of the second fundamental form of $X: M \to S_1^3$ is given by $\omega_i^0 = h_{ij}\omega_j, h_{ij} = h_{ji}, i, j = 1, 2$. The induced metric of $N: M \to H^3$ is $ds_*^2 = \langle dN, dN\rangle = h_{ik}h_{jk}\omega_i\omega_j$. Choose the local tangent frame $\{e_1, e_2\}$ near p, such that $h_{ij} = \lambda_i\delta_{ij}$. Then $ds_*^2 = \lambda_1^2(\omega_1)^2 + \lambda_2^2(\omega_2)^2$. So, when $\lambda_1\lambda_2 \neq 0$, i.e. $K \neq 1$, $N(M)$ is an immersed surface into H^3. Its space-like unit normal vector is X and the second fundamental form is $\mathrm{II} = -\langle dX, dN\rangle = -\lambda_1(\omega_1)^2 - \lambda_2(\omega_2)^2$. By the Gauss equation, $K^* = -1 + \dfrac{1}{\lambda_1\lambda_2} = \dfrac{K}{1-K}$.

By Theorem 3.3, (4.1), (4.2) and Theorem 4.1, we get the following duality.

THEOREM 4.2. *Let* M *be a connected 2-dimensional manifold. Let* $X: M \to H^3$ *be an immersed surface with* $K \neq -1$ *and without umbilics and let* $N: M \to S_1^3$ *be a space-like surface with* $K \neq 1$ *and without umbilics. Suppose that* $N: M \to S_1^3$ *is the generalized Gauss map of* $X: M \to H^3$ *and vice versa. Then, the normal Gauss map of* $X: M \to H^3$ *is conformal if and only if one of* $N: M \to S_1^3$ *is conformal. And, at this time,* $dV_N = \left(\dfrac{N_0 - N_3}{X_3 - X_0}\right)^2 dV_X$.

REMARK. Like (Lawson 1970) for minimal surfaces in S^3, we call the generalized Gauss map $N: M \to S_1^3$ the polar variety of the immersed surface $X: M \to H^3$ with conformal normal Gauss map and vice versa.

5 WEIERSTRASS REPRESENTATION FORMULA

In this section, we give a Weierstrass representation formula for space-like surfaces in S_1^3 with conformal normal Gauss map. At first, we describe the normal Gauss map and the de Sitter Gauss map of the space-like surfaces in S_1^3. Take upper half-space model R_+^3 of S_1^3.

The normal Gauss map of the space-like surface $x: M \to S_1^3$ is given by

$$\widetilde{N} = \eta_1\frac{\partial}{\partial x_1}(e) + \eta_2\frac{\partial}{\partial x_2}(e) + \eta_3\frac{\partial}{\partial x_3}(e): M \to H^2(-1) \subset L^3.$$

By means of the stereographic projection from the north pole $(0, 0, 1)$ of $H^2(-1)$ to the (x_1, x_2)-plane identified with C, we get

$$g^S = \frac{\eta_1 + i\eta_2}{1 - \eta_3}: M \to C \cup \{\infty\}\backslash\{|z| = 1\},$$

which is also called the normal Gauss map of the space-like surface $x: M \to S_1^3 \cdot \widetilde{N}$ can be written as

$$\widetilde{N} = \left(-\frac{2Re(g^S)}{|g^S|^2 - 1}, -\frac{2\,\mathrm{Im}(g^S)}{|g^S|^2 - 1}, \frac{1 + |g^S|^2}{|g^S|^2 - 1}\right).$$

Next, we describe the definition of the de Sitter Gauss map for space-like surfaces in S_1^3 (in (Lee 2005), it is still called hyperbolic Gauss map), which is the analogue of Epstein and Bryant's hyperbolic

Gauss map for surfaces in H^3 (Epstein 1986, Bryant 1987, Shi 2004). The time-like geodesic is either the Euclidean equilateral half-hyperbola consisting of two branches which is orthonormal to the coordinate plane $\{(x_1, x_2, 0)|(x_1, x_2) \in R^2\}$ or the Euclidean straight line which is orthonormal to the above coordinate plane. For the space-like surface $x = (x_1, x_2, x_3)\colon M \to S_1^3$, at each point $x \in M$, the oriented time-like geodesic in S_1^3 passing through x with time-like tangent vector N meets $\{(x_1, x_2, 0)|(x_1, x_2) \in R^2\} \cup \{\infty\}$ at two points. Since the geodesic is oriented, we may speak of one of the two points as the initial point and the other one as the final point. Call the final point the image of the de Sitter Gauss map for $x(M)$ at the point x. Denote the de Sitter Gauss map by G^S. On the coordinate plane $\{(x_1, x_2, 0)|(x_1, x_2) \in R^2\}$, we introduce the natural complex coordinate $z = x_1 + ix_2$. Using the Euclidean geometry, as similar as done in the Theorem 5.1 of (Shi 2004), we get

$$G^S = x_1 + ix_2 + x_3 g^S. \tag{5.1}$$

Let $x = (x_1, x_2, x_3)\colon M \to H^3$ be an immersed surface with unit normal vector

$$N = x_3\eta_1 \frac{\partial}{\partial x_1} + x_3\eta_2 \frac{\partial}{\partial x_2} + x_3\eta_3 \frac{\partial}{\partial x_3}.$$

By the duality given in section 4, the generalized Gauss map of $x\colon M \to H^3$ is given, when $\eta_3 > 0$, by

$$N = \left(\frac{\eta_1}{\eta_3}x_3 - x_1,\ \frac{\eta_2}{\eta_3}x_3 - x_2,\ \frac{x_3}{\eta_3}\right)\colon M \to S_1^3, \tag{5.2}$$

and when $\eta_3 < 0$, by

$$N = \left(x_1 - \frac{\eta_1}{\eta_3}x_3,\ x_2 - \frac{\eta_2}{\eta_3}x_3,\ -\frac{x_3}{\eta_3}\right)\colon M \to S_1^3 \tag{5.3}$$

and in the Minkowski model of the de Sitter 3-space, their time-like unit normal vector is $X\colon M \to H^3$. Again by the duality given in section 4, a straightforward computation shows us that the normal Gauss map of $N\colon M \to S_1^3$ is given by

$$\widetilde{N} = \frac{\eta_1}{\eta_3}\frac{\partial}{\partial x_1}(e) + \frac{\eta_2}{\eta_3}\frac{\partial}{\partial x_2}(e) + \frac{1}{\eta_3}\frac{\partial}{\partial x_3}(e)\colon M \to H^2(-1).$$

So,

$$g^S = \frac{\frac{\eta_1}{\eta_3} + i\frac{\eta_2}{\eta_3}}{1 - \frac{1}{\eta_3}} = \frac{\eta_1 + i\eta_2}{\eta_3 - 1} = -g^H, \tag{5.4}$$

where $g^H\colon M \to C \cup \{\infty\}$ is exactly the normal Gauss map of $x\colon M \to H^3$ (Kokubu 1997, Shi 2004, 2006). From this, we also prove the Theorem 4.2.

By (5.1)–(5.4) and the Theorem 5.1 of (Shi 2004), we get that when $\eta_3 > 0$, i.e. $|g^S| > 1$,

$$G^S = -G^H, \tag{5.5}$$

and when $\eta_3 < 0$, i.e. $|g^S| < 1$,

$$G^S = G^H, \tag{5.6}$$

where G^H is exactly the hyperbolic Gauss map of $x: M \to H^3$ (Epstein 1986, Bryant 1987, Shi 2004).

In the following, we write respectively g^S and G^S as g and G.

By (5.2)–(5.6) and the Weierstrass representation for surfaces in H^3 with conformal normal Gauss map (Shi 2004), we get the Weierstrass representation formula for space-like surfaces in S_1^3 with conformal normal Gauss map.

THEOREM 5.1. *Let M be a simply connected Riemann surface. Given the map $G\colon M \to C \cup \{\infty\}$ and the nonconstant conformal map $g\colon M \to C \cup \{\infty\} \backslash \{|z| = 1\}$.*

(1) *When the holomorphic map $g\colon M \to C \cup \{\infty\} \backslash \{|z| = 1\}$ satisfies $|g| > 1$ and*

$$\frac{G_z}{g_z} > 0, \tag{5.7}$$

$$|g|^2|G_{\bar z}| > |G_z|, \tag{5.8}$$

$$G_{z\bar z} + \frac{\bar g_{\bar z}}{(|g|^4 - 1)\bar g} G_z - \frac{|g|^2 \bar g g_z}{|g|^4 - 1} G_{\bar z} = 0, \tag{5.9}$$

put

$$x_1 = \operatorname{Re}\left\{G - \frac{1 + |g|^2}{\bar g g_z} G_z\right\}, \tag{5.10}$$

$$x_2 = \operatorname{Im}\left\{G - \frac{1 + |g|^2}{\bar g g_z} G_z\right\}, \tag{5.11}$$

$$x_3 = \frac{1 + |g|^2}{|g|^2 g_z} G_z. \tag{5.12}$$

Then $x = (x_1, x_2, x_3)\colon M \to S_1^3$ is a space-like surface with de Sitter Gauss map G and holomorphic normal Gauss map g and Gauss curvature K satisfying $\sqrt{1 - K} = \frac{1+|g|^2}{|g|^2-1}$. And the conformal structure on M is induced by the negative definite second fundamental form. Conversely, any surface $x: M \to S_1^3$ with $\sqrt{1 - K} = \frac{1+|g|^2}{|g|^2-1} (= \eta_3)$ can be given by (5.10), (5.11), (5.12), *and the de Sitter Gauss map G and the normal Gauss map g must satisfy* (5.7), (5.8), (5.9), *where the conformal structure on M is induced by the negative definite second fundamental form.*

(2) *When the antiholomorphic map $g\colon M \to C \cup \{\infty\} \backslash \{|z| = 1\}$ without holomorphic points satisfies $|g| < 1$ and*

$$\frac{G_{\bar z}}{|g|^2 g_{\bar z}} > 0, \tag{5.13}$$

$$\frac{|g|^2 |G_z|}{|G_{\bar z}|} < 1, \tag{5.14}$$

$$G_{z\bar z} + \frac{\bar g_z}{(|g|^4 - 1)\bar g} G_{\bar z} - \frac{|g|^2 \bar g g_{\bar z}}{|g|^4 - 1} G_z = 0, \tag{5.15}$$

put

$$x_1 = \mathrm{Re}\left\{G - \frac{1+|g|^2}{\bar{g}g_{\bar{z}}}G_{\bar{z}}\right\}, \tag{5.16}$$

$$x_2 = \mathrm{Im}\left\{G - \frac{1+|g|^2}{\bar{g}g_{\bar{z}}}G_{\bar{z}}\right\}, \tag{5.17}$$

$$x_3 = \frac{1+|g|^2}{|g|^2 g_{\bar{z}}}G_{\bar{z}}. \tag{5.18}$$

Then $x = (x_1, x_2, x_3)\colon M \to S_1^3$ is a space-like surface with de Sitter Gauss map G and antiholomorphic normal Gauss map g and Gauss curvature K satisfying $\sqrt{1-K} = \frac{1+|g|^2}{1-|g|^2}$. And the conformal structure on M is induced by the negative definite second fundamental form. Conversely, any surface $x\colon M \to S_1^3$ with $\sqrt{1-K} = \frac{1+|g|^2}{1-|g|^2}(= -\eta_3)$ can be given by (5.16), (5.17), (5.18), *and the de Sitter Gauss map G and the normal Gauss map g must satisfy* (5.13), (5.14), (5.15), *where the conformal structure on M is induced by the negative definite second fundamental form.*

6 GRAPHS AND EXAMPLES

In this section,we give the examples of surfaces in S_1^3 with conformal normal Gauss map within the translational surfaces and the Euclidean ruled surfaces.

In H^3, the graph $(u, v, f(u, v))$ with conformal normal Gauss map satisfies the following fully nonlinear equation of Monge-Ampère type (Shi 2004, 2006)

$$f\big(f_{uu}f_{vv} - f_{uv}^2\big) + \big[\big(1+f_v^2\big)f_{uu} - 2f_u f_v f_{uv} + \big(1+f_u^2\big)f_{vv}\big] = 0. \tag{6.1}$$

Take upper half-space model of S_1^3. Consider the space-like graph $(u, v, f(u, v))$ in S_1^3 with $f_u^2 + f_v^2 < 1$. Its Gauss curvature is given by

$$K = 1 - \frac{f^2\big(f_{uu}f_{vv} - f_{uv}^2\big) - f\big[\big(1-f_v^2\big)f_{uu} + 2f_u f_v f_{uv} + \big(1-f_u^2\big)f_{vv}\big] + \big(1 - f_u^2 - f_v^2\big)}{\big(1 - f_u^2 - f_v^2\big)^2}.$$

So $K = 1 - \eta_3^2$ is equivalent to

$$f\big(f_{uu}f_{vv} - f_{uv}^2\big) - \big[\big(1-f_v^2\big)f_{uu} + 2f_u f_v f_{uv} + \big(1-f_u^2\big)f_{vv}\big] = 0, \tag{6.2}$$

where $f_u^2 + f_v^2 < 1$. This is the fully nonlinear equation of Monge-Ampère type which the space-like graph in S_1^3 with $K = 1 - \eta_3^2$ must satisfy.

REMARK. There exists a nice duality between the solutions of minimal surface equation

$$\big(1+f_v^2\big)f_{uu} - 2f_u f_v f_{uv} + \big(1+f_u^2\big)f_{vv} = 0$$

in R^3 and the ones of maximal surface equation

$$\big(1-f_v^2\big)f_{uu} + 2f_u f_v f_{uv} + \big(1-f_u^2\big)f_{vv} = 0$$

in Lorentz-Minkowski 3-space L^3 (Alías and Palmer 2001). Here, by the duality given by (5.2) (or (5.3)), we know that if $f(u, v)$ is a solution of (6.1), then the local graph of the space-like surface $(-ff_u - u, -ff_v - v, f\sqrt{1 + f_u^2 + f_v^2})$ (or $(ff_u + u, ff_v + v, f\sqrt{1 + f_u^2 + f_v^2})$) in S_1^3 satisfies (6.2). Conversely, if $f(u, v)$ is a solution of (6.2) with $f_u^2 + f_v^2 < 1$, then the local graph of the surface $(ff_u - u, ff_v - v, f\sqrt{1 - f_u^2 - f_v^2})$ (or $(u - ff_u, v - ff_v, f\sqrt{1 - f_u^2 - f_v^2})$) in H^3 satisfies (6.1).

Next, as similar as done in section 6 of (Shi 2004), we get the following Theorem.

THEOREM 6.1. *The nontrivial translational space-like surfaces with form* $f(u, v) = \phi(u) + \psi(v)$ *in* S_1^3 *with conformal normal Gauss map are given, up to a linear translation of variables, by*

$$f(u, v) = \sqrt{a^2 + u^2} \pm \sqrt{b^2 + v^2} \tag{6.3}$$

with $f_u^2 + f_v^2 < 1$, *where a and b are nonzero constants. The parameter form of these translational surfaces are locally given by*

$$x(u, v) = (a \sinh u, b \sinh v, a \cosh u + b \cosh v). \tag{6.4}$$

REMARK. We may check that the isometric transformation

$$\big(x_1, x_2, x_3\big) \to \big(x_1 \cos\theta - x_2 \sin\theta + a, x_1 \sin\theta + x_2 \cos\theta + b, x_3\big) \tag{6.5}$$

preserves the concept of the ruled surfaces and the conformality of the normal Gauss map of the space-like surface in S_1^3.

Considered as surfaces in 3-dimensional Minkowski space L^3, the space-like ruled surfaces in S_1^3 can be represented as $x(u, v) = \alpha(v) + u\beta(v)\colon D \to S_1^3$, where $D(\subset R^2)$ is a parameter domain and $\alpha(v)$ and $\beta(v)$ are two vector valued functions into L^3 corresponding to two curves in L^3. When β is locally nonconstant, without loss of generality we can assume that either $\langle\beta, \beta\rangle = 1$, $\langle\beta', \beta'\rangle = \pm 1$, and $\langle\alpha', \beta'\rangle = 0$ or $\langle\beta, \beta\rangle = 1$, $\langle\beta', \beta'\rangle = 0$, and $\langle\alpha', \beta\rangle = 0$, where $\langle\cdot, \cdot\rangle$ is the scalar product in L^3. As similar as done in Theorem 2 of (Shi 2006), we have

THEOREM 6.2. *Up to an isometric transformation* (6.5) *in* S_1^3, *every space-like ruled surface in* S_1^3 *with conformal normal Gauss map is locally a part of one of the following,*

(1) *ordinary Euclidean space-like planes in* S_1^3,

(2) $(u \cosh v, c \cdot \sinh v, u \sinh v)$, *for a constant* $c \neq 0$,

(3) $(c_2 \sinh v + u \cosh v, c_1 \sinh v, c_2 \cosh v + u \sinh v)$, *for constants* $c_1 \neq 0$ *and* $c_2 \neq 0$.

We should note that in the proof of Theorem 6.2, only when $\langle\beta', \beta'\rangle = -1$, we may get the nontrivial cases (2) and (3).

Locally, the ruled surfaces (2) and (3) in Theorem 6.2 can be represented as the graph $(u, v, f(u, v))$ as follows, COROLLARY. $f(u,v) = \pm\dfrac{c_1c_2+uv}{\sqrt{c_1^2+v^2}}$ *is a solution of equation (6.2), where* $c_1 \neq 0$ *and* c_2 *are constants.*

REMARK. In H^3, the translational surfaces

$$(a\cos u, b\cos v, a\sin u + b\sin v) \tag{6.6}$$

and the ruled surfaces

$$(u\cos v, c\cdot\sin v, u\sin v) \tag{6.7}$$

and

$$(-c_2\sin v + u\cos v, c_1\cdot\sin v, c_2\cos v + u\sin v) \tag{6.8}$$

with conformal normal Gauss map have been obtained (Shi 2004, 2006), where a, b, c, c_1 and c_2 are nonzero constants. Using (5.2) (or (5.3)) and Theorem 4.2, we may check that up to a isometric transformation (6.5) in S_1^3 ($\theta = \pm\frac{\pi}{2}$), (6.4) in Theorem 6.1 and (2) and (3) in Theorem 6.2 are, respectively, the polar varieties of (6.6), (6.7) and (6.8) and vice versa.

REMARK. Every geodesic of H^3, corresponding respectively to $u = 0$, $u = \pi$, $v = 0$ and $v = \pi$ on surfaces (6.6) and to $v = \frac{\pi}{2}$ on surfaces (6.7) and to $v = \pm\frac{\pi}{2}$ on surfaces (6.8) follow which $K = -1$ is mapped to a single point in S_0 by the generalized Gauss map.

7 TIME-LIKE SURFACES IN S_1^3 WITH CONFORMAL NORMAL GAUSS MAP

In this section, we state the similar results as above for time-like surfaces in S_1^3 without proofs.

Take upper-half space model of S_1^3. Let M be a 2-dimensional Lorentz surface and $x\colon M \to S_1^3$ be the time-like immersion with local coordinates u_1, u_2. The first and the second fundamental forms are given, respectively, by $\mathrm{I} = g_{ij}du_idu_j$ and $\mathrm{II} = h_{ij}du_idu_j$. The space-like unit normal vector is

$$N = x_3\eta_1\frac{\partial}{\partial x_1} + x_3\eta_2\frac{\partial}{\partial x_2} + x_3\eta_3\frac{\partial}{\partial x_3},$$

where $\eta_1^2 + \eta_2^2 - \eta_3^2 = 1$. We have the Weingarten formula

$$\frac{\partial\eta_A}{\partial u_k} = \frac{1}{x_3}\left(\eta_3\frac{\partial x_A}{\partial u_k} - g^{jl}h_{kl}\frac{\partial x_A}{\partial u_j}\right).$$

Left-translating N to $T_e(R_+^3)$, we obtain

$$\widetilde{N}\colon M \to S_1^2(1) \subset T_e(R_+^3), \qquad \widetilde{N} = L_{x^{-1}*}(N) = \eta_1\frac{\partial}{\partial x_1}(e) + \eta_2\frac{\partial}{\partial x_2}(e) + \eta_3\frac{\partial}{\partial x_3}(e),$$

which is called the normal Gauss map of time-like surface $x\colon M \to S_1^3$ (Aiyama and Akutagawa 2000). Call $\mathrm{IV} = \langle d\widetilde{N}, d\widetilde{N}\rangle$ the fourth fundamental form of the time-like surface $x\colon M \to S_1^3$. We have

$$\mathrm{IV} = \big(\eta_3^2 g_{ij} - 2\eta_3 h_{ij} + g^{kl}h_{ik}h_{jl}\big)du_idu_j.$$

Of course, we may also define the high-dimensional version of the fourth fundamental form for time-like hypersurfaces in $S_1^{n+1}(1)$.

THEOREM 7.1. *Let M be a 2-dimensional Lorentz surface and $x: M \to S_1^3$ be a time-like immersed surface without umbilics. Then the normal Gauss map of $x(M)$ is conformal if and only if the Gauss curvature $K = 1 + \eta_3^2$.*

In the Minkowski model of the de Sitter 3-space S_1^3, the generalized Gauss map $N: M \to S_1^3$ of the time-like surface $X: M \to S_1^3$ is a branched time-like immersion with branch points where $K = 1$.

THEOREM 7.2. *Let M be a connected 2-dimensional Lorentz surface. Let $X: M \to S_1^3$ be a time-like surface with $K \neq 1$ and without umbilics. If the normal Gauss map of $X: M \to S_1^3$ is conformal, then the normal Gauss map of its generalized Gauss map $N: M \to S_1^3$ is also conformal and vice versa.*

The time-like graph $(u, v, f(u, v))$ in S_1^3 with conformal normal Gauss map also satisfies the fully nonlinear equation of Monge-Ampère type (6.2) with $f_u^2 + f_v^2 > 1$.

THEOREM 7.3. *The nontrivial translational time-like surfaces with form $f(u, v) = \phi(u) + \psi(v)$ in S_1^3 with conformal normal Gauss map are given, up to a linear translation of variables, by*

(1) $f(u, v) = \sqrt{u^2 + a^2} \pm \sqrt{v^2 + b^2}$,

(2) $f(u, v) = \sqrt{u^2 - a^2} \pm \sqrt{v^2 - b^2}$,

(3) $f(u, v) = \sqrt{u^2 + a^2} \pm \sqrt{v^2 - b^2}$,

(4) $f(u, v) = \sqrt{u^2 - a^2} - \sqrt{v^2 + b^2}$,

(5) *Flaherty's time-like surfaces in S_1^3* (Milnor 1987) $f(u, v) = \pm u + \psi(v)$ *and* $f(u, v) = \pm v + \varphi(u)$, *where a and b are nonzero constants and $\psi'(v) \neq 0$ and $\varphi'(u) \neq 0$.*

REMARK. We may check that the isometric transformation (6.5) preserves the concept of the ruled surfaces and the conformality of the normal Gauss map of the time-like surfaces in S_1^3.

We may prove that the normal Gauss map of the time-like surfaces (2) and (3) in Theorem 6.2 are also conformal. In addition, for the time-like ruled surface $x(u, v) = \alpha(v) + u\beta(v)$ in S_1^3, we may also assume the remained four cases:

(i) $\langle\beta, \beta\rangle = -1$, $\langle\beta', \beta'\rangle = 1$, and $\langle\alpha', \beta'\rangle = 0$,

(ii) β is constant null vector,

(iii) β is constant and $\langle\beta, \beta\rangle = -1$, $\langle\alpha', \beta\rangle = 0$,

(iv) $\langle\beta, \beta\rangle = 0$, $\langle\beta', \beta'\rangle = 1$, and $\langle\alpha', \beta'\rangle = 0$, where $\langle\cdot, \cdot\rangle$ is the scalar product in L^3. Hence, we have

THEOREM 7.4. *Up to an isometric transformation* (6.5) *in S_1^3, every time-like ruled surface in S_1^3 with conformal normal Gauss map is locally a part of one of the following,*

(1) *ordinary Euclidean time-like planes in* S_1^3,

(2) *ordinary Euclidean generalized cylinder* $x(u,v) = \alpha(v) + u\beta$, *where* $\beta = (0,0,1)$ *and* $\alpha(v)$ *is arbitrary curve in* L^3 *with* $\langle\alpha',\alpha'\rangle > 0$ *and* $\langle\alpha',\beta\rangle = 0$,

(3) $(u\cosh v, c\cdot\sinh v, u\sinh v)$, *for a constant* $c \neq 0$,

(4) $(c_2\sinh v + u\cosh v, c_1\sinh v, c_2\cosh v + u\sinh v)$, *for constants* $c_1 \neq 0$ *and* $c_2 \neq 0$,

(5) $(u\sinh v, c\cdot\cosh v, u\cosh v)$, *for a constant* $c \neq 0$,

(6) $(c_2\cosh v + u\sinh v, c_1\cosh v, c_2\sinh v + u\cosh v)$, *for constants* $c_1 \neq 0$ *and* $c_2 \neq 0$,

(7) *Flaherty's time-like surfaces in* S_1^3 (Milnor 1987), $x(u,v) = \alpha(v) + u\beta$, *where* $\beta = (1,0,1)$ *and* $\alpha(v)$ *is arbitrary curve in* L^3 *with* $\langle\alpha',\beta\rangle \neq 0$.

We should note that in the proof of Theorem 7.4, only for case (i) and (ii), we may get the surfaces (5), (6), (7) in Theorem 7.4. For case (iv), we may assume $\beta(v) = (\rho(v)\cos\theta(v), \rho(v)\sin\theta(v), \rho(v))$ with $\rho^2(\theta')^2 = 1$. Next, we get a contradictory system of equations.

Locally, the ruled surfaces (5) and (6) in Theorem 7.4 can be represented as the graph $(u, v, f(u,v))$ as follows,

COROLLARY. $f(u,v) = \pm\dfrac{c_1c_2 - uv}{\sqrt{v^2 - c_1^2}}$ *is a solution of equation (6.2), where* $c_1 \neq 0$ *and* c_2 *are constants.*

REMARK. The totally umbilical time-like surfaces in S_1^3 given by $(x_1 - a)^2 + (x_2 - b)^2 - (x_3 - c)^2 = R^2$, where constants $c \neq 0$ and $R > 0$, are Euclidean ruled surfaces but $K \neq 1 + \eta_3^2$. Their normal Gauss maps are also conformal.

REMARK. Up to a isometric transformation (6.5) in S_1^3 ($\theta = \pm\frac{\pi}{2}$), the time-like surfaces (3) and (4) in Theorem 7.4 are, respectively, the polar varieties of the time-like surfaces (5) and (6) in Theorem 7.4 and vice versa. The similar result also holds for the time-like surfaces in Theorem 7.3. Generally, if $f(u,v)$ is a solution of (6.2) with $f_u^2 + f_v^2 > 1$, then the local graph of the time-like surface $(ff_u - u, ff_v - v, f\sqrt{f_u^2 + f_v^2 - 1})$ (or $(u - ff_u, v - ff_v, f\sqrt{f_u^2 + f_v^2 - 1})$) in S_1^3 also satisfies (6.2).

REMARK. When we do not assume that $f > 0$, (6.3) and

$$f(u,v) = \pm\frac{c_1c_2 + uv}{\sqrt{c_1^2 + v^2}} \quad \text{and} \quad f(u,v) = \pm u + \psi(v) \quad \text{and} \quad f(u,v) = \pm v + \varphi(u)$$

with $\psi'(v) \neq 0$ and $\varphi'(u) \neq 0$ are all nontrivial entire solutions of the equation (6.2) defined on R^2. In addition, the cone $f(u,v) = \sqrt{u^2 + v^2}$ is also the special solution of the equation (6.2), but its graph is the light-like surface. By Omori-Yau's Maximum Principle (Omori 1967, Yau 1975), there exist no entire solution $f(u,v)$ of (6.2) satisfying $f_u^2 + f_v^2 > 1$ and $f > 0$ on R^2. Does there exist nontrivial entire solutions of equation (6.2) defined on R^2 satisfying $f_u^2 + f_v^2 < 1$ and $f > 0$?

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RESUMO

Neste artigo, introduzimos a quarta forma fundamental de uma hipersuperfície em H^{n+1} de uma hipersuperfície tipo-espaço em S_1^{n+1}, e discutimos a conformalidade da aplicação normal de Gauss de tais hipersuperfícies. Em particular, investigamos o caso de superfícies com aplicação normal de Gauss conforme em H^3 e S_1^3, e provamos um teorema de dualidade. Apresentamos uma representação de Weierstrass para superfícies tipo-espaço em S_1^3 com aplicação de Gauss conforme. Enunciamos também resultados semelhantes para superfícies tipo-tempo em S_1^3. São dados alguns exemplos de superfícies em S_1^3 com aplicações de Gauss conformes, e é deduzida uma equação totalmente não-linear do tipo Monge-Ampère para gráficos em S_1^3 com aplicações de Gauss conformes.

Palavras-chave: quarta forma fundamental, aplicação normal de Gauss conforme, aplicação de Gauss generalizada, propriedade de dualidade, aplicação de Gauss de Sitter, equação de Monge-Ampère.

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